IN THE COFFEE CUP TRICK, THE MOTION OF THE CUP CAN BE DESCRIBED BY A CLOSED IN $SO(3)$. THE TRICK SHOW THAT IT IS HOMOTOPIC TO THE CONSTANT PATH. MY ARM/BODY GIVES US THE DESIRED HOMOTOPY

THE CUP'S PATH LIES IN $SO(2)$, BUT THE HOMOTOPY LIES IN $SO(3)$.

$SO(2) \approx S^1$, $\pi_1(S^1) \cong \mathbb{Z}$ (TO BE PROVED LATER). THE CLOSED PATH REPRESENTS 2 ROTATIONS, HENCE $2 \times$ GENERATOR.

$$\begin{array}{ccc} \alpha \in \mathbb{Z} \cong \pi_1 SO(2) & \longrightarrow & \pi_1(SO(3)) \cong \mathbb{Z}/2 \\ 2\alpha & \longmapsto & 0 \end{array}$$

<u>WHAT THE SPACE $SO(3)$??</u>

WILL COME BACK TO THIS.

<u>DEF</u> $\mathbb{R}P^n$, REAL PROJECTIVE n-SPACE, IS THE SET OF LINES THRU THE ORIGIN IN $\mathbb{R}^{n+1}$.

LET $U \subset \mathbb{R}^{n+1}$ BE ANY OPEN SUBSET.

LET $U' \subset \mathbb{R}P^n$ BE THE SET OF

LET $U' \subset \mathbb{RP}^n$ BE THE SET OF LINES INTERSECTING. THESE ARE ALL OPEN IN $\mathbb{RP}^n$.

EACH $x \in \mathbb{RP}^n$ DETERMINES A PAIR OF ANTIPODAL POINTS IN S^n.

$$\text{LET } S^n_+ := \left\{ (x_0, x_1 \ldots x_n) \in \mathbb{R}^{n+1} : \begin{array}{l} \sum x_i^2 = 1 \\ x_0 \geq 0 \end{array} \right\} \approx \underset{\substack{\| \\ n\text{-DIMENSIONAL BALL OR DISK}}}{\overset{\substack{B^n \\ \|}}{D^n}}$$

= "NORTHERN HEMISPHERE"

$$S^{n-1} \approx S^n_0 := \{ (\cdots) \in S^n : x_0 = \}$$

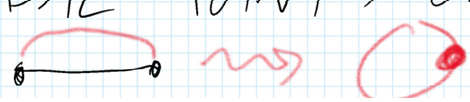
= "EQUATOR"

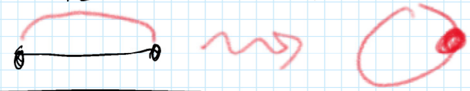
THE EQUATION $x_0 = 0$ DEFINES A HYPERPLANE, A LINEAR n-DIMENSIONAL SUBSPACE OF $\mathbb{R}^{n+1}$.

A LINE THRU 0 NOT IN IT INTERSECTS S^n_+ IN A UNIQUE POINT.

A LINE THRU 0 IN S^n_0 INTERSECTS S^n_+ IN TWO ANTIPODAL POINTS ON ITS BOUNDARY.

$\boxed{\text{IT FOLLOWS THAT } \mathbb{RP}^n \text{ IS A TOPOLOGICAL QUOTIENT OF } D^n \text{ OBTAINED BY INDENTIFY ANTIPODAL POINTS ON ITS BOUNDARY.}}$

RP^1 

BOUNDARY. RP^1 

FACT: $\mathbb{RP}^2$ IS <u>NOT</u> HOMEOMORPHIC TO A SUBSPACE OF $\mathbb{R}^3$.

IT CANNOT BE EMBEDDED IN $\mathbb{R}^3$.

HOWEVER THERE IS A MAP $\mathbb{RP}^2 \hookrightarrow \mathbb{R}^3$ THAT IS LOCALLY 1-1. THIS IS <u>BOY'S SURFACE</u> (LOOK IT UP!)

<u>THEOREM</u> $SO(3) \approx \mathbb{RP}^3$

PROOF: LET $\gamma \in SO(3)$ WITH $\gamma \neq e$

γ IS A ROTATION OF $\mathbb{R}^3$. IT HAS AN AXIS IN S^2 ABOUT WE ARE ROTATING COUNTERCLOCKWISE THRU ANGE θ, $0 < \theta \leq \pi$.

WE CAN DEFINE A MAP

$$D^3 \xrightarrow{f} SO(3)$$

$$0 \longmapsto e = \text{INDENTITY ELEMENT}$$

For $x \in S^2$, AND $0 < m \leq 1$, $mx \in D^3$

$f(mx)$ = ROTATION ABOUT x THRU ANGLE πm

THIS MAP IS ONTO

THRU ANGLE πM

THIS MAP IS ONTO

NOTE $f(x) = f(-x)$ FOR $|x| = 1$

THE MAP IS 2-1 ON THE BOUNDARY, SO IT FACTORS THRU THE QUOTIENT $\mathbb{RP}^3$.

$$\begin{tikzcd} D^3 \arrow[rr, "f \text{ ONTO}"] \arrow[dr] & & SO(3) \\ & \mathbb{RP}^3 \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

$\bar f$ 1-1 AND ONTO

HENCE $\bar{f}$ IS A HOMEOMORPHISM.

QED

IT IS ALSO KNOWN THAT

$SO(4) \approx SO(3) \times S^3$ (QUATERNIONS USED HERE)

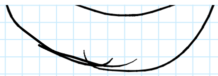
$SO(n) = ???$ FOR $n \geq 5$.

IT IS A COMPACT MANIFOLD OF DIMENSION $\binom{n}{2}$.

MORE ABOUT $SO(3) \approx \mathbb{RP}^3$

x_0 = NORTH POLE

CONSIDER PATH p FROM x_0 THRU THE CENTER TO $-x_0$, THE SOUTH POLE



TO $-x_0$, THE SOUTH POLE
IT IS A CLOSED PATH.

$$\begin{array}{ccc} SO(2) & \longrightarrow & SO(3) \\ \cong & & \cong \\ S^1 & \longrightarrow & \mathbb{RP}^3 \end{array}$$

THIS IS THE CLOSED PATH ABOVE.

LET $\alpha \in \pi_1 \mathbb{RP}^3$ BE THE CLASS OF THIS PATH.

TWICE THIS PATH IS HOMOTOPIC TO ONE ON THE BOUNDARY OF D^3

NORTH POLE $\longrightarrow$ ROCHESTER $\longrightarrow$ SOUTH POLE $\longrightarrow$ ANTI ROCHESTER $\longrightarrow$ NORTH POLE

NULL HOMOTOPIC.