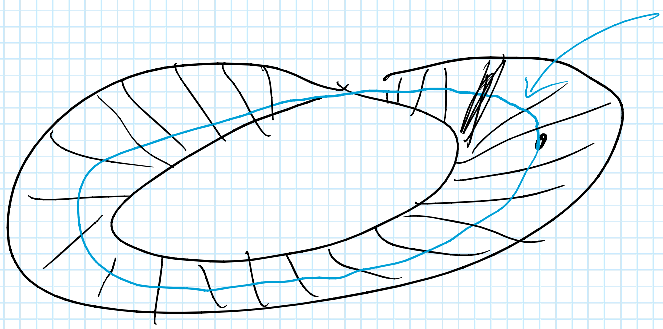
Wednesday, September 9, 2020 8:24 AM

ARCHIMEDES PALIMPSEST

CONSIDER THE MÖBIUS STRIP X



$A \approx S^1$

x_0

$A \underset{g}{\overset{f}{\rightleftarrows}} X$

CLAIM f INDUCES AN ISOMORPHISM IN π_1. $gf : A \to A$ IS THE IDENTITY 1_A.

$fg \simeq 1_X$

DEF A MAP $X \xrightarrow{f} Y$ IS A <u>HOMOTOPY EQUIVALENCE</u> IF THERE IS A MAP $Y \xrightarrow{g} X$ SUCH THAT

$$gf : X \to X \quad \text{AND} \quad fg : Y \to Y$$

ARE HOMOTOPIC TO THE IDENTITY.

EXAMPLE $\quad pt \underset{g}{\overset{f}{\rightleftarrows}} \mathbb{R}^n \qquad gf = 1_{pt}$

$fg \simeq 1_{\mathbb{R}^n}$

$pt \longmapsto 0$

POINTED ANALOG:

A MAP $f : (X, x_0) \to (Y, y_0)$ IS A POINTED HOMOTOPY EQUIVALENCE IF THERE IS A MAP $g : (Y, y_0) \to (X, x_0)$ WITH

WITH . . . . ,

PROP A POINTED HOMOTOPY EQUIV. INDUCES AN ISOMORPHISM IN π_1 — .

DIGRESSION INTO CATEGORY THEORY. INVENTED IN 1945 BY EILENBERG + MAC LANE.

DEF A CATEGORY $\mathcal{C}$ CONSISTS OF

1. A COLLECTION OF OBJECTS
2. FOR EACH PAIR OF OBJECTS X, Y, THERE IS SET OF MORPHISMS $X \longrightarrow Y$ SUCH THAT

   a) $X \xrightarrow{f} Y \xrightarrow{g} Z$ DETERMINES A COMPOSITE MORPHISM $X \xrightarrow{gf} Z$

   b) IF $X = Y$, THEN THERE IS AN IDENTITY MORPHISM $X \xrightarrow{1_X} X$ SUCH

$$\begin{tikzcd} W \arrow[d, "f"] \arrow[dr, "1_X \circ f = f"] & \\ X \arrow[r, "1_X"] \arrow[dr, "g = g \circ 1_X"'] & X \arrow[d, "g"] \\ & Y \end{tikzcd}$$

   c) COMPOSITION IS ASSOCIATIVE

$$\begin{tikzcd} W \arrow[r, "gf"] \arrow[d, "f"'] & Y \arrow[d, "h"] \\ X \arrow[ur, "g"] \arrow[r, "hg"'] & Z \end{tikzcd}$$

$$(hg)f = h(gf)$$

EXAMPLE ① $\mathcal{C} = \mathcal{S}et$, THE CATEGORY

EXAMPLE ① $\mathcal{C} = \mathcal{Set}$, THE CATEGORY OF SETS AND MAPS BETWEEN THEM.

② $\mathcal{C} = \mathcal{T}op$, THE CATEGORY OF TOPOLOGICAL SPACES AND CONTINUOUS MAPS

③ $\mathcal{T}op_0$ = THE CATEGORY OF POINTED SPACES AND POINTED CONTINUOUS MAPS.

④ $\mathcal{Ab}$ = CATEGORY OF ABELIAN GROUPS AND GROUP HOMOMORPHISMS.

THE OBJECTS IN A CATEGORY NEED NOT BE SETS WITH STRUCTURE.

⑤ $\mathcal{H}_0$ = HOMOTOPY CATEGORY OF (POINTED) TOPOLOGICAL SPACES

OBJECTS ARE TOP. SPACES

MORPHISMS ARE HOMOTOPY CLASSES OF CONT. MAPS.

DEF GIVEN TWO CATEGORIES $\mathcal{C}$ AND $\mathcal{D}$, A FUNCTOR

DEF. GIVEN TWO CATEGORIES $\mathcal{C}$ AND $\mathcal{D}$, A <u>FUNCTOR</u>

$\mathcal{C} \xrightarrow{F} \mathcal{D}$ IS A RULE THAT ASSIGNS TO EACH OBJECT X IN $\mathcal{C}$ AN OBJECT $F(X)$ IN $\mathcal{D}$ AND TO EACH MORPHISM $X \xrightarrow{f} Y$ IN $\mathcal{C}$ A MORPHISM $F(X) \xrightarrow{F(f)} F(Y)$ IN $\mathcal{D}$ SUCH THAT $F(1_X) = 1_{F(X)}$ AND

$$F(gf) = F(g)F(f).$$

EXAMPLE:

① $\mathcal{C} = Top_0$ = CATEGORY OF POINTED TOP. SPACES

$\mathcal{D} = Grp$ = CATEGORY OF GROUPS

$F = \pi_1(-)$

② $\mathcal{C} = Top_0 \xrightarrow{F_1} \mathcal{D} = Top \xrightarrow{F_2} \mathcal{E} = Set$

"FORGETFUL" FUNCTORS

$$Ab \xrightarrow{F_3} Grp \xrightarrow{F_4} Set$$

$Ab \longrightarrow Grp \longrightarrow Set$

(3) $Set \xrightarrow{F} Ab \xrightarrow{U} Set$

$X \longmapsto$ FREE ABELIAN GROUP GENERATED BY X.

G = FORGETFUL FUNCTOR

OBSERVATION

LET X BE A SET ~~so~~ $F(X)$ IS IN Ab

FOR AN ABELIAN GROUP A, $U(A)$ IS A SET

A MAP FROM $X \xrightarrow{f} U(A)$

THIS DETERMINES A HOMOMORPHISM FROM $F(X) \xrightarrow[F(f)]{} F(U(A)) \longrightarrow A$

DEF A PAIR OF FUNCTORS

$$X \in \mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D} \ni Y$$

F IS THE LEFT ADJOINT OF G

G IS THE RIGHT ADJOINT OF F

IS <u>ADJOINT</u> IF

$F \dashv G$

$\mathcal{C}(X, U(Y)) :=$ SET OF MORPHISMS IN $\mathcal{C}$ FROM X TO $U(Y)$

$\Phi^{-1}_{X,Y} \uparrow \quad \cong \downarrow \phi_{X,Y}$

$\mathcal{D}(F(X), Y) :=$ SET OF MORPHISMS IN $\mathcal{D}$ FROM $F(X)$ TO Y

$\mathcal{D}(F(X), Y) :=$ SET OF MORPHISMS IN $\mathcal{D}$ FROM $F(X)$ TO Y

$\dashv$ IS THE KAN TURNSTILE

EXAMPLE $\mathcal{C} = \mathcal{S}et$, $\mathcal{D} = \mathcal{A}b$

F = FREE ABELIAN GROUP FUNCTOR

U = FORGETFUL FUNCTOR

$$\boxed{\mathcal{S}et(X, U(Y)) \cong \mathcal{A}b(F(X), Y)}$$

X IS A SET

Y IS AN AB. GROUP

$$\begin{tikzcd} X \in \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \ni Y \arrow[l, bend left, "U"] \end{tikzcd}$$

$$\mathcal{D}(F(X), Y) \cong \mathcal{C}(X, U(Y))$$

a) SUPPOSE $Y = F(X)$ IN $\mathcal{D}$

$$\underset{\substack{\cup \\ 1_{F(X)}}}{\mathcal{D}(F(X), F(X))} \xrightarrow[\cong]{\Phi_{X,F(X)}} \mathcal{C}(X, U(F(X)))$$

UNIT OR COUNIT

$X \longrightarrow U\,F(X)$

$\cup$

$x \longmapsto x \in F(X)$

b) SUPPOSE $X = U(Y)$

$$\mathcal{D}(F(U(Y)), Y) \cong \mathcal{C}(U(Y), U(Y))$$

$$F(U(Y)) \to Y \longleftarrow\!\!\shortmid\; 1_{U(Y)}$$