BORSUK-ULAM THEOREM

Wednesday, September 23, 2020 8:41 AM

LET $f: S^2 \longrightarrow \mathbb{R}^2$, $S^2 \subset \mathbb{R}^3$. THEN $\exists x \in S^2$ WITH $f(x) = f(-x)$

PROOF: ASSUME $\nexists$ SUCH POINT, THEN

LET $g(x) = \dfrac{f(x) - f(-x)}{|f(x) - f(-x)|}$ = UNIT VECTOR $\in S^1$

THEN $g(-x) = -g(x)$. $g: S^2 \longrightarrow S^1$ IS A MAP COMMUTING WITH THE ANTIPODAL ACTION OF C_2. WILL SHOW SUCH A MAP CANNOT EXIST.

DEFINE

$$[0,1] \xrightarrow{h} S^2 \xrightarrow{g} S^1$$

$$s \longmapsto (\cos 2\pi s, \sin 2\pi s, 0)$$

FOR $0 \le s \le 1/2$, $gh(s + 1/2) = -gh(s)$

$$\begin{tikzcd}
 & & (\mathbb{R}, 0) \arrow[d, "p"] & \{(2n+1)/2,\ n \in \mathbb{Z}\} \arrow[d] \\
([0,1], 0) \arrow[urr, dashed, "\tilde{h}"] \arrow[r, "h"] & S^2 \arrow[r, "g"] & (S^1, (1,0)) & (-1,0)
\end{tikzcd}$$

$$1/2 \longmapsto (-1,0)$$

$\tilde{h}(s + 1/2) = \tilde{h}(s) + \dfrac{q}{2}$ WHERE $q = 2n+1$ IS AN ODD INTEGER

$$\tilde{h}(1) = \tilde{h}(1/2) + q/2 = \tilde{h}(0) + q/2 + q/2 = \tilde{h}(0) + q$$

$\tilde{h}$ IS A PATH FROM 0 TO AN ODD INTEGER q. gh REPRESENTS A NON TRIVIAL ELT IN $\pi_1(S^1)$

BUT THE EQUATOR $h([0,1])$ BOUNDS

BUT THE EQUATOR $h([0,1])$ BOUNDS A DISK IN S^2, SO gh MUST BE NULL HOMOTOPIC. CONTRADICTION

QED

HAM SANDWICH THEOREM.

GIVEN 3 DISJOINT COMPACT SUBSETS OF $\mathbb{R}^3$ (BREAD, HAM + CHEESE), $\exists$ A PLANE (THE KNIFE) WHICH BISECTS EACH SUBSET, I.E. CUTS IT INTO TWO SUBSETS OF EQUAL VOLUME.

PROOF: LET THE SUBSETS BE K_1, K_2 AND K_3. GIVEN A UNIT VECTOR $x \in \mathbb{R}^3$, $\exists$ A PLANE $\perp$ TO x WHICH BISECTS K_1. (DETAILS OMITTED) THESE PLANES VARY CONTINUOUSLY WITH.

WILL DEFINE A MAP $f: S^2 \to \mathbb{R}^2$ AS FOLLOWS

$S^2 \ni x \longmapsto$ PLANE BISECTING K_1 $\longmapsto$ ((MEASURE OF K_2 ABOVE PLANE) − MEASURE OF K_2 BELOW , SAME FOR K_3)

NOTE x AND $-x$ DEFINE THE SAME

BISECTING PLANE OF K_i

THIS MEANS $f(-x) = -f(x)$
FOR ALL $x \in S^2$

BORSUK-ULAM SAYS
$\exists\, x \in S^2$ WITH $f(-x) = f(x)$
THIS MEANS $f(x) = -f(x) = (0,0)$

THE CORRESPONDING PLANE
BISECTS K_1, K_2 AND K_3
AS DESIRED. QED

NEW TOPIC: VAN KAMPEN THEOREM

SUPPOSE $X = A \cup B$ WHERE X, A, B AND $A \cap B$ ARE PATH CONNECTED

LET $x_0 \in A \cap B$

$$\begin{tikzcd}
 & \pi_1(A) \arrow[dr, "f"] \arrow[drr, bend left, "f'"] & & \\
\pi_1(A \cap B) \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \pi_1(X) \arrow[r, dashed, "h"] & G \\
 & \pi_1(B) \arrow[ur, "g"'] \arrow[urr, bend right, "g'"'] & &
\end{tikzcd}$$

G = ANY GROUP
$f'\alpha = g'\beta$

THEN $\pi_1(X)$ IS THE PUSHOUT OF THIS DIAGRAM

RECALL PUSHOUTS IN A CATEGORY $\mathcal{C}$

SUPPOSE WE HAVE A DIAGRAM

V —— f'

SUPPOSE WE HAVE A DIAGRAM:

$$\begin{tikzcd} & X_1 \arrow[dr, "f"] \arrow[drr, bend left, "f'"] & & \\ X_0 \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & P \arrow[r, dashed, "h"] & P' \\ & X_2 \arrow[ur, "g"'] \arrow[urr, bend right, "g'"'] & & \end{tikzcd}$$

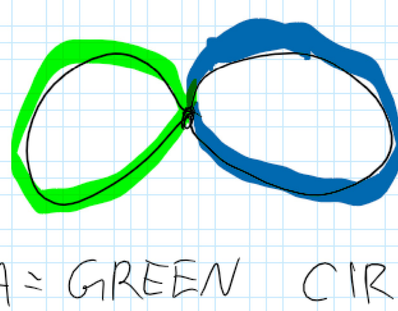
THE PUSHOUT P (IF IT EXISTS) RECIEVES MAPS f AND g S.T. $f\alpha = g\beta$ AND GIVEN ANY OTHER SUCH OBJECT P' WITH $f'\alpha = g'\beta$, $\exists!\, h : P \longrightarrow P'$ WITH $f' = hf$ AND $g' = gf$.

IT IS KNOWN THAT PUSHOUTS EXISTS IN Top = TOP. SPACES

Grp = GROUPS.

EXAMPLES

$X = S^1 \vee S^1$



A = GREEN CIRCLE

B = BLUE CIRCLE

$A \cap B$ = point

$Z = $

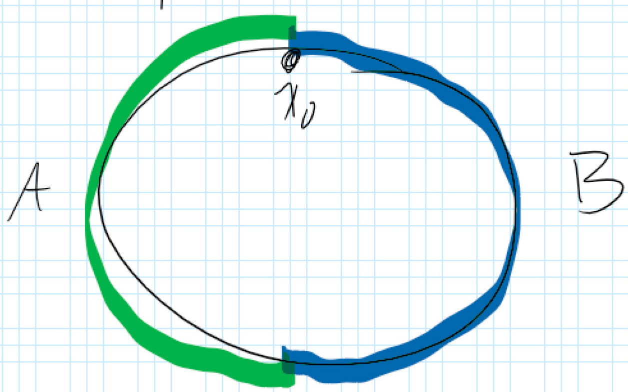
$A \cap B = \text{point}$

$$\begin{tikzcd} & \mathbb{Z} = \pi_1 A \arrow[dr] & \\ 0 = \pi_1(A \cap B) \arrow[ur] \arrow[dr] & & \text{PUSHOUT} \\ & \mathbb{Z} = \pi_1 B \arrow[ur] & \end{tikzcd}$$

PUSHOUT = FREE GROUP ON TWO GENERATORS

FALSE EXAMPLE

$X = S^1$



$A \cap B = 2$ pts BAD

$\pi_1 A$, $\pi_1 B$ AND $\pi_1(A \cap B)$ ARE ALL TRIVIAL

THE PUSHOUT IS TRIVIAL.

VAN KAMPEN THEOREM DOES NOT APPLY

EXAMPLE

A = MOBIUS STRIP $\simeq S^1$

= SURFACE BOUNDED BY S^1

$B = D^2 =$ "

$X = A \underset{\text{BOUNDARY}}{\cup} B \approx \mathbb{RP}^2$ $\quad x_0 \in S^1$

$A \cap B = S^1$ = COMMON BOUNDARY

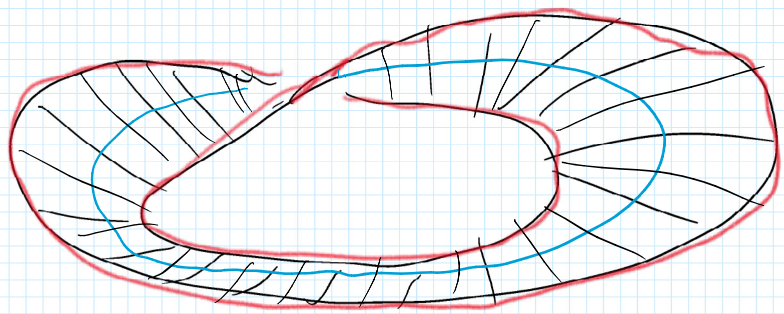
$\mathbb{Z} = \alpha \to \pi_1(A) = \mathbb{Z} \ni \mu$ $\quad b'$

$$\begin{tikzcd}
 & \pi_1(A) = \mathbb{Z} \ni \mu \arrow[dr, "f"] \arrow[drr, bend left, "f'"] & & \\
\lambda \in \mathbb{Z} = \pi_1(A \cap B) \arrow[ur, "2 = \alpha"] \arrow[dr, "\beta"'] & & \pi_1(\mathbb{R}P^2) & G \\
 & \pi_1(B) = 0 \arrow[ur, "g = 0"'] \arrow[urr, bend right, "g' = 0"'] & &
\end{tikzcd}$$

$\alpha(\lambda) = 2\mu$

$f\alpha(\lambda) = g\beta(\lambda) = 0$

$f\alpha(\lambda) = f(2\mu)$

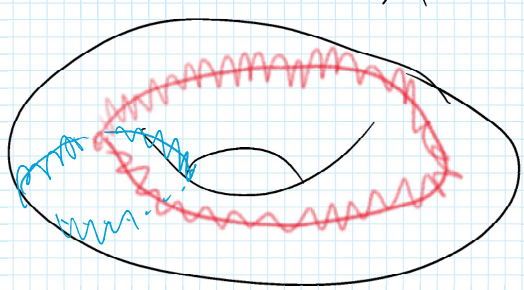


A∩B

$S^1 \simeq A$

IT FOLLOWS THAT $\pi_1 \mathbb{R}P^2 \cong \mathbb{Z}/2$.

EXAMPLE $\quad X = $ TORUS



A = NEIGHBORHOOD OF $S^1 \vee S^1$

$B = X - \text{int } A$

$= D^2$

$A \cap B = S^1 = \partial D^2 = \partial A$

$$\begin{tikzcd}
 & \pi_1(A) = F_2 \arrow[dr] & \\
\mathbb{Z} = \pi_1(A \cap B) \arrow[ur, "\alpha"] \arrow[dr] & & \pi_1(X) \\
 & \pi_1(B) = 0 \arrow[ur] &
\end{tikzcd}$$

$\pi_1(X) = F_2 /$ NORMAL SUBG GENERATED BY IMAGE OF α

$= ???$

$0 \qquad =$