Wednesday, December 2, 2020 1:57 PM

RECALL THE HOPF INVARIANT OF A MAP $S^{4m-1} \to S^{2m}$ IS AN INTEGER DEFINED IN TERMS OF LINKING #S

EXAMPLES:

① $m=1$ $S^3 \xrightarrow{\eta} S^2$ HOPF MAP
$HI(\eta)=1$

② FOR ANY $m>0$ THERE IS A MAP $g: S^{4m-1} \to S^{2m}$ WITH $HI(g)=2$

③ $m=2$ $S^7 \xrightarrow{\nu} S^4$ CONSTRUCTION IS SIMILAR TO THAT OF η WITH QUARTERNIONS INSTEAD OF COMPLEX #S. $\mathbb{R}^4$ HAS A NONCOMMUTATIVE DIVISION ALGEBRA STRUCTURE.

BASIS $\mathbb{H} \cong \mathbb{R}^4$ $\{1, i, j, k\}$

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = k = -ji \quad \text{ETC}$$

$$\mathbb{H}^2 \supset S^7 \longrightarrow S^4 = \mathbb{H} \cup \{\infty\}$$

$$(q_1, q_2) \longmapsto \begin{cases} \infty & \text{IF } q_2 = 0 \\ q_1/q_2 & \text{IF } q_2 \neq 0 \end{cases}$$

$\exists$ A SIMILAR MAP $S^{15} \xrightarrow{\sigma} S^8$

CONSTRUCTION USING OCTONIONS (CAYLEY #S) NONASSOCIATIVE MULTIPLICATION

CONSTRUCTION USING NONASSOCIATIVE

OCTONIONS (CAYLEY #S) MULTIPLICATION

BOTH $\nu : S^7 \to S^4$ AND $\sigma : S^{15} \to S^8$

HAVE HOPF INVARIANT ONE.

THE PREIMAGE UNDER ν OF ANY PT IS S^3

" σ " S^7

<u>QUESTION</u> ARE THERE MAPS

$S^{4m-1} \to S^{2m}$ WITH $HI = 1$

FOR ANY OTHER m?

RELATED QUESTION: CAN $\mathbb{R}^{2m}$ BE MADE INTO A DIVISION ALGEBRA FOR $m \neq 1, 2, 4$.

ANSWER: <u>NO</u> THEOREM PROVED BY FRANK ADAMS IN 1961.

(BEGINNING OF MODERN HOMOTOPY THEORY)

<u>EASY FACT</u> FOR EACH m, THE SET OF POSSIBLE HOPF INVARIANTS IS A SUBGROUP

OF $\mathbb{Z}$ (SUBMONOID OF $\mathbb{N}$).

PROOF: LET $f_1, f_2 : S^{4m-1} \to S^{2m}$

$$S^{4m-1} \xrightarrow{\text{PINCH}} S^{4m-1} \vee S^{4m-1} \xrightarrow{f_1 \vee f_2} S^{2m}$$

$$HI(f_1 \vee f_2 \circ \text{PINCH}) = HI(f_1) + HI(f_2)$$

RECALL THE EULER CHARACTERISTIC

LET M BE A COMPACT n-DIMENSIONAL MANIFOLD WITH BOUNDARY. A <u>TANGENT VECTOR FIELD</u> ON M IS A FUNCTION ASSIGNING A TANGENT VECTOR TO EACH $x \in M$.

<u>THEOREM</u> M AS ABOVE HAS A NONZERO (AT EVERY POINT) TANGENT VECTOR FIELD IFF $\chi(M) = 0$.

COR S^{2m-1} HAS SUCH A VECTOR FIELD, BUT S^{2m} DOES NOT

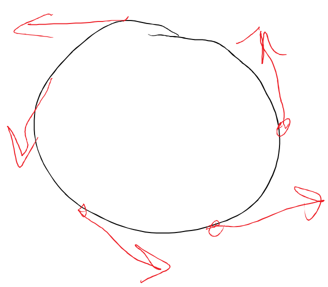
NOTE $\chi(S^{2m-1}) = 0$

$\chi(S^{2m}) = 2$

HAIRY BALL THEOREM IS A SPECIAL CASE.

PROOF OF COROLLARY:

ODD DIMENSIONAL CASE



$$S^{2m-1} \subset \mathbb{C}^m$$

$$\cup\!\!\!\!\mid$$

$x = (z_1, z_2, \dots z_m)$ $\quad i = \sqrt{-1}$

THEN ix IS $\perp$ TO x,

ASSIGN TO x THE UNIT TANGENT VECTOR IN THE DIRECTION OF ix.

EVEN DIMENSIONAL CASE:

LEMMA LET $g: S^k \longrightarrow S^k$
$x \longmapsto -x$

BE THE ANTIPODAL MAP. THEN $H_k(g)$ IS MULTIPLICATION BY $(-1)^{k+1}$. FOR EVEN k, g IS NOT HOMOTOPIC TO IDENTITY MAP.

PROOF: WE KNOW THIS BY C_2-METHODS. QED

LEMMA SUPPOSE S^k HAS A NONZERO VECTOR FIELD φ. THEN g IS HOMOTOPIC TO 1_{S^k}.

PROOF REPLACE $\varphi(x)$ BY THE UNIT VECTOR $\varphi(x)/|\varphi(x)|$.

CAN DEFINE A MAP

$$[0,1] \times S^k \xrightarrow{\tilde{\varphi}} S^k$$

$$(0, x) \longmapsto x$$

$$(1, x) \longmapsto x$$ MOVED IN DIRECTION OF $\varphi(x)$ THRU A SPHERICAL ANGLE OF π.

THIS GIVES THE DESIRED HOMOTOPY. QED

FOR k EVEN THERE CAN NOT BE SUCH A HOMOTOPY, SO THERE IS NO NONZERO VECTOR FIELD.

VECTOR BUNDLES

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AN $\mathbb{R}^n$-BUNDLE ξ OVER A SPACE X IS "A FAMILY OF n-DIMENSIONAL VECTOR SPACES PARAMETRIZED BY X."

MORE PRECISELY, WE HAVE MAP

$$E \xrightarrow{p} X$$

TOTAL SPACE OF ξ

SUCH THAT

1. $\forall x \in X$, $p^{-1}(x) \approx \mathbb{R}^n$
2. EACH $x \in X$ HAS A NEIGHBORHOOD U WITH $p^{-1}(U) \approx U \times \mathbb{R}^n$ AND MAP TO X IS PROJECTION TO FIRST FACTOR
3. GIVEN TWO SUCH NEIGHBORHOODS U_1 AND U_2 WITH $U_1 \cap U_2 \neq \emptyset$,

$$\begin{tikzcd}
(U_1 \cap U_2) \times \mathbb{R}^n \arrow[r, "\approx"] \arrow[rr, bend right, "h_{12}"'] & p^{-1}(U_1 \cap U_2) & (U_1 \cap U_2) \times \mathbb{R}^n \arrow[l, "\approx"']
\end{tikzcd}$$

For $u \in U_1 \cap U_2$ and $x \in \mathbb{R}^n$

$$h_{12}(u, x) = (u, \eta_{12}(x))$$

$\eta_{12}(x) \in GL_n(\mathbb{R})$

THIS DATA GIVES A MAP

$U_1 \cap U_2 \xrightarrow{\eta_{12}} GL_n(\mathbb{R})$ TRANSITION FUNCTION

THESE MAPS ARE COMPATIBLE IN A SUITABLE WAY.

EXAMPLES

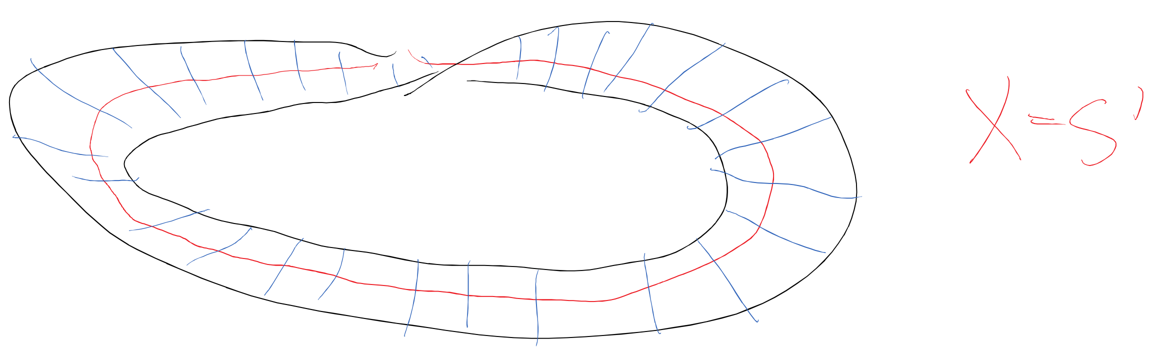
① TRIVIAL CASE

$E = X \times \mathbb{R}^n$

$\eta_{12} =$ CONSTANT I-VALUED MAP.

② $X = S^1$, $n = 1$ (LINE BUNDLE)

$E =$ INTERIOR OF MÖBIUS STRIP



EACH BLUE LINE $\cong \mathbb{R}$

THIS LINE BUNDLE IS NONTRIVIAL, $E \ncong S^1 \times \mathbb{R}$

③ $M = n$-MANIFOLD

$\overset{}{\underset{e}{\hookrightarrow}} \mathbb{R}^{n+k} \qquad k >> 0$

EACH POINT x HAS A TANGENT n-PLANE $T_x \subset \mathbb{R}^{n+k}$

$$E = \left\{ (x, v) \in M \times \mathbb{R}^{n+k} : v \in T_x \right\}$$

THIS IS THE TANGENT BUNDLE OF M.

RELATED CONSTRUCTION

EACH $x \in M$ HAS A NORMAL k-PLANE N_x. THIS LEADS TO AN $\mathbb{R}^k$-BUNDLE WITH TOTAL SPACE

$$\{ (x, v) \in M \times \mathbb{R}^{n+k} \quad | \quad \}$$

$$\{(x, w) \in M \times \mathbb{R}^{n+k} : w \in N_x\}$$

THIS IS THE NORMAL BUNDLE OF M IN $\mathbb{R}^{n+k}$

BOTH BUNDLES DEPEND ON THE EMBEDDING $e : M \hookrightarrow \mathbb{R}^{n+k}$.

THE TANGENT BUNDLE IS INDEPENDENT (UP TO ISOMORPHISM) OF e.

THE NORMAL BUNDLE FOR A GIVEN k IS ALSO INDEPENDENT OF e.

<u>EXERCISE</u>: DEFINE ISOMORPHISM OF VECTOR BUNDLES OVER X.