RECALL POSTNIKOV CONSTRUCTION
FOR A SPACE X AND INTEGER $n \geq 0$, THERE IS A SPACE $P^n X$ WITH A MAP $X \xrightarrow{g_n} P^n X$ S.T.

$$\pi_i(P^n X) = \begin{cases} \pi_i X & \text{FOR } i \leq n \\ 0 & \text{FOR } i > 0 \end{cases}$$

AND $\pi_i(g_n)$ IS AN ISOMORPHISM FOR $i \leq n$. THE CONSTRUCTION OF $P^n X$ IS NOT UNIQUE, BUT ANY TWO SUCH ARE HOMOTOPY EQUIVALENT.

PROP FOR $n \geq n'$,

$$P^{n'}(P^n X) \simeq P^{n'} X$$

WE HAVE MAPS

$$\begin{tikzcd}
X \arrow[d] \arrow[dr] \arrow[drr] & & & & \\
P^1 X & P^2 X \arrow[l] & P^3 X \arrow[l] & P^4 X \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

THIS IS THE POSTNIKOW TOWER FOR X.

EXAMPLES LET $X = S^n$ $n \geq 1$

WE KNOW $\pi_i(S^n) = \begin{cases} 0 & \text{FOR } 0 \leq i < n \\ \mathbb{Z} & \text{FOR } i = n \\ ??? & i > n \end{cases}$

HENCE $\pi_i(P^n S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq 0 \end{cases}$

HENCE $\pi_i(P^n S^n) = \begin{cases} \quad & n \\ 0 & i \neq 0 \end{cases}$

$P^n S^n$ HAS A SINGLE NON TRIVIAL HOMOTOPY GROUP, $\mathbb{Z}$.

SUCH AS SPACE IS CALLED AN <u>EILENBERG-MAC LANE</u> SPACE.

<u>NOTATION</u> LET $n > 0$ AND LET A BE AN ABELIAN GROUP.

$K(A, n)$ = A SPACE WITH

$$\pi_i \, K(A, n) = \begin{cases} A & \text{FOR } i = n \\ 0 & i \neq n \end{cases}$$

SUCH CAN BE CONSTRUCTED FOR ANY A.

RECALL FOR A DISCRETE GROUP G, THERE IS A SPACE BG WITH

$$\pi_i(BG) = \begin{cases} G & i = 1 \\ 0 & i > 1 \end{cases}$$

G NEED NOT BE ABELIAN

<u>EXAMPLE</u> $G = \mathbb{Z}$ = INTEGERS

$BG \simeq S^1$

WE KNOW $\pi_1(S^1) \cong \mathbb{Z}$.

$\mathbb{R}$ IS THE UNIVERSAL COVERING SPACE OF S^1.

FROM COVERING SPACE THEORY WE KNOW THAT WHEN X HAS A UNIVERSAL COVERING SPACE $\tilde{X}$, $\pi_i(\tilde{X}) = \pi_i(X)$ FOR $i > 1$.

$$\begin{tikzcd} & & \tilde{X} \arrow[d] \\ n > 1 & S^n \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

e.g. $\pi_i(S^1) = \pi_i(\mathbb{R})$ FOR $i > 1$

$= 0$ SINCE $\mathbb{R}$ IS CONTRACTIBLE

S^1 IS ALSO A TOPOLOGICAL GROUP, SO IT HAS A CLASSIFYING SPACE BS^1.

$BS^1 \simeq \mathbb{C}P^\infty$.

RELATED CONSTRUCTION:

$SP^n X$ = nTH SYMMETRIC PRODUCT OF X

$$= X^{\times n} / \Sigma_n$$

Σ_n = nTH SYMMETRIC GROUP

IT ACTS ON $X^{\times n}$ BY PERMUTING CO-ORDINATES

EXAMPLE

$X = S^2 = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$

EXERCISE $SP^n S \cong \mathbb{C}P^n$

POINTS IN $\mathbb{C}P^n$ CAN WRITTEN AS $[z_0, z_1, \cdots z_n]$ $z_i \in \mathbb{C}$ WHERE z_i ARE NOT ALL 0

AND $[\lambda z_0, \lambda z_1, \cdots \lambda z_n]$ $\lambda \neq 0 \in \mathbb{C}$

$$\begin{gathered}[\lambda z_0, \lambda z_1, \cdots \lambda z_n] \\ \| \\ [z_0, z_1, \cdots z_n]\end{gathered}$$

$\mathbb{C}P^n$ = SPACE COMPLEX THRU ORIGIN IN $\mathbb{C}^{n+1}$

WILL DEFINE A MAP

$$x = [z_{1,0}, z_{1,1}], [z_{2,0}, z_{2,1}] \cdots\cdots [z_{n,0}, z_{n,1}]$$

CONSIDER THE EXPRESSION (t = DUMMY VARIABLE)

$$\prod_{1 \le i \le n} \left(z_{i,0} + t z_{i,0} \right) = \sum_{0 \le j \le n} w_j t^j$$

$$\lambda(x) := [w_0, w_1, \cdots w_n] \in \mathbb{C}P^n$$

THIS MAP

$$\begin{tikzcd}
(\mathbb{C}P^1)^{\times n} \arrow[rr] \arrow[dr] & & \mathbb{C}P^n \\
& SP^n(\mathbb{C}P^1) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

$\cong$

$$SP^n(\mathbb{C}P^1)$$

FOR ANY POINTED SPACE X WE HAVE A MAP

$$SP^n X \longrightarrow SP^{n+1} X$$

$$(x_1, \dots x_n) \longmapsto (x_1, \dots x_n, x_0)$$

$x_i \in X$ BASE POINT

LET $SP^\infty X = \operatorname{colim} SP^n X$

= INFINITE SYMMETRIC PRODUCT

WE CAN THINK OF $SP^\infty X$ AS THE FREE ABELIAN MONOID GENERATED BY X

A POINT $w \in SP^\infty X$ COMES FROM $SP^n X$ FOR SOME $n < \infty$

IT IS A COLLECTION OF n (NOT NECESSARILY DISTINCT) POINTS OF X.

THINK OF THIS AS A (NOT CONTINUOUS) FUNCTION

$$X \xrightarrow{\hat{w}} \mathbb{N} = \text{NATURAL \#s}$$

WHERE $\hat{w}$ VANISHES ON ALL BUT A FINITE # OF POINTS IN X

WHERE $\tilde{w}$ VANISHES ON ALL BUT A FINITE # OF POINTS IN X.

$\tilde{w}(x)$ = # OF TIME x APPEARS IN THE POINT $w \in SP^\infty X$.

WHY DESCRIBE IT THIS WAY?

GIVEN $w_1, w_2 \in SP^\infty$ CORRESPONDING TO FUNCTIONS $\tilde{w}_1, \tilde{w}_2 : X \longrightarrow \mathbb{N}$ WE CAN ADD THEM AND GET $\tilde{w}_1 + \tilde{w}_2 : X \longrightarrow \mathbb{N}$.

THIS MAKES $SP^\infty X$ INTO AN ABELIAN MONOID.

WE CAN FORMALLY CONSIDER DIFFERENCES OF SUCH FUNCTIONS AND GET $\mathbb{Z}$-VALUED FUNCTIONS ON X WHICH ALSO VANISH AT ALL BUT A FINITE # OF POINTS.

CALL THE RESULTING SPACE $SP^\infty_{\mathbb{Z}} X$. IT IS A TOPOLOGICAL ABELIAN GROUP, THE

"FREE ABELIAN GROUP GENERATED BY X".

RECAP: $SP^{\infty}_{\mathbb{Z}} X$ IS THE SET OF $\mathbb{Z}$-FUNCTION ON X WHICH VANISH AT ALL BUT A FINITE # OF POINTS.

VARIATION REPLACE $\mathbb{Z}$ BY AN ABELIAN GROUP A. CALL THIS $SP^{\infty}_{A} X$

THEOREM LET $X = S^n$

$$SP^{\infty}_{\mathbb{Z}} S^n \simeq P^n S^n = K(\mathbb{Z}, n)$$

$$SP_A S^n = K(A, n)$$

= SPACE WITH $\pi_n = A$ AND OTHER $\pi_i = 0$.

LET $[X, Y]$ DENOTE THE SET OF HOMOTOPY CLASS OF MAP $X \longrightarrow Y$.

SUPPOSE $Y = K(A, n)$. SINCE $K(A, n)$ IS AN ABELIAN GP, SO IS $[X, K(A, n)]$

THEOREM $[X, K(A,n)] \simeq H^n(X; A)$

THEOREM $[X, K(A,n)] \cong H^n(X;A)$.

VERY DIFFERENT INTERPRETATION OF $H^*(X;A)$

THIS POINT OF VIEW LEADS TO ADDITION STRUCTURE IN $H^*(X;A)$.

<u>REMARK</u> THE SPACES $K(A,n)$ ARE VERY WELL UNDERSTOOD.
e.g. WE KNOW $H^*(K(A,n);A')$ IS EXPLICITLY KNOWN.

SUPPOSE WE HAVE AN ELEMENT $\theta \in H^m(K(A,n);A')$. IT CORRESPONDS TO A MAP

$$X \xrightarrow{x} K(A,n) \xrightarrow{\theta} K(A',m)$$

$x \in H^n(X;A)$

$\theta x \in H^m(X;A')$

i.e. WE HAVE A MAP

$$H^n(X;A) \xrightarrow{\theta} H^m(X;A')$$

(IT NEED NOT BE A HOMOMORPHISM)

IT IS NATURAL IN X

$$W \xrightarrow{f} X \xrightarrow{x} K(A,n) \xrightarrow{\theta} K(A',m)$$

$x \in H^n(X;A)$

xf CORRESPONDS TO

$$H^n(W;A) \xleftarrow{f^*} H^n(X;A)$$

$$(xf) \longleftarrow (x)$$

$$\theta x \in H^m(X;A')$$

$$\theta x f \in H^m(Y;A')$$

θ PLAYS NICELY WITH f.

SOME CATEGORY TERMINOLOGY

SUPPOSE WE HAVE FUNCTORS

$$\mathcal{C} \underset{F_2}{\overset{F_1}{\rightrightarrows}} \mathcal{D}$$

DEF

A <u>NATURAL TRANSFORMATION</u>

$\theta : F_1 \longrightarrow F_2$ CONSISTS OF

1) FOR EACH X IN $\mathcal{C}$,

A MORPHISM $\theta_X : F_1(X) \to F_2(X)$

IN $\mathcal{D}$ SUCH THAT

2) GIVEN A MORPHISM $X_1 \xrightarrow{g} X_2$ IN $\mathcal{C}$, WE HAVE A COMMUTATIVE DIAGRAM IN $\mathcal{D}$

$$\begin{tikzcd}
F_1(X_1) \arrow[r, "\theta_{X_1}"] \arrow[d, "F_1(g)"'] & F_2(X_1) \arrow[d, "F_2(g)"] \\
F_1(X_2) \arrow[r, "\theta_{X_2}"] & F_2(X_2)
\end{tikzcd}$$

MOTIVATING EXAMPLE.

$\mathcal{C}^{op}$ = CATEGORY OF TOP SPACES

$\mathcal{D}$ = CATEGORY OF ABELIAN GROUPS

$F_1 = H^n(- ; A)$

$F_2 = H^m(- ; A')$

θ IS AS ABOVE