Wednesday, August 26, 2020 12:38 PM

CONSIDER $O(n)$, THE n-DIMENSIONAL ORTHOGONAL GROUP, THE SET OF INVERTIBLE $n \times n$ MATRICES M FOR WHICH $M^{-1} = M^t$ = TRANSPOSE. EACH ROW VECTOR HAS LENGTH 1, AND ANY 2 ARE ORTHOGONAL. APPLYING SUCH AN M TO THE STANDARD BASIS OF $\mathbb{R}^n$ GIVES ANOTHER ORTHONORMAL BASIS. THERE IS A BIJECTION

$$O(n) \longleftrightarrow \left\{ \begin{matrix} \text{ORTHONORMAL} \\ \text{BASES OF } \mathbb{R}^n \end{matrix} \right\}$$

IF $M \in O(n)$, THEN $\det(M) = \pm 1$. WE GET A GROUP HOMOMORPHISM

$$SO(n) \to O(n) \xrightarrow{\det} \{\pm 1\}$$

$SO(n)$ = SPECIAL ORTHOGONAL GROUP = GROUP OF ROTATIONS IN $\mathbb{R}^n$.

WE ALSO HAVE HOMOMORPHISMS

$$\begin{array}{ccccc} \text{rotation in } \mathbb{R}^n & SO(n) & \to & O(n) & M \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ \text{FIX LAST BASIS VECTOR} & SO(n+1) & \to & O(n+1) & \left[\begin{array}{c|c} M & 0 \\ \hline 0 & 1 \end{array}\right] \end{array}$$

WE CAN TOPOLOGIZE $O(n)$ AS FOLLOWS. $O(n) \subset \mathbb{R}^{(n^2)}$

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IT HAS THE SUBSPACE TOPOLOGY.

IT IS CLOSED AND COMACT.

$$n=2: \quad SO(2) = \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} : 0 \le \theta \le 2\pi \right\}$$

$$SO(3) = ???$$

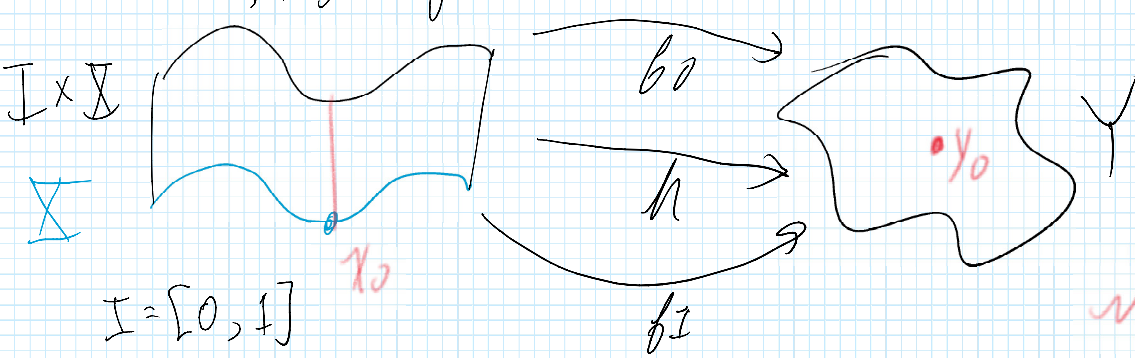
SOME TOPOGICAL DEFINITIONS

<u>DEF</u> LET $X \overset{f_0}{\underset{f_1}{\rightrightarrows}} Y$ BE TWO CONTINUOUS MAPS. THEY ARE <u>HOMOTOPIC</u> IF THERE IS A

$$[0,1] \times X \xrightarrow{h} Y$$

SUCH THAT

$$h(0,x) = f_0(x) \text{ AND } h(1,x) = f_1(x)$$



POINTED HOMOTOPY

Want h to send the red interval to y_0

h IS A HOMOTOPY BETWEEN f_0 AND f_1

ALTERNATE DEFINITION

LET $MAP(X,Y)$ BE THE OF ALL CONTINUOUS MAPS $X \to Y$ WITH THE COMPACT-OPEN TOPOLOGY.

THEN $f_0, f_1 \in MAP(X,Y)$ AND h DEFINES A PATH FROM f_0 TO f_1, IE A. MAP

$$
\begin{aligned}
I &\xrightarrow{h} MAP(X,Y) \\
0 &\longmapsto f_0 \\
1 &\longmapsto f_1
\end{aligned}
$$

TECHNICAL VARIATION:

LET $x_0 \in X$ AND $y_0 \in Y$. CALL THEM <u>BASE POINTS</u>. LET

$$MAP_0(X,Y) = \{ f : X \to Y : f(x_0) = y_0 \}$$

$$\cap$$

$MAP(X,Y)$ = SPACE OF <u>POINTED MAP</u> $X \to Y$.

CONSIDER CLOSED PATHS IN X THAT START / END AT x_0.

NOTATION:

A BASE POINT PRESERVING MAP $X \longrightarrow Y$ IS WRITTEN $(X, x_0) \to (Y, y_0)$

[VARIATION: LET $A \subset X$ AND $B \subset Y$ THEN $(X, A) \xrightarrow{g} (Y, B)$ DENOTES A MAP $X \xrightarrow{g} Y$ WITH $g(A) \subset B$.]

A CLOSED PATH IN X IS A MAP.

$$(I, \partial I) \longrightarrow (X, x_0)$$

$I = [0,1]$, $\partial I = \{0,1\}$

A HOMOTOPY BETWEEN SUCH PATHS p_0 and p_1 is a map

$$I \times (I, \partial I) \xrightarrow{h} (X, x_0)$$

[IN GENERAL .]

IN GENERAL

$$(X, A) \times (Y, B) := (X \times Y,\ X \times B \cup A \times Y)$$

$$h(0, x) = p_0(x)$$
$$h(1, x) = p_1(x)$$

DEF $\pi_1(X, x_0)$ = SET OF SUCH HOMOTOPY CLASSES

NOTE: HOMOTOPY DEFINES AN EQUIVALENCE RELATION ON THE SET (OR SPACE) $MAP(X, Y)$ AND ON $MAP((X, A), (Y, B))$. THE EQUIVALENCE CLASSES ARE CALLED HOMOTOPY CLASSES.

THEOREM $\pi_1(X, x_0)$ HAS A NATURAL GROUP STRUCTURE.

"NATURAL" MEANS THAT A MAP

$(X, x_0) \xrightarrow{f} (Y, y_0)$ INDUCES

$\pi_1(X, x_0) \xrightarrow{\pi_1(f)} \pi_1(Y, y_0)$ WHICH IS

A GROUP HOMOMORPHISM.

THE BINARY OPERATION IN $\pi_1(X, x_0)$

IS AS FOLLOWS.

LET $\alpha, \beta : (I, \partial I) \longrightarrow (X, x_0)$

BE CLOSED PATHS. DEFINE

$\alpha * \beta : (I, \partial I_0) \longrightarrow (X, x_0)$ BY

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & \text{for } 0 \le t \le 1/2 \\ \beta(2t-1) & \text{for } 1/2 \le t \le 1 \end{cases}$$

HOMOTOPIC TO

EXERCISE: SHOW $\alpha \simeq \alpha'$ AND $\beta \simeq \beta'$

$\Rightarrow \alpha * \beta \simeq \alpha' * \beta'$.

THIS MEANS OUR BINARY OPERATION

THIS MEANS OUR BINARY OPERATION ON $\mathrm{MAP}((I, \partial I), (X, x_0))$ INDUCES ONE ON $\pi_1(X, x_0)$

THE IDENTITY IN THIS GROUP IS REPRESENTED BY THE CONSTANT PATH e, i.e. FOR ANY CLOSED PATH p,

$$e * p \simeq p \simeq p * e.$$