WE ARE DISCUSSING $\underline{S}^n$ WITH

Monday, November 16, 2020 2:00 PM

THE ANTIPODAL ACTION OF $G = C_2$ AS A G-CW COMPLEX

S^n = SPHERE WITH TRIVIAL

$\underline{S}^0$ = G-SET C_2

WE NEED TO DEFINE A G-SET K_n AND A G-MAP

$$\begin{array}{ccc} K_n \times S^{n-1} & \xrightarrow{f_n} & \underline{S}^{n-1} \\ \downarrow & & \downarrow \\ K_n \times D^n & \longrightarrow & \underline{S}^n \end{array}$$

$K_n = C_2$ SO $K_n \times S^{n-1} = S^{n-1} \amalg S^{n-1}$

TO DEFINE f_n, SEND ONE COPY OF S^{n-1} TO THE SPACE $\underline{S}^{n-1}$ (IGNORING THE G-ACTION) VIA THE IDENTITY MAP. THE OTHER COPY OF S^{n-1} SENDS $x \in S^{n-1}$ to $-x \in \underline{S}^{n-1}$.

THIS LEADS TO THE EXPECTED $\underline{S}^n$.

WHAT THE RESULTING CELLULAR CHAIN COMPLEX? IT IS A

CHAIN OF $\mathbb{Z}G$-MODULES, WHERE $\mathbb{Z}G$ IS THE GROUP RING FOR $G = C_2$, MEANING A FREE ABELIAN GROUP ON THE SET $\{[e],[\gamma]\}$ WHERE $G = \{e,\gamma\}$

$$[e][\gamma] = [\gamma][e] = [\gamma]$$

$$[e][e] = [\gamma][\gamma] = [e]$$

WILL ABBREVIATE $[e]$ AND $[\gamma]$ BY 1 AND γ

$$\mathbb{Z}G = \mathbb{Z}[\gamma]/(\gamma^2 - 1)$$

$$= \mathbb{Z}[\gamma]/(\gamma+1)(\gamma-1)$$

NOT AN INTEGRAL DOMAIN

$C(\underline{S}^n)$ IS A CHAIN COMPLEX OF FREE $\mathbb{Z}G$-MODULES WITH $C_i = \mathbb{Z}G$ FOR $0 \leq i \leq n$.

SINCE $K_i = G$

WHAT IS THE BOUNDARY OPERATOR?

THEOREM: THE MAP $d_i : C_i \longrightarrow C_{i-1}$

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$$\begin{array}{ccc} C_i & \longrightarrow & C_{i-1} \\ \| & & \| \\ \mathbb{Z}G & & \mathbb{Z}G \end{array}$$

IS MULTIPLICATION BY $1 + (-1)^i \gamma$.

CONSEQUENCES

NOTE $d_{i-1} d_i = (1 + (-1)^{i-1}\gamma)(1 + (-1)^i \gamma)$

$$= 1 - \gamma^2 = 0$$

$$\ker d_i = \{ a + b\gamma : a, b \in \mathbb{Z},\ (a+b\gamma)(1+(-1)^i\gamma) = 0 \}$$

$$(a+b\gamma)(1+(-1)^i\gamma) = a + b\gamma + (-1)^i a\gamma + (-1)^i b$$

$$= (a + (-1)^i b) + (b + (-1)^i a)\gamma$$

THIS VANISHES FOR $b = (-1)^{i+1} a$

$$\ker d_i = (1 + (-1)^{i+1}\gamma) = \operatorname{im} d_{i+1}$$

$$\begin{array}{ccccccc} 0 & & 1 & & 2 & & 3 \\ \mathbb{Z}G & \xleftarrow{1-\gamma} & \mathbb{Z}G & \xleftarrow{1+\gamma} & \mathbb{Z}G & \xleftarrow{1-\gamma} & \mathbb{Z}G \end{array}$$

IN HOMOLOGY WE GET

$n = 3$

$$\begin{array}{cccc} \operatorname{coker}(1-\gamma) & 0 & 0 & \ker(1-\gamma) \\ \| & & & \| \\ \mathbb{Z} & & & \mathbb{Z} \end{array}$$

$\mathbb{Z}G$-MODULE WITH TRIVIAL

ACTION

$n=2$

$$\begin{array}{ccccc} 0 & & 1 & & 2 \\ \mathbb{Z}G & \xleftarrow{1-\gamma} & \mathbb{Z}G & \xleftarrow{1+\gamma} & \mathbb{Z}G \end{array}$$

$$\begin{array}{ccc} \operatorname{coker}(1-\gamma) & 0 & \ker(1+\gamma) \\ \| & & \| \\ \mathbb{Z} & & \{m - m\gamma \in \mathbb{Z}G\} \\ & & \| \\ & & \mathbb{Z}_- = \mathbb{Z}G/(1+\gamma) \end{array}$$

ACTION OF G BY SIGN $\longrightarrow$ $\mathbb{Z}_-$

IT FOLLOWS THAT

$$H_i(\underline{S}^n) = \begin{cases} \mathbb{Z} & \text{FOR } i=0 \\ 0 & \text{FOR } 0<i<n \\ \mathbb{Z} & \text{FOR } i=n \text{ AND } n \text{ ODD} \\ \mathbb{Z}_- & \text{FOR } i=n \text{ AND } n \text{ EVEN} \end{cases}$$

IF WE IGNORE THE GROUP ACTION, WE GET THE EXPECTED VALUE OF $H_* S^n$.

(ANY VALUE OF d_1 MUST GIVE THIS ANSWER)

PROOF OF THEOREM: FOR $i=1$

$d_1(p) = 1-\gamma$

BY DIRECT CALCULATION

WILL COMPUTE d_i FOR $i>1$ BY INDUCTION ON i, IT MUST SATISFY $\operatorname{im} d_i = \ker d_{i-1}$

THIS IMPLIES d_i = MULTIPLICATION BY $1+(-1)^i\gamma$.

QED.

WILL USE THIS TO COMPUTE $H_* \mathbb{RP}^n$. RP^n IS THE ORBIT SPACE OF $\underline{S}^n$ BY DEFINITION.

THIS MEANS WE CAN GET A CHAIN COMPLEX FOR RP^n BY TAKING THE "ORBIT COMPLEX" OF $C(\underline{S}^n)$, I.E. WE APPLY THE FUNCTOR

$\boxed{\mathbb{Z} = \mathbb{Z}G/(1-\gamma)}$

$$\mathbb{Z} \otimes_{\mathbb{Z}G} C(\underline{S}^n) = C(\underline{S}^n)/(1-\gamma)$$

$$
\begin{array}{lccccccc}
 & 0 & & 1 & & 2 & & 3 \\
C(\underline{S}^n) & \mathbb{Z}G & \xleftarrow{1-\gamma} & \mathbb{Z}G & \xleftarrow{1+\gamma} & \mathbb{Z}G & \xleftarrow{1-\gamma} & \mathbb{Z}G \leftarrow \cdots
\end{array}
$$

$$
\begin{array}{lccccccc}
C(\mathbb{R}P^n) & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} \xleftarrow{2} \\
H_*(\mathbb{R}P^n) & \mathbb{Z} & & \mathbb{Z}/2 & & 0 & & \mathbb{Z}/2 \cdots
\end{array}
$$

$$
H_n(\mathbb{R}P^n) = \ker\left(1+(-1)^n\right) \subset \mathbb{Z}
$$

$$
= \begin{cases} 0 & \text{FOR } n \text{ EVEN} \\ \mathbb{Z} & \text{FOR } n \text{ ODD} \end{cases}
$$

REPLACE G BY C_n FOR $n>2$.

C_n ACTS ON $S^{2k-1} \subset \mathbb{C}^k$

VIA MULTIPLICATION BY $e^{2\pi i/n}$

ORBIT SPACE IS CALLED

A LENSE SPACE L^{2k-1}

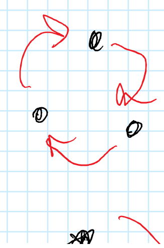
THE SPACE $\underline{S}^{2k-1}$ (WITH THIS

C_n ACTION) HAS A CW

STRUCTURE WITH

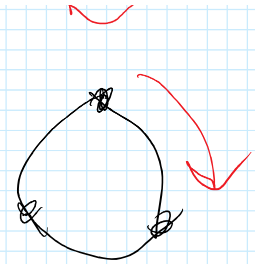
$X^0 = G$

$X^1 =$



$n=3$

$X^1 =$



$X^2 = X^1$ WITH n 2-DISKS, EACH BY IDENTITY MAP ON BOUNDARY

WE GET A CELLULAR CHAIN COMPLEX WITH

$$C_i = \mathbb{Z}G = \mathbb{Z}[\gamma]/(\gamma^n - 1)$$
$$= \mathbb{Z}[\gamma]/(1-\gamma)(1+\gamma+\gamma^2+\dots\gamma^{n-1})$$

LET $T = 1+\gamma+\dots+\gamma^{n-1} \in \mathbb{Z}G$

THEOREM. IN THE CELLULAR CHAIN COMPLEX

$$d_i = \begin{cases} 1-\gamma & \text{FOR } i \text{ ODD} \\ T & \text{FOR } i \text{ EVEN} \end{cases}$$

NOTE $\left.\begin{aligned} \ker(1-\gamma) &= \operatorname{im} T \\ \ker T &= \operatorname{im}(1-\gamma) \end{aligned}\right\}$ IN $\mathbb{Z}G$

WE GET THE EXPECTED

VALUE OF $H_*(\underline{S}^{2k-1})$

$\mathbb{Z} = \mathbb{Z}G/(1-\gamma)$ AS BEFORE

TRIVIAL G-ACTION

AS BEFORE WE CAN GET A CELLULAR CHAIN COMPLEX FOR L^{2k-1} BY APPLYING $\mathbb{Z} \otimes_{\mathbb{Z}G} (-)$ TO $C(\underline{S}^{2k-1})$.

DIFFERENCE WITH $G = C_2$: WE HAVE A FREE ACTION OF $G = C_n$ $(n>2)$ ON S^m ONLY IF m IS ODD.

THE RESULTING CHAIN COMPLEX HAS THE FORM

$n=5$

	0	1	2	3	4	5
	$\mathbb{Z} \xleftarrow{0}$	$\mathbb{Z} \xleftarrow{n}$	$\mathbb{Z} \xleftarrow{0}$	$\mathbb{Z} \xleftarrow{n}$	$\mathbb{Z} \xleftarrow{0}$	$\mathbb{Z}$
H_*	$\mathbb{Z}$	$\mathbb{Z}/n$	0	$\mathbb{Z}/n$	0	$\mathbb{Z}$

so $H_i(L^{2k-1}) = \begin{cases} \mathbb{Z} & \text{FOR } i=0 \\ \mathbb{Z}/n & \text{FOR } i \text{ ODD} \end{cases}$

$$\text{SO } H_i\left(L^{2k-1}\right) = \begin{cases} \mathbb{Z} & \text{FOR } i=0 \\ \mathbb{Z}/n & \text{FOR } i \text{ ODD AND } 0<i<2k-1 \\ 0 & \text{FOR } i \text{ EVEN AND } i>0 \\ \mathbb{Z} & \text{FOR } i=2k-1 \end{cases}$$

NEW TOPIC : THE POSTNIKOV CONSTRUCTION

LET X BE A PATH CONNECTED SPACE, AND $n>0$.

WILL CONSTRUCT A SPACE $P^n X$ WITH A MAP $X \xrightarrow{f_n} P^n X$ SUCH THAT

$$\pi_i\left(P^n X\right) = \begin{cases} \pi_i X & \text{FOR } i \le n \\ 0 & \text{FOR } i > n \end{cases}$$

AND $\pi_i(f_n)$ IS AN ISO FOR $i \le n$.

$P^n X$ IS CALLED THE <u>nTH POSTNIKOV SECTION OF X</u>.

HOW TO DO IT.

<u>CHOOSE</u> A SET OF MAPS $S^{n+1} \longrightarrow X$ GIVING A SET

OF GENERATORS OF $\pi_{n+1} X$

$$\begin{tikzcd} \coprod S^{n+1} \arrow[r] \arrow[d] & X \arrow[d, "g_1"] \\ \coprod D^{n+2} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X' \end{tikzcd}$$

WE ARE ATTACHING SOME $(n+2)$-CELLS TO X TO KILL ITS π_{n+1}

CAN SHOW $\pi_{n+1}(X') = 0$, $\pi_i(X') = \pi_i(X)$ AND g_1 INDUCES AN ISOMORPHISM FOR $i \leq n$.

NEXT WE KILL $\pi_{n+2} X'$ IN THE SAME WAY AND GET A SPACE X'' WITH

$$\pi_i(X'') = \begin{cases} \pi_i X & \text{FOR } i \leq n \\ 0 & \text{FOR } i = n+1, n+2 \\ ? & \text{FOR } i \geq n+3 \end{cases}$$

THEN KILL $\pi_{n+3} X''$ IN THE SAME, AND SO ON.

THE RESULTING SPACE IS $P^n X$.