EULER CHARACTERISTIC

THEOREM LET X BE A FINITE CW-COMPLEX WITH c_i CELLS IN DIMENSION i FOR $0 \le i \le n$ THEN

TOPOLOGICAL INVARIANT

$$\sum_{i \ge 0} (-1)^i c_i = \sum_{i \ge 0} (-1)^i \operatorname{rank} H_i(X; k)$$

FOR ANY FIELD k.

REMARKS

① THE RELEVANT EXAMPLES OF k ARE $\mathbb{Q}$ AND $\mathbb{Z}/p$. THE ALTERNATING SUM IS SAID TO BE INDEPENDENT OF k.

② THE LEFT EXPRESSION IS SAID TO BE A TOPOLOGICAL INVARIANT INDEPENDENT OF THE CHOICE OF CELL STRUCTURE.

DEF THE NUMBER IN THE THEOREM IS DENOTED BY $\chi(X)$, THE EULER CHARACTERISTIC OF X.

PROOF: SUPPOSE A CHAIN COMPLEX C OF FINITE DIMENSIONAL k-VECTOR SPACES. LET

c_i = RANK OF C_i

h_i = " $H_i(C)$.

WE WILL SHOW

$$\sum_{i\geq 0}(-1)^i c_i = \sum_{i\geq 0}(-1)^i h_i$$

CAN APPLY THIS TO $C(X)\otimes k$, WHERE $C(X)$ IS THE CELLULAR CHAIN COMPLEX OF X

NOTATION

z_i := RANK OF $\ker : C_i \xrightarrow{d_i} C_{i-1}$
= RANK OF VECTOR SPACE OF i-CYCLES

b_i := RANK OF $\mathrm{im}\ C_{i+1} \xrightarrow{d_{i+1}} C_i$
= RANK OF VECTOR SPACE OF i-BOUNDARIES

SINCE $H_i(C) = \ker(d_i)/\mathrm{im}(d_{i+1})$

$$h_i = z_i - b_i \quad (1)$$

$\forall i$, WE HAVE A SES

$$0 \to Z_i \longrightarrow C_i \longrightarrow B_{i-1} \longrightarrow 0$$

$$\begin{array}{ccc} \| & & \| \\ \ker d_i & & \operatorname{im} d_i \end{array}$$

$$z_i \qquad c_i \qquad b_{i-1}$$

$$c_i = z_i + b_{i-1} \qquad (2)$$

IT FOLLOWS THAT

$$\sum_{i \geq 0} (-1)^i h_i = \sum_{i \geq 0} (-1)^i (z_i - b_i)$$

$$= \sum (-1)^i z_i - \sum (-1)^i b_i$$

$$= \sum (-1)^i (z_i + b_{i-1})$$

$$= \sum (-1)^i c_i$$

APPLY THIS TO $C(X) \otimes k$, AND THE THEOREM FOLLOWS. QED

COROLLARY. LET X' AND X'' BE FINITE CW-COMPLEXES. THEN

FINITE CW-COMPLEXES. THEN

$$\chi(X' \times X'') = \chi(X')\,\chi(X'')$$

PROOF: LET $C' = C(X')$ AND $C'' = C(X'')$, THEIR CELLULAR CHAIN COMPLEXES. THEN $X' \times X''$ HAS A CW-STRUCTURE WITH

$$C = C(X' \times X'') \cong C' \otimes C''$$

LET c_i, c_i' AND c_i'' BE THE # RANKS OF C_i, C_i' AND C_i''

THEN

$$c_i = \sum_{0 \le j \le i} c_j' \, c_{i-j}''$$

DEFINE POLYNOMIALS

t = DUMMY VARIABLE

$$\left.\begin{aligned} g(t) &= \sum c_i t^i \\ g'(t) &= \sum c_i' t^i \\ g''(t) &= \sum c_i'' t^i \end{aligned}\right]$$

POINCARE SERIES

THEN $g(t) = g'(t)\, g''(t)$

SO $g(-1) = g'(-1)\, g''(-1)$

NOTE $g(-1) = \sum (-1)^i c_i$, ETC.

QED

COROLLARY LET X BE A FINITE CW COMPLEX AND $\tilde{X} \to X$ A FINITE COVERING OF DEGREE d. THEN $\chi(\tilde{X}) = d\,\chi(X)$.

PROOF $\tilde{X}$ HAS A CW STRUCTURE WITH d CELLS FOR EACH CELL IN X IN EACH DIMENSION. RESULT FOLLOWS.

QED

EXAMPLES

$$\chi(S^n) = \begin{cases} 2 & \text{IF } n \text{ IS EVEN} \\ 0 & \text{IF } n \text{ IS ODD} \end{cases}$$

WE KNOW THERE IS A DOUBLE

COVERING $S^n \longrightarrow \mathbb{RP}^n$, SO

$$\chi(\mathbb{RP}^n) = \begin{cases} 1 & \text{IF } n \text{ IS EVEN} \\ 0 & \text{IF } n \text{ IS ODD} \end{cases}$$

THIS IS CONSISTENT WITH OUR KNOWLEDGE OF $H_*(\mathbb{RP}^n; \mathbb{Z}/2)$

$$H_i(\mathbb{RP}^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{FOR } 0 \leq i \leq n \\ 0 & \text{FOR } i > n. \end{cases}$$

RP^n HAS A CW STRUCTURE WITH ONE i-CELL FOR $0 \leq i \leq n$. $C(RP^n)$

$$\begin{array}{ccccccccccc} 0 & & 1 & & 2 & & 3 & & 4 & & n \\ \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \longleftarrow \cdots & \mathbb{Z} \end{array}$$

TENSORING WITH $\mathbb{Z}/2$ MAKES ALL BOUNDARY OPERATORS TRIVIAL.

$$\mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \longleftarrow \cdots\cdots \mathbb{Z}/2$$

$$\sum_{i=0}^{n} (-1)^i = \underbrace{\overbrace{1-1+1}^{n=2}-1}_{n=3} \cdots\cdots + (-1)^n$$

PLATONIC SOLIDS AS FINITE CW COMPLEXES

FINITE CW COMPLEXES

	C_0	C_1	C_2	$\chi = C_0 - C_1 + C_2$
TETRAHEDRON	4	6	4	$4-6+4=2$
CUBE	8	12	6	$8-12+6=2$
OCTAHEDRON	6	12	8	$6-12+8=2$
DODECAHEDRON	20	30	12	$20-30+12=2$
ICOSAHEDRON	12	30	20	$12-30+20=2$

COULD THERE BE OTHER "PLATONIC SOLIDS", IE 2 DIM CW COMPLEXES $\approx S^2$

WHERE EACH VERTEX HAS q EDGES
AND " FACE " p EDGES

LET $V = C_0 = \#$ OF VERTICES
$E = C_1 =$ " EDGES
$F = C_2 =$ " FACES

$$\chi = V - E + F = 2$$

$$E = qV/2 \qquad V = \left(\frac{2}{q}\right) E$$

$$F = pV/2 \qquad F = \left(\frac{2}{p}\right) E$$

$$2 = \chi = \left(\frac{2}{q}\right)E - E + \left(\frac{2}{p}\right)E = \boxed{\left(\frac{2}{p} + \frac{2}{q} - 1\right)} E$$

WHERE $p, q \geq 3$. HENCE

$$\frac{2}{p} + \frac{2}{q} > 1$$

THE ONLY VALUES OF (p, q) SATISFYING THIS ARE

$(3,3)$, $(3,4)$, $(3,5)$, $(4,3)$, $(5,3)$

THESE ARE THE FIVE PLATONIC SOLIDS WE KNOW ABOUT.

VARIATION

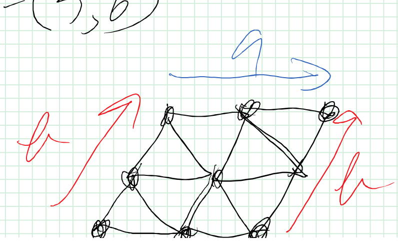
SUPPOSE SURFACE IS $\simeq$ TORUS SO $\chi(X) = 0$. THIS MEANS

$$\frac{2}{p} + \frac{2}{q} = 1 \qquad p, q \geq 3$$

$(p, q) = (3,6)$, $(6,3)$ OR $(4,4)$

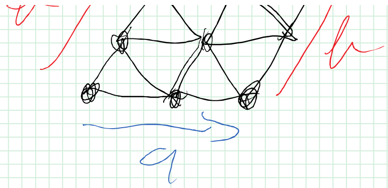
WE CANNOT SOLVE FOR E

$(p, q) = (3,6)$



$V = 4$

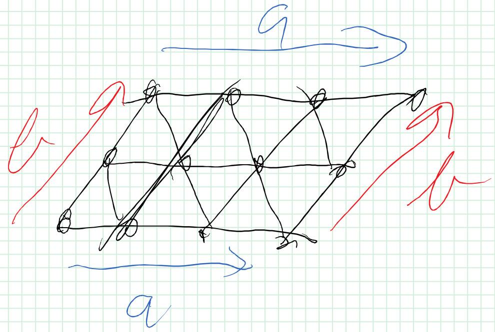
$E = 12$ $\qquad \chi = 4 - 12 + 8$



$E = 12$

$F = 8$

$\chi = 4 - 12 + 8$
$= 0$

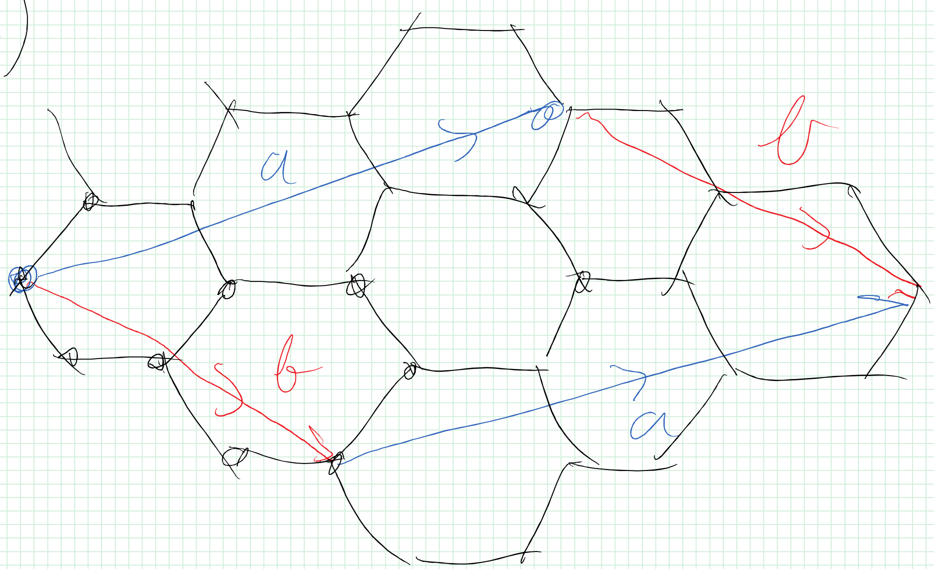


$V = 6$

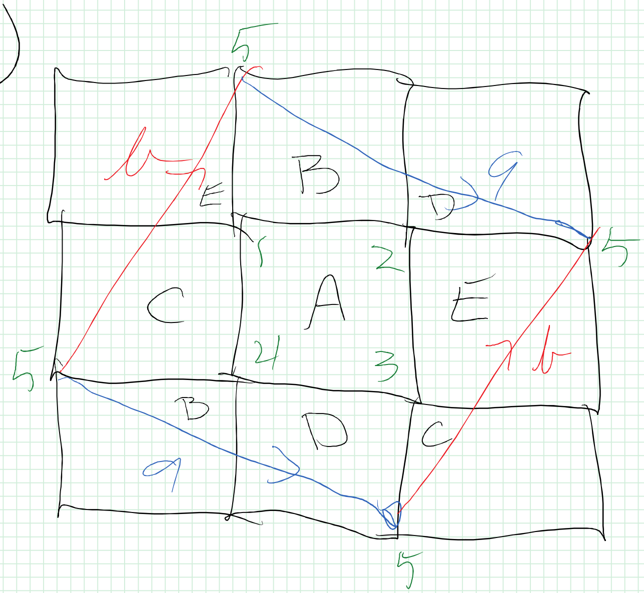
$E = 18$

$F = 12$

$(p, q) = (6, 3)$



$(p, q) = (4, 4)$



$V = 5$

$E = 10$

$F = F$