RECALL

Monday, November 2, 2020 2:02 PM

LEMMA GIVEN A SHORT EXACT SEQUENCE OF R-MODULES

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{B} M'' \longrightarrow 0,$$

THERE IS A SHORT EXACT SEQUENCE OF PROJECTIVE RESOLUTIONS

$$0 \longrightarrow P_0' \longrightarrow P_0 \longrightarrow P_0'' \longrightarrow 0$$

WE HAVE

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\alpha"] & M \arrow[r, "B"] & M'' \arrow[r] & 0 \\
0 \arrow[r] & P_0' \arrow[r] \arrow[u, "d_0'"] \arrow[ur, "f"] & P_0' \oplus P_0'' \arrow[r] \arrow[u, "d_0"] & P_0'' \arrow[r] \arrow[u, "d_0''"'] \arrow[ul, dotted, "g"'] & 0
\end{tikzcd}$$

$\exists g$ BECAUSE B IS ONTO AND P_0'' IS PROJECTIVE

$f = \alpha d_0'$. THEN $d_0 = f \oplus g$.

FOR THE NEXT STEP, CONSIDER THE SHORT EXACT SEQUENCE

$$0 \longrightarrow \ker d_0' \longrightarrow \ker d_0 \longrightarrow \ker d_0'' \longrightarrow 0$$

CONSTRUCT P_1', P_1'' AND $P_1 = P_1' \oplus P_1''$ AS ABOVE AND GET

$$\begin{tikzcd}
0 \arrow[r] & \ker d_0' \arrow[r] & \ker d_0 \arrow[r] & \ker d_0'' \arrow[r] & 0 \\
0 \arrow[r] & P_1' \arrow[r] \arrow[u, "d_1'"] & P_1 = P_1' \oplus P_1'' \arrow[r] \arrow[u] & P_1'' \arrow[r] \arrow[u, "d_1''"'] & {}
\end{tikzcd}$$

REPEAT THIS AD INFINITUM.

QED

THIS WORKS FOR ANY RING R. IN PRACTICE R WILL BE A PID, AND THE RESOLUTIONS WILL HAVE LENGTH ONE.

$\boxed{\text{UCT}}$

UNIVERSAL COEFFICIENT THEOREM

LET C BE A CHAIN OF FREE R-MODULES FOR A PID R (e.g. $\mathbb{Z}$), AND LET N BE AN R-MODULE. THEN THERE IS A SHORT EXACT SEQUENCE

$$0 \to \underset{\substack{\uparrow \\ \text{NAIVE GUESS}}}{H_n(C)\otimes_R N} \longrightarrow H_n(C\otimes_R N) \overset{\dashleftarrow}{\longrightarrow} \underset{\substack{\uparrow \\ \text{ERROR TERM}}}{\operatorname{Tor}_1^R(H_{n-1}(C),R)} \to 0$$

THIS IS SPLIT $(H_n(C\otimes_R N)\cong \sim \oplus \sim)$ BUT NOT NATURALLY SPLIT.

PROOF: IN ANY CHAIN COMPLEX WE HAVE

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

$Z_n := \ker d_n =$ GROUP OF n-CYCLES

$\cup$

$B_n := \operatorname{im} d_{n+1} =$ " " n-BOUNDARIES

$H_n(C) := Z_n / B_n$.

NOT $\mathbb{Z}$

DEFINE CHAIN COMPLEXES Z AND B WHOSE nTH COMPONENTS ARE Z_n AND B_{n-1} WITH TRIVIAL BOUNDARY OPERATORS. WE HAVE A SHORT EXACT SEQUENCE $\forall n$

$$\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r] \arrow[d, phantom, "\shortparallel"] & C_n \arrow[r] & B_{n-1} \arrow[r] \arrow[d, phantom, "\shortparallel"] & 0 \\
& \ker d_n & & \operatorname{im} d_n &
\end{tikzcd}$$

AND HENCE A SES OF CHAIN COMPLEXES

① $0 \longrightarrow Z \longrightarrow C \longrightarrow B \longrightarrow 0$

WE HAVE BY DEFINITION A SES

② $0 \longrightarrow B_n \xrightarrow{d_{n+1}} Z_n \longrightarrow H_n(C) \longrightarrow 0$

WHICH IS A PROJECTIVE RESOLUTION OF $H_n(C)$.

NOTE $H_n(Z) = Z_n$ AND $H_n(B) = B_{n-1}$.

① LEADS TO A LES IN H_*

$$\begin{tikzcd}
 & & B_n \\
 & H_{n+1}(C) \arrow[r, "0"] & H_{n+1}(B) \arrow[u, phantom, "\shortparallel"] \arrow[dll, "\partial_{n+1}"'] \\
H_n(Z) \arrow[r] & H_n(C) \arrow[r, "0"] & {}
\end{tikzcd}$$

∂_{n+1} IS THE INCLUSION $B_n \hookrightarrow Z_n$

THUS WE GET THE SES ②.

APPLY $-\otimes_R N$ TO ② AND GET

$$B_n \otimes_R N \xrightarrow{d'_{n+1}\otimes N} Z_n \otimes_R N$$

THIS IS THE CHAIN COMPLEX FOR COMPUTING $\mathrm{Tor}^R_*(H_n(C), N)$,

IE $\ker d'_{n+1} \otimes N = \mathrm{Tor}^R_1(H_n(C), N)$

$\operatorname{coker} d_{n+1} \otimes N = \mathrm{Tor}^R_0(-,-) = H_n(C) \otimes_R N$

---

④ $0 \to Z \to C \to B \to 0$

THESE ARE CHAIN COMPLEXES OF FREE R-MODULE, SO $-\otimes_R N$ PRESERVES EXACTNESS

③ $0 \to Z\otimes_R N \to C\otimes_R N \to B\otimes_R N \to 0$

WE WANT TO KNOW $H_*(C\otimes_R N)$

SINCE Z AND B HAVE TRIVIAL BOUNDARY OPERATORS, SO DO

$Z\otimes_R N$ AND $B\otimes_R N$

$$H_n(Z\otimes_R N) = Z_n\otimes_R N$$

$$H_n(B\otimes_R N) = B_n\otimes_R N$$

③ HAS A LES IN H_*

$$\begin{tikzcd}
 & Z_n\otimes_R N & & B_{n-1}\otimes_R N \\
\cdots \arrow[r] & H_n(Z\otimes_R N) \arrow[u, equal] \arrow[r] & H_n(C\otimes_R N) \arrow[r] & H_n(B\otimes_R N) \arrow[u, equal] \arrow[dll, "\partial_n"] \\
 & H_{n-1}(Z\otimes_R N) \arrow[r] \arrow[d, equal] & \cdots & \\
 & Z_{n-1}\otimes_R N & &
\end{tikzcd}$$

IT FOLLOWS THAT THERE IS A SES

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker}\partial_{n+1} \arrow[r] \arrow[d, phantom, "\shortparallel"] & H_n(C \otimes_R N) \arrow[r] & \ker\partial_n \arrow[r] \arrow[d, phantom, "\shortparallel"] & 0 \\
& \operatorname{Tor}_0^R(H_n(C), N) & & \operatorname{Tor}_1^R(H_{n+1} C, N) &
\end{tikzcd}$$

QED

SIMILARLY WE CAN APPLY $\operatorname{Hom}_R(-, N)$ TO ④ AND GET SES OF COCHAIN COMPLEXES

$$0 \leftarrow \operatorname{Hom}_R(Z, N) \leftarrow \operatorname{Hom}_R(C, N) \leftarrow \operatorname{Hom}(B, N) \leftarrow 0$$

AND LES IN COHOMOLOGY LEADING TO A SES

$$0 \leftarrow \operatorname{Hom}_R(H^n(C), N) \leftarrow H^n(\operatorname{Hom}_R(C, N)) \leftarrow \operatorname{Ext}_R^1(H_{n-1}(C), N) \leftarrow 0$$

THIS IS THE UCT FOR COHOMOLOGY.

EXAMPLE: $R = \mathbb{Z}$,

AND C IS $\mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \leftarrow 0$

AND C IS

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, phantom, "\shortparallel"] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, phantom, "\shortparallel"] & \mathbb{Z} \arrow[l, "\boxed{2}"'] \arrow[d, phantom, "\shortparallel"] & 0 \arrow[l] \arrow[d, phantom, "\shortparallel"] \\
C_0 & C_1 & C_2 & C_3
\end{tikzcd}$$

$$H_i(C) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/2 & i=1 \\ 0 & i=2 \end{cases}$$

APPLYING $(- \otimes \mathbb{Z}/2)$

$$C \otimes \mathbb{Z}/2 \qquad 0 \leftarrow \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \leftarrow 0$$

$$\Rightarrow H_i(C \otimes \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & i=0 \\ \mathbb{Z}/2 & i=1 \\ \underline{\mathbb{Z}/2} & i=2 \end{cases}$$

$\cong$ $\cong$

$$H_i(C) \otimes \mathbb{Z}/2 = \begin{cases} \mathbb{Z}/2 & i=0 \\ \mathbb{Z}/2 & i=1 \\ \underline{0} & i=2 \end{cases}$$

$$\mathrm{Tor}_1(H_i(C), \mathbb{Z}/2) = \begin{cases} \mathrm{Tor}_1(\mathbb{Z}, \mathbb{Z}/2) = 0 & i=0 \\ \mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2 & i=1 \\ \mathrm{Tor}_1(0, \mathbb{Z}/2) = 0 & i=2 \end{cases}$$

THE UCT SAYS THERE IS A SES

$$0 \to H_i(C) \otimes \mathbb{Z}/2 \to H_i(C \otimes \mathbb{Z}/2) \to \mathrm{Tor}_1(H_{i-1}(C), \mathbb{Z}/2) \to 0$$

$i=0$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
$i=1$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
$i=2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$

COULD ALSO STUDY

$$\mathrm{Hom}(C, \mathbb{Z}/2)$$

$$\begin{tikzcd}
0 & 1 & 2 \\
\mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/2
\end{tikzcd}$$