# NEW TOPIC : HOMOLOGY

Wednesday, October 7, 2020 1:37 PM

WE HAVE DEFINED

$\pi_1(X, x_0)$ = GROUP COMPUTABLE IN PRACTICE

$\pi_k(X, x_0)$ = ABELIAN GROUP FOR $k > 1$

HARD TO COMPUTE

DEFINED IN TERMS OF MAPS

$(I^k, \partial I^k) \to (X, x_0)$ WITH

BINARY OPERATION SIMILAR TO $k = 1$ CASE

WILL DEFINE

ABSOLUTE $H_k(X)$ = $k$TH HOMOLOGY GROUP OF $X$

= ABELIAN GROUP FOR $k \geq 0$

HARDER TO DEFINE BUT EASIER TO COMPUTE WITH.

IT HAS A RELATIVE VERSION

$H_k(X, A)$ FOR $A \subseteq X$

WITH $H_k(X) = H_k(X, \emptyset)$

IT WILL BE SHOWN THE EILENBERG - STEENROD AXIOMS

$H_k$ IS A FUNCTOR FROM

$$\begin{tikzcd}
\text{CAT OF TOP SPACES} = \mathcal{T}op \arrow[r, "\text{ABSOLUTE}", "H_k"'] \arrow[d] & \mathcal{A}b = \text{CAT OF ABELIAN GROUPS} \\
\text{PAIRS OF} \arrow[ur, "\text{RELATIVE}"] &
\end{tikzcd}$$

1) <u>HOMOTOPY AXIOM</u>

SUPPOSE $X \xrightarrow{f, g} Y$ THAT ARE HOMOTOPIC

THEN $H_k(X) \xrightarrow{f_*, g_*} H_k(Y)$ ARE EQUAL

SIMILARLY FOR $(X, A) \xrightarrow{f, g} (Y, B)$

## 2) EXACTNESS AXIOM

GIVEN $A \xhookrightarrow{i} X$, WE GET

$$H_{k+1}(X,A) \xrightarrow{\partial_{k+1}} H_k(A) \xrightarrow{i_*} H_k(X) \xrightarrow{j_*} H_k(X,A) \xrightarrow{\partial_k} H_{k-1}(A) \to \cdots$$

$$H_k(X) = H_k(X, \emptyset).$$

$$(X, \emptyset) \xrightarrow{j} (X, A)$$

TO BE DEFINED LATER

THIS IS A LONG EXACT SEQUENCE

## 3) EXCISION AXIOM

LET $B \subset A \subset X$ WITH

CLOSURE$(B) \subseteq$ INTERIOR OF $A$

$$H_k(X-B, A-B) \xrightarrow{\cong} H_k(X, A)$$

## 4) DIMENSION AXIOM

$$H_k(\text{point}) = \begin{cases} \mathbb{Z} & \text{FOR } k=0 \\ 0 & \text{FOR } k>0 \end{cases}$$

TWO WAYS TO PROCEED:

① PROVE SOME THINGS ASSUMING THESE AXIOMS

② CONSTRUCT A FUNCTOR THAT SATISFIES THEM.

WILL PROCEED WITH ② FOR NOW.

WE NEED SOME ALGEBRAIC DEFINITIONS.

DEF A CHAIN COMPLEX $C$ (OF ABELIAN GROUPS) IS A DIAGRAM

$$C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \xleftarrow{d_3} C_3 \longleftarrow \cdots$$

EACH $C_i$ IS AN ABELIAN GRP

" $d_i$ IS HOMOMORPHISM

WITH $d_i d_{i+1} = 0$ FOR $i \geq 1$.

(WEAKER THAN EXACTNESS)

$x \in C_n$ IS AN n-CHAIN

IF $d_n(x) = 0$, $x$ IS AN n-CYCLE

IF $x \in \operatorname{Im} d_{n+1}$, $x$ IS AN n-BOUNDARY

$d_n$ = nTH BOUNDARY OPERATOR

ITS nTH HOMOLOGY GROUP IS

$$C_n \supset \ker(d_n) \supset \operatorname{im}(d_{n+1})$$

$$\ker(d_n)/\operatorname{im}(d_{n+1}) =: H_n(C)$$

A CHAIN MAP $C \xrightarrow{f} C'$ IS A COMMUTATIVE DIAGRAM

$$\begin{array}{ccccccccc}
C_0 & \xleftarrow{d_1} & C_1 & \xleftarrow{d_2} & C_2 & \xleftarrow{d_3} & C_3 & \longleftarrow & \cdots \\
{\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & \\
C_0' & \xleftarrow[d_1']{} & C_1' & \xleftarrow[d_2']{} & C_2' & \xleftarrow[d_3']{} & C_3' & \longleftarrow & \cdots
\end{array}$$

$$f_n d_{n+1} = d_{n+1}' f_{n+1} \quad \text{FOR } n \geq 0$$

EXERCISE: SHOW THAT $f$ ABOVE INDUCES HOMOMORPHISMS

$$H_n(C) \xrightarrow[f_*]{H_n(f)} H_n(C')$$

FOR $n \geq 0$.

**THEOREM** SUPPOSE WE HAVE A SHORT EXACT SEQUENCE OF CHAIN COMPLEXES

$$0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$$

i.e.

$$0 \to C'_n \xrightarrow{f_n} C_n \xrightarrow{g_n} C''_n \to 0$$

IS EXACT, $\ker f_n = 0$, $\operatorname{im} f_n = \ker g_n$ AND $g_n$ IS ONTO $\forall n \geq 0$

THEN THERE IS A LONG EXACT SEQUENCE

CONNECTING HOMOMORPHISM

$$\cdots \to H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(C'') \xrightarrow{\partial_n} H_{n-1}(C') \to \cdots$$

PROOF

$$\begin{tikzcd}
 & & u \in C_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d_{n+1}"] & C''_{n+1} \ni w \arrow[d, "d''_{n+1}"] & \\
0 \arrow[r] & v \in C'_n \arrow[r, "f_n"] \arrow[d, "d'_n"] & C_n \arrow[r, "g_n"] \arrow[d, "d_n"] & C''_n \arrow[r] \arrow[d, "d''_n"] & 0 \\
0 \arrow[r] & z \in C'_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "d'_{n-1}"] & C_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "d_{n-1}"] & C''_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C'_{n-2} \arrow[r, "f_{n-2}"] & C_{n-2} & &
\end{tikzcd}$$

$y$ OR $y + d_{n+1}(u)$

$x$ OR $x + d''_{n+1}(w) = x + d''_{n+1} g_{n+1}(u)$

$d_n(y)$

$0$

HOW TO DEFINE $\partial_n$:

$$\alpha \in H_n(C''_n) = \ker(d''_n) / \operatorname{im}(d''_{n+1})$$

CHOOSE A REPRESENTATIVE

$x \in \ker(d''_n) < C''_n$ OF $\alpha$

$x \in \ker(d_n'') \subset C_n''$ OF $\alpha$

CHOOSE $y \in C_n$ WITH $g_n(y) = x$

NOTE $g_{n-1} d_n(y) = d_n'' g_n(y) = d_n''(x) = 0$

SO $d_n(y) = \ker g_{n-1} = \operatorname{im} f_{n-1}$

AND $d_n(y) = f_{n-1}(z)$ FOR A UNIQUE $z$.

NOTE $f_{n-2} d_{n-1}'(z) = d_{n-1} f_{n-1}(z) = d_{n-1}(d_n(y))$

$= 0 \in C_{n-2}$

THIS MEANS $d_{n-1}'(z) = 0$

$z$ IS AN $(n-1)$-CYCLE REPRESENTING SOME $\gamma \in H_{n-1}(C')$

WE NEED TO SHOW $\gamma$ IS INDEPENDENT OF THE TWO CHOICES MADE.

A DIFFERENT REPRESENTATIVE $\tilde{x}$ OF $\alpha$ WOULD HAVE THE FORM

$x + d_{n+1}''(w)$ FOR $w \in C_{n+1}''$

$w = g_{n+1}(u)$ FOR $u \in C_{n+1}$

COULD REPLACE $y$ BY $y + d_{n+1}(u)$,

THEN $d_n(y + d_{n+1}(u))$

$$= d_n(y) + d_n d_{n+1}(u) = d_n(y)$$

A DIFFERENT CHOICE OF $y$

HAS THE FORM $y + f_n(v)$ $v \in C_n'$

$$d_n(y + f_n(v)) = d_n(y) + d_n f_n(v)$$

$$= d_n(y) + f_{n-1} d_n'(v)$$

THIS IS IN ker $g_{n-1}$.

WE WOULD REPLACE $z$ BY

$z + d_{n-1}'(v)$

IT DIFFERS FROM $z$ BY A BOUNDARY, SO IT REPRESENTS THE SAME $\gamma \in H_{n-1}(C')$

$\gamma$ IS WELL DEFINED

DIAGRAM CHASING

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