Monday, November 9, 2020 2:03 PM

$1729 = 1^3 + 12^3 = 9^3 + 10^3$

$= 1 + 1728 = 729 + 1000$

G = GROUP

$\mathcal{B}G$ = 1 OBJECT CATEGORY FOR G

$N\mathcal{B}G$ = NERVE = SIMPLICIAL

$BG = |N\mathcal{B}G|$ = GEOMETRIC REALIZATION

THIS IS THE CLASSIFYING SPACE FOR G.

THEOREM

$$H^* BG \cong \boxed{\mathcal{E}xt^*_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})}$$

$\mathbb{Z}G$ = INTEGER GROUP RING OF G

= FREE ABELIAN GROUP ON THE UNDERLYING SET OF G

FOR $\gamma \in G$, $[\gamma]$ DENOTES THE CORRESPONDING GENERATOR OF $\mathbb{Z}G$. AN ELEMENT IN $\mathbb{Z}[G]$ IS AN INTEGER LINEAR COMBINATION OF SUCH $[\gamma]$. THE MULTIPLICATION IN $\mathbb{Z}G$ IS GIVEN $[\gamma'][\gamma''] = [\gamma'\gamma'']$

RECALL

LET $E_n G = \overbrace{G * G * \cdots * G}^{n+1 \text{ FACTORS}}$

FACTS ABOUT $E_n G$

① IT IS $(n-1)$-CONNECTED

i.e. $\pi_k E_n G = 0$ for $k \leq n-1$

$\bar{H}_k E_n G = 0$ ″

CAN BE PROVED BY INDUCTION

ON n USING

LEMMA: IF X IS $(k-1)$-CONNECTED AND Y IS $(\ell-1)$-CONNECTED THEN $X * Y$ IS $(k+\ell)$-CONNECTED

e.g: $S^m * S^n = S^{m+n+1}$

(2) $E_n G$ HAS A FREE ACTION OF G DEFINED AS FOLLOWS

A POINT IN $E_n G$ HAS THE FORM

$$(t_0, \gamma_0, t_1, \gamma_1, t_2, \gamma_2, \ldots, t_n, \gamma_n) \in \Delta^n \times G^{n+1}/\sim$$

$\gamma_i \in G$ and $t_i \in [0,1]$ WITH

$$\sum t_i = 1$$

ACTION OF G IS BY LEFT MULTIPLICATION ON EACH G-VALUED CO-ORDINATE.

LET $B_n G = E_n G / G$ = ORBIT SPACE

EXAMPLE

(1) $G = C_2$

$E_n G \cong S^n = G^{*(n+1)}$ $\qquad EG = S^\infty$

WITH ANTIPODAL ACTION

$B_n G \cong \mathbb{RP}^n$

(2) $G = S^1$ $\qquad E_n G \cong S^{2n+1}$

$B_n G \cong \mathbb{CP}^n$

WE HAVE MAPS $E_n G \to E_{n+1} G$ AND $B_n G \to B_{n+1} G$

$BG = \operatorname{colim} B_n G$ = UNION OF

ALL $B_n G$.

$EG = \operatorname{colim} EG$

= CONTRACTIBLE SPACE WITH FREE ACTION OF G

EG IS WEAKLY CONTRACTIBLE

RECALL A MAP $X \xrightarrow{f} Y$ IS A <u>HOMOTOPY EQUIVALENCE</u> IF $\exists$ $h : Y \to X$ SUCH THAT

h IS A HOMOTOPY INVERSE OF f

$X \xrightarrow{f} Y \xrightarrow{h} X$ IS HOMOTOPIC TO 1_X

$Y \xrightarrow{h} X \xrightarrow{f} Y$ " 1_Y.

IT FOLLOWS THAT $H_*(f)$ IS AN ISOMORPHISM $H_*(X) \to H_*(Y)$

$\pi_*(X_\alpha) \to \pi_*(Y_\alpha)$

FOR EACH PATH CONNECTED COMPONENT OF X AND Y.

DEF A MAP $X \xrightarrow{f} Y$ IS A <u>WEAK EQUIVALENCE</u> IF IT INDUCES ISOMORPHISMS AS ABOVE.

(f IS NOT REQUIRED TO HAVE A HOMOTOPY INVERSE.)

<u>VARIATION</u>: SUPPOSE X AND Y EACH HAVE A G-ACTION COMPATIBLE WITH F. WE

CAN DEFINE G-HOMOTOPY EQUIVALENCE TO BE AS ABOVE WHERE f, h AND THE TWO HOMOTOPIES TO RESPECT THE G-ACTION.

A G-MAP $X \xrightarrow{f} Y$ IS A WEAK G-EQUIVALENCE IF

$$X^H \xrightarrow{f^H} Y^H$$

IS A WEAK EQUIVALENCE FOR EACH SUBGROUP $H \leq G$, WHERE

$$X \supset X^H = \{x \in X : \eta(x) = x \ \forall \eta \in H\}$$

QUESTION

IS THE MAP $EG \longrightarrow pt$ A WEAK G-EQUIVALENCE?

$G = C_2$

NO

$$S^\infty \longrightarrow pt$$

ANTIPODAL ACTION

TRIVIAL ACTION

NOTE $(S^\infty)^G \doteq$ EMPTY SET

USE EQUIVARIANT METHODS TO COMPUTE $H_* \mathbb{R}P^n$:

RECALL A CW-COMPLEX X HAS SETS K_n AND MAPS

FOR $n \geq 0$,

$$\begin{tikzcd} K_n \times S^{n-1} \arrow[r, "f_n"] \arrow[d] & X^{n-1} \arrow[d] \\ K_n \times D^n \arrow[r] & X^n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

(PREVIOUSLY CONSTRUCTED)

$X^0 = K_0$

X^n = PUSHOUT

SUPPOSE EACH SET K_n HAS A G-ACTION, AND EACH f_n IS A G-MAP.

WE GET A CELLULAR CHAIN COMPLEX $C(X)$ AS BEFORE, WHERE $C_n(X)$ = FREE AB GROUP ON K_n

IT IS NOW A $\mathbb{Z}G$-MODULE, AND $C(X)$ IS A CHAIN COMPLEX OF $\mathbb{Z}G$-MODULES.

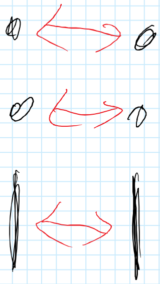
THIS MAKES $H_* X$ A $\mathbb{Z}G$-MODULE.

EXAMPLE FOR $G = C_2$

$X = S^n$ WITH ANTIPODAL ACTION.

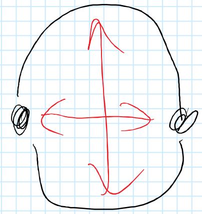
$X^0 = K_0 = G = 2$ ELEMENTS PERMUTED BY $e \neq \gamma \in G$

$K_i = G$ FOR $0 \leq i \leq n$.



$D^1 \amalg D^1$

$$\begin{tikzcd} K_1 \times S^0 \arrow[r] \arrow[d] & K_0 = X^0 \arrow[d] \\ K_1 \times D^1 \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & X^1 \end{tikzcd}$$



SIMILARLY EACH X^i IS S^i WITH ANTIPODAL ACTION.

QUESTION: DESCRIBE $C(X)$. $C_i(X) = \mathbb{Z}G$ FOR $0 \leq i \leq n$

WHAT IS BOUNDARY OPERATOR?