EXCISION AXIOM

GIVEN $Z \subset A \subset X$ WITH CLOSURE$(Z) \subset$ INTERIOR(A), THEN THE MAP $H_*(X-Z, A-Z) \longrightarrow H_*(X,A)$ IS AN ISOMORPHISM.

EQUIVALENT FORMULATION

LET $A, B \subset X$ WITH $\text{INT}(A) \cup \text{INT}(B) = X$. THEN

$$H_*(A, A \cap B) \xrightarrow{\lambda} H_*(X, A)$$

IS AN ISOMORPHISM.

TO SEE THE EQUIVALENCE, LET $B = X - Z$, SO $A \cap B = A - Z$.

IN THE SECOND FORMULATION LET $C(A+B) \hookrightarrow C(X)$ BE THE SUBCHAIN COMPLEX GENERATED BY SIMPICES $\Delta^n \xrightarrow{\sigma} X$ WITH $\sigma(\Delta^n) \subset A$ OR B.

CLAIM THE CHAIN INCLUSION ABOVE IS A CHAIN HOMOTOPY EQUIVALENCE,

$$H_* C(A+B) \cong H_*(X).$$ PROOF LATER

CONSEQUENC: CONSIDER THE SHORT EXACT SEQUENCE OF CHAIN COMPEXES

$$0 \to C(A\cap B) \longrightarrow C(A)\oplus C(B) \longrightarrow C(A+B) \to 0$$

THE LONG EXACT IN H_* IS

$$\begin{tikzcd}
 & & & H_n(X) \arrow[d, phantom, "\cong\ \text{CLAIM}"] \\
\cdots \arrow[r] & H_n(A\cap B) \arrow[r] & H_n(A)\oplus H_n(B) \arrow[r] & H_n C(A+B) \arrow[dlll, rounded corners] \\
 H_{n-1}(A\cap B) \arrow[r] & \cdots & &
\end{tikzcd}$$

THIS IS THE MAYER-VIETORIS SEQUENCE.

FOR THE EXCISION AXIOM CONSIDER

$$\begin{tikzcd}
0 \arrow[r] & C(A\cap B) \arrow[r] \arrow[d] & C(B) \arrow[r] \arrow[d] & C(B, A\cap B) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & C(A) \arrow[r] & C(X) \arrow[r] & C(X, A) \arrow[r] & 0 \\
0 \arrow[r] & C(A) \arrow[r] \arrow[u, "="', "\alpha"] & C(A+B) \arrow[r] \arrow[u, "\simeq"', "\beta"] & C(A+B, A) \arrow[r] \arrow[u, "\gamma"] \arrow[d, equal] & 0 \\
 & & \text{CLAIM} & C(A+B)/C(A) &
\end{tikzcd}$$

β IS CHE BY THE CLAIM

THE EQUIV FORMULATION OF THE EXCISION AXIOM SAYS $H_*(\delta)$ IS AN ISOMORPHISM.

WE KNOW BY THE FIVE LEMMA THAT $H_*(\gamma)$ IS AN ISOMORPHISM.

EXERCISE: SHOW $H_*(\delta)$ IS AN ISOMORHISM

(???)

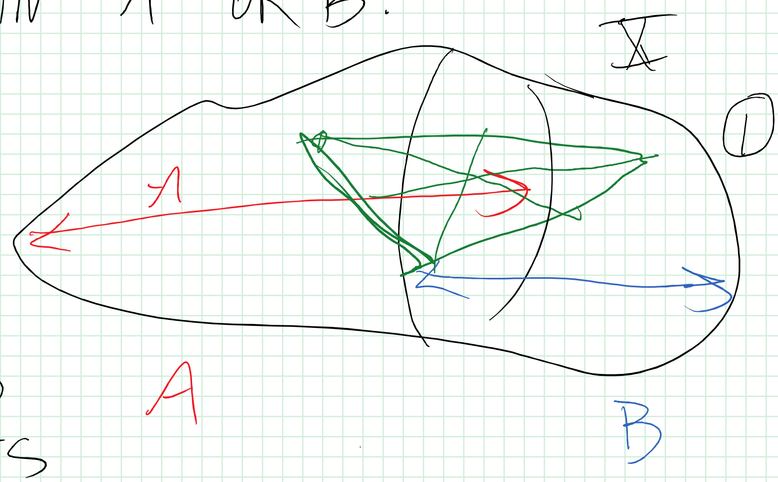
PROOF OF CLAIM (USES BARYCENTRIC SUBDIVISION). WE NEED A CHAIN MAP

$$C(X) \xrightarrow{P} C(A+B)$$

$A, B \subset X$

$\mathrm{int}(A) \cup \mathrm{int}(B) = X$

LET $\sigma : \Delta^n \to X$ HAVE IMAGE NOT CONTAINED IN A OR B.



USING REPEATED BARYCENTRIC SUBDIVISION, WE CAN SHOW $\sigma(\Delta^n)$ IS UNION OF IMAGES OF SMALLER SIMPLICES EACH CONTAINED IN A OR IN B.

DEF A SIMPLEX $\sigma : \Delta^n \to X$ IS <u>SMALL</u> IF ITS IMAGE IS IN A OR B.

A <u>SMALL CHAIN</u> IS A LINEAR COMBINATION OF SMALL SIMPLICES

$$\underset{\text{ALL CHAINS}}{C(X)} \underset{\beta}{\overset{P}{\rightleftarrows}} \underset{\text{SMALL CHAINS}}{C(A+B)}$$

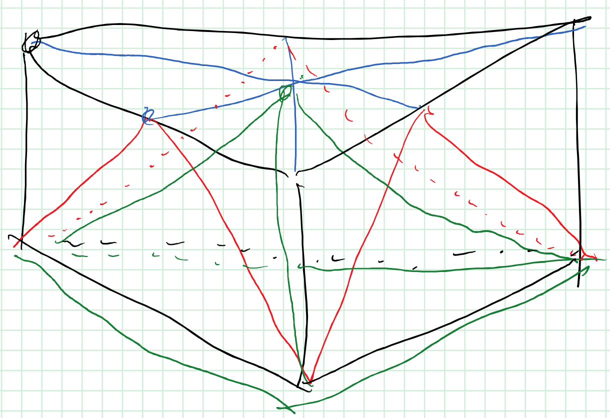
$$P\beta = 1_{C(A+B)}$$

$$\beta P \neq 1_{C(X)}$$

THE IMAGE OF β IS THE SMALL CHAINS IN X BY DEFINITION

WE NEED TO SHOW βP IS CHAIN HOMOTOPIC TO $1_{C(X)}$. WE NEED MAPS $D_n : C_n(X) \longrightarrow C_{n+1}(X)$ WITH SUITABLE PROPERTIES.

FOR (1) CONSIDER



$\Delta^2 \times I$ $\cong$ UNION OF 7 Δ^3

$\Delta^n \times I \cong$ UNION OF $(1+(n+1)!)$ COPIES OF Δ^{n+1}

GIVEN $\sigma : \Delta^n \longrightarrow X$ NOT SMALL

$$\begin{array}{c} \Delta^n \\ \uparrow \\ \Delta^n \times I \approx \bigcup_{1+(n+1)!} \Delta^{n+1} \end{array}$$

THIS GIVES US $D_n(\sigma) \in C_{n+1}(X)$

THIS WILL LEAD TO $D_n(\sigma)$ BEING A SMALL CHAIN IN $C_{n+1}(X)$

DETAILS ARE ON PAGES 121-123 OF HATCHER.

THIS IS THE CHAIN HOMOTOPY THAT SHOWS $C(A+B)$ IS CHAIN HOMOTOPY EQUIVALENT TO $C(X)$.

QED

MODULO THE "EXERCISE", THIS PROVE THE EXCISION AXIOM AND THE MAYER-VIETORIS SEQUENCE.

FUTURE GOALS

① HOW TO DESCRIBE $H_*(X \times Y)$ IN TERMS OF $H_*(X)$ AND $H_*(Y)$??? NAIVE GUESS

$$H_n(X \times Y) \cong \bigoplus_{0 \leq i \leq n} H_i(X) \otimes H_{n-i}(Y)$$

SOMETIMES BUT NOT ALWAYS

RELATED QUESTION:

DESCRIBE $H_*(C' \otimes C'')$

IN TERMS OF $H_*(C')$ AND $H_*(C'')$.

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(2) SIMPLER ALGEBRAIC QUESTION

LET C BE A CHAIN COMPLEX

AND A AN ABELIAN GROUP

THEN $C \otimes A$ IS ALSO A CHAIN

COMPLEX DEFINED BY

$$(C \otimes A)_n = C_n \otimes A$$

NAIVE GUESS:

$$H_n(C \otimes A) = H_n(C) \otimes A$$

COUNTER EXAMPLE

$$C: 0 \leftarrow \mathbb{Z} \xleftarrow{2} \mathbb{Z} \leftarrow 0 \leftarrow \cdots\cdots$$

$$\mathbb{Z} = C_0, \quad \mathbb{Z} = C_1, \quad 0 = C_2$$

$H_0(C) = \mathbb{Z}/2$ AND $H_1(C) = 0$

$A = \mathbb{Z}/2$ $\qquad C_0 \otimes A \qquad C_1 \otimes A$

$A = \mathbb{Z}/2$

$$C_\cdot \otimes A \qquad 0 \leftarrow \overset{C_0 \otimes A}{\overset{\parallel}{\mathbb{Z}/2}} \xleftarrow{0} \overset{C_1 \otimes A}{\overset{\parallel}{\mathbb{Z}/2}} \leftarrow 0 \leftarrow \cdots$$

$$H_i(C \otimes A) = \begin{cases} \mathbb{Z}/2 & i = 0 \\ \mathbb{Z}/2 & i = 1 \end{cases}$$

$$\neq H_i(C) \otimes A$$