

RECALL POSTNIKOV CONSTRUCTION
 FOR A SPACE X AND INTEGER
 $n > 0$, THERE IS A SPACE $P^n X$
 WITH A MAP $X \xrightarrow{p^n} P^n X$ S.T.

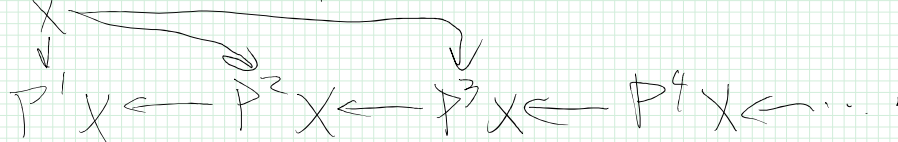
$$\pi_i(P^n X) = \begin{cases} \pi_i X & \text{FOR } i \leq n \\ 0 & \text{FOR } i > n \end{cases}$$

AND $\pi_i(p^n)$ IS AN ISOMORPHISM
 FOR $i \leq n$. THE CONSTRUCTION
 OF $P^n X$ IS NOT UNIQUE, BUT
 ANY TWO SUCH ARE HOMOTOPY
 EQUIVALENT.

PROP FOR $n > n'$,

$$P^{n'}(P^n X) \cong P^{n'} X$$

WE HAVE MAPS



THIS IS THE POSTNIKOV TOWER
 FOR X .

EXAMPLES LET $X = S^n \rightarrow Y$

$$\text{WE KNOW } \pi_i(S^n) = \begin{cases} 0 & \text{FOR } 0 \leq i < n \\ \mathbb{Z} & \text{FOR } i = n \\ \text{???} & i > n \end{cases}$$

$$\text{HENCE } \pi_i(P^n S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$$

$$\pi_n(\mathbb{R}^n) = \begin{cases} 0 & i \neq 0 \end{cases}$$

$P^n S^n$ HAS A SINGLE
NON TRIVIAL HOMOTOPY GROUP, \mathbb{Z} .
SUCH AS SPACE IS CALLED
AN EILENBERG-MACLANE SPACE.

NOTATION LET $n > 0$ AND LET
A BE AN ABELIAN GROUP.

$$K(A, n) = \text{A SPACE WITH}$$

$$\pi_i K(A, n) = \begin{cases} A & \text{FOR } i = n \\ 0 & i \neq n \end{cases}$$

SUCH CAN BE CONSTRUCTED FOR
ANY A.

RECALL FOR A ^{DISCRETE} GROUP G, THERE
IS A SPACE BG WITH

$$\pi_i(BG) = \begin{cases} G & i = 1 \\ 0 & i > 1 \end{cases}$$

G NEED NOT BE ABELIAN

EXAMPLE $G = \mathbb{Z} = \text{INTEGERS}$

$$BG \cong S^1$$

WE KNOW $\pi_1(S^1) \cong \mathbb{Z}$.

\mathbb{R} IS THE UNIVERSAL COVERING
SPACE OF S^1 .

FROM COVERING SPACE THEORY
 WE KNOW THAT WHEN X HAS
 A UNIVERSAL COVERING SPACE \tilde{X} ,
 $\pi_i(\tilde{X}) = \pi_i(X)$ FOR $i > 1$.

$$n > 1 \quad S^n \begin{array}{c} \dashrightarrow \tilde{X} \\ \longrightarrow X \end{array}$$

e.g. $\pi_i(S^1) = \pi_i(\mathbb{R})$ FOR $i > 1$
 $= 0$ SINCE \mathbb{R} IS
 CONTRACTIBLE

S^1 IS ALSO A TOPOLOGICAL
 GROUP, SO IT HAS A
 CLASSIFYING SPACE BS^1
 $BS^1 \cong \mathbb{C}P^\infty$

RELATED CONSTRUCTION :

$$SP^n X = \text{nTH SYMMETRIC PRODUCT OF } X \\ = X^{X^n} / \Sigma_n$$

$\Sigma_n = \text{nTH SYMMETRIC GROUP}$
 IT ACTS ON X^{X^n} BY
 PERMUTING CO-ORDINATES

EXAMPLE

$$X = S^2 = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$$

EXERCISE $SP^n \cong CP^n$

POINTS IN CP^n CAN BE WRITTEN

AS $[z_0, z_1, \dots, z_n]$ $z_i \in \mathbb{C}$

WHERE z_i ARE NOT ALL 0

AND $[\lambda z_0, \lambda z_1, \dots, \lambda z_n]$ $\lambda \neq 0 \in \mathbb{C}$
||
 $[z_0, z_1, \dots, z_n]$

$CP^n =$ SPACE COMPLEX THRU
ORIGIN IN \mathbb{C}^{n+1}

WILL DEFINE A MAP

$$\gamma = [z_{1,0}, z_{1,1}], [z_{2,0}, z_{2,1}] \dots [z_{n,0}, z_{n,1}]$$

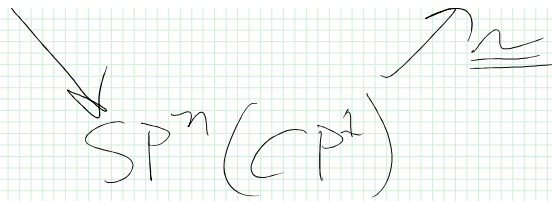
CONSIDER THE EXPRESSION
($t =$ DUMMY VARIABLE)

$$\prod_{1 \leq i \leq n} (z_{i,0} + t z_{i,1}) = \sum_{0 \leq j \leq n} w_j \cdot t^j$$

$$\gamma(\gamma) := [w_0, w_1, \dots, w_n] \in CP^n$$

THIS MAP

$$\begin{array}{ccc} (CP^1)^{\times n} & \longrightarrow & CP^n \\ & \searrow & \nearrow \cong \\ & CP^n / CP^1 & \end{array}$$



FOR ANY POINTED SPACE
 X WE HAVE A MAP

$$SP^n X \longrightarrow SP^{n+1} X$$

$$(\chi_1, \dots, \chi_n) \longmapsto (\chi_1, \dots, \chi_n, \chi_0)$$

$$\chi_i \in X$$

BASE POINT

$$\text{LET } SP^\infty X = \text{colim } SP^n X$$

= INFINITE
 SYMMETRIC
 PRODUCT

WE CAN THINK OF $SP^\infty X$ AS
 THE FREE ABELIAN MONOID
 GENERATED BY X

A POINT $w \in SP^\infty X$ COMES
 FROM $SP^n X$ FOR SOME $n < \infty$

IT IS A COLLECTION OF n
 (NOT NECESSARILY DISTINCT)
 POINTS OF X . (NOT CONTINUOUS)

THINK OF THIS AS A \wedge FUNCTION

$$X \xrightarrow{\tilde{w}} \mathbb{N} = \text{NATURAL } \#$$

WHERE \tilde{w} VANISHES ON ALL

WHERE \tilde{w} VANISHES ON ALL BUT A FINITE # OF POINTS IN X .

$\tilde{w}(x) = \#$ OF TIME x APPEARS IN THE POINT $w \in SP^\infty X$.

WHY DESCRIBE IT THIS WAY?

GIVEN $w_1, w_2 \in SP^\infty$

CORRESPONDING TO FUNCTIONS

$$\tilde{w}_1, \tilde{w}_2 : X \rightarrow \mathbb{N}$$

WE CAN ADD THEM AND

$$\text{GET } \tilde{w}_1 + \tilde{w}_2 : X \rightarrow \mathbb{N}.$$

THIS MAKES $SP^\infty X$ INTO AN ABELIAN MONOID.

WE CAN FORMALLY CONSIDER DIFFERENCES OF SUCH FUNCTIONS AND GET

\mathbb{Z} -VALUED FUNCTIONS ON X .

WHICH ALSO VANISH AT ALL BUT A FINITE # OF POINTS.

CALL THE RESULTING SPACE

$SP_{\mathbb{Z}}^\infty X$. IT IS A TOPOLOGICAL

ABELIAN GROUP, THE

"FREE ABELIAN GROUP GENERATED BY X"

RECAP: $SP_{\mathbb{Z}}^{\infty} X$ IS THE SET OF \mathbb{Z} -FUNCTION ON X WHICH VANISH AT ALL BUT A FINITE # OF POINTS.

VARIATION REPLACE \mathbb{Z} BY AN ABELIAN GROUP A .

CALL THIS $SP_A^{\infty} X$

THEOREM LET $X = S^n$

$$SP_{\mathbb{Z}}^{\infty} S^n \cong P^n S^n = K(\mathbb{Z}, n)$$

$$SP_A S^n = K(A, n)$$

= SPACE WITH $\pi_n = A$ AND OTHER $\pi_i = 0$.

LET $[X, Y]$ DENOTE THE SET OF HOMOTOPY CLASS OF MAP $X \rightarrow Y$.

SUPPOSE $Y = K(A, n)$. SINCE $K(A, n)$ IS AN ABELIAN GP,

SO IS $[X, K(A, n)]$

THEOREM $[X, K(A, n)] \cong H^n(X, A)$

THEOREM $[X, K(A, n)] \cong H^n(X; A)$.

VERY DIFFERENT INTERPRETATION
OF $H^*(X; A)$

THIS POINT OF VIEW LEADS
TO ADDITION STRUCTURE IN
 $H^*(X; A)$.

REMARK THE SPACES $K(A, n)$
ARE VERY WELL UNDERSTOOD.
e.g. WE KNOW $H^*(K(A, n); A')$
IS EXPLICITLY KNOWN.

SUPPOSE WE HAVE AN ELEMENT

$\theta \in H^m(K(A, n); A')$. IT

CORRESPONDS TO A MAP

$$X \xrightarrow{x} K(A, n) \xrightarrow{\theta} K(A', m)$$

$$x \in H^n(X; A)$$

$$\theta x \in H^m(X; A')$$

i.e. WE HAVE A MAP

$$H^m(X; A) \xrightarrow{\theta} H^m(X; A')$$

(IT NEED NOT BE A HOMOMORPHISM)

IT IS NATURAL IN X

$$W \xrightarrow{f} X \xrightarrow{\gamma} K(A, m) \xrightarrow{\theta} K(A', m)$$

$$\gamma \in H^m(X; A)$$

γ CORRESPONDS TO

$$H^m(W; A) \xleftarrow{\beta^*} H^m(X; A)$$

$$(\gamma \beta) \longleftarrow (\gamma)$$

$$\theta \gamma \in H^m(X; A')$$

$$\theta \gamma \beta \in H^m(Y; A')$$

θ PLAYS NICELY WITH β .

SOME CATEGORY TERMINOLOGY
SUPPOSE WE HAVE FUNCTORS

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F_1} & \mathcal{D} \\ & \xrightarrow{F_2} & \mathcal{D} \end{array}$$

DEF

A NATURAL TRANSFORMATION

$\theta: F_1 \rightarrow F_2$ CONSISTS OF

1) FOR EACH X IN \mathcal{C} ,

A MORPHISM $\theta_X: F_1(X) \rightarrow F_2(X)$

IN \mathcal{D} SUCH THAT

2) GIVEN A MORPHISM

$X_1 \xrightarrow{g} X_2$ IN \mathcal{C} , WE HAVE
A COMMUTATIVE DIAGRAM IN \mathcal{D}

$$\begin{array}{ccc} F_1(X_1) & \xrightarrow{\theta_{X_1}} & F_2(X_1) \\ F_1(g) \downarrow & & F_2(g) \downarrow \\ F_1(X_2) & \xrightarrow{\theta_{X_2}} & F_2(X_2) \end{array}$$

MOTIVATING EXAMPLE.

$\mathcal{C}^{op} =$ CATEGORY OF TOP SPACES

$\mathcal{D} =$ CATEGORY OF ABELIAN
GROUPS

$$F_1 = H^m(-; A)$$

$$F_2 = H^m(-; A')$$

θ IS AS ABOVE