DEF. THE SINGULAR CHAIN COMPLEX

Wednesday, October 14, 2020 ~~1:43 PM~~

$C(X)$ OF A SPACE X DEFINED BY

$C_n(X) =$ FREE ABELIAN GROUP GENERATED BY ALL MAPS $\Delta^n \xrightarrow[\sigma]{} X$

$$\Delta^{n-1} \xrightarrow{f_i} \Delta^n \xrightarrow{\sigma} X$$

$f_i = i$TH FACE FOR $0 \le i \le n$

$$\underbrace{(x_0, \cdots x_{n-1})}_{n \text{ COORDS}} \longmapsto \underbrace{(x_0, \cdots x_{i-1}, 0, x_i, \cdots x_{n-1})}_{(n+1) \text{ COORDS}}$$

WE DEFINE

$$d_n[\sigma] = \sum_{0 \le i \le n} (-1)^i [\sigma f_i] \in C_{n-1}(X)$$

EXERCISE: SHOW $d_{n-1} d_n = 0$.

EXAMPLE $X = pt.$ THERE IS ONE MAP $\Delta^n \to X$, SO $C_n(X) = \mathbb{Z}$ FOR ALL $n \ge 0$.

LET s_n BE THE CORRESPONDING GENERATOR OF $C_n(X)$.

$$d_n(s_n) = \sum_{i=0}^{n} (-1)^i s_{n-1} = (1-1+1-1\cdots) s_{n-1}$$

$$= \begin{cases} 0 & \text{FOR } n \text{ ODD} \\ 1 & \text{FOR } n \text{ EVEN} \end{cases}$$

$= \begin{cases} 0 & \text{FOR } n \text{ ODD} \\ \partial_{n-1} & \text{FOR } n \text{ EVEN} \end{cases}$

$$\begin{array}{ccccccccccc} 0 & & 1 & & 2 & & 3 & & 4 & & \\ \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{1} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{1} & \mathbb{Z} & \leftarrow & \cdots \end{array}$$

SO $\quad H_i(\text{point}) = \begin{cases} \mathbb{Z} & \text{FOR } i = 0 \\ 0 & \text{FOR } i > 0 \end{cases}$

THIS SATISFIES THE EILENBERG-STEENROD DIMENSION AXIOM

THEOREM (~1950) $H_*(X)$ DEFINED AS $H_* C(X)$ SATISFIES THE 4 E-S AXIOMS.

HOW TO DEFINE $H_*(X, A)$ FOR $A \subset X$.

$$0 \to C(A) \to C(X) \to C(X)/C(A) \to 0$$
$$C(X)/C(A) =: C(X, A)$$

$$H_*(X, A) := H_*(C(X)/C(A))$$

IT FOLLOW THAT WE HAVE A LONG EXACT SEQUENCE IN HOMOLOGY

$$\cdots \to H_*(A) \to H_*(X) \to H_*(X, A) \to$$

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to \cdots$$

THIS IS THE EXACTNESS AXIOM.

WE WILL PROVE THE HOMOTOPY AND EXCISION AXIOMS LATER.

ASSUMING THE AXIOMS HOLD, WE WILL NOW COMPUTE $H_*(S^n)$ FOR ALL $n \geq 0$.

THEOREM

a) $H_i(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & i=0 \\ 0 & i>0 \end{cases}$

FOR $n>0$

b) $H_i(S^n) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} & i=n \\ 0 & \text{ELSE} \end{cases}$

PROOF: (a) IS A COROLLARY OF

LEMMA $H_*(X_0 \amalg X_1) = H_*(X_0) \oplus H_*(X_1)$

(BOTH X_0 AND X_1 ARE OPEN/CLOSED IN $X_0 \amalg X_1$

PROOF ANY $\Delta^n \longrightarrow X_0 \amalg X_1$ OR ANY PATH CONNECTED SPACE
HAS IMAGE IN EITHER X_0 OR X_1.
THIS MEANS $C(X_0 \amalg X_1) \cong C(X_0) \oplus C(X_1)$

so $H_*(X_0 \amalg X_1) \cong H_*(X_0) \oplus H_*(X_1)$

QED.

PROOF FOR $n>0$ BY INDUCTION ON n. ASSUME $H_*(S^{n-1})$ IS AS CLAIMED. CONSIDER THE PAIR (S^n, D^n) WHERE $D^n \subset S^n$ IS THE NORTHERN HEMISPHERE.
THE HOMOTOPY AXIOM IMPLIES THAT $H_*(D^n) \cong H_*(pt)$. MORE GENERALLY A HOMOTOPY EQUIVALENCE INDUCES AN ISOMORPHISM IN $H_*(-)$.
WILL USE THE EXCISION AXIOM

$$(D^n)' \longrightarrow D^n \xrightarrow{\text{NORTH}} S^n$$

$(D^n)'$ = INTERIOR SMALLER DISK, $D^n = A$, $S^n = X$

$$H_*(X, A) \cong H_*(X - B, A - B)$$

$$H_*(X,A) = H_*(S^n, D^n_{\text{NORTH}}) \cong H_*(D_{\text{SOUTH}}, S^{n-1} \times [0,1])$$

$$D_{\text{SOUTH}} \cong (D^n) \cong \mathbb{R}^n \simeq pt.$$

$$\text{COLLAR} \cong S^{n-1} \times [0,1] \simeq S^{n-1}$$

THE PAIRS (S^n, D^n) AND $(D^n_{\text{SOUTH}}, S^{n-1})$ HAVE LONG EXACT SEQUENCES

$$\begin{array}{ccccccccc}
 & 0 & & \mathbb{Z} & & \mathbb{Z} & & 0 & i=n \\
\cdots \to & H_i(D^n) & \to & H_i(S^n) & \to & H_i(S^n, D^n) & \to & H_{i-1} D^n & \to \cdots \\
 & \uparrow & & \uparrow & & \uparrow \cong & & \uparrow & \\
\to & H_i S^{n-1} & \to & H_i D^n_{\text{SOUTH}} & \to & H_i(D^n_{\text{SOUTH}}, S^{n-1}) & \to & H_{i-1}(S^{n-1}) & \to 0 \\
 & & & \| & & \| & & & \\
 & & & H_i(pt) & & & & & \\
 & & & 0 & \to & \mathbb{Z} & \to & \mathbb{Z} & \to 0 \quad i=n
\end{array}$$

KNOWN BY INDUCTION

$$i=0 \quad \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \longrightarrow 0 \longrightarrow 0$$

THE BOTTOM SEQUENCE GIVES

$$H_i(D^n, S^{n-1}) = \begin{cases} \mathbb{Z} & i = n \end{cases}$$

$$H_i(D^n, S^{n-1}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq 0 \end{cases}$$

THE TOP SEQUENCE THEN GIVES $H_i(S^n)$ IS AS CLAIMED.

QED.

WALTER LEOPOLD

MAYER-VIETORIS SEQUENCE (1930)

LET $X = A \cup B$ WITH MILD HYPOTHESIS ON $A \cap B$. THERE IS A LONG EXACT SEQUENCE

$$x \longmapsto i_1(x),\ i_2(x)$$

$$\cdots \to H_i(A \cap B) \to H_i(A) \oplus H_i(B) \to H_i(X) \to H_{i-1}(A \cap B) \to \cdots$$

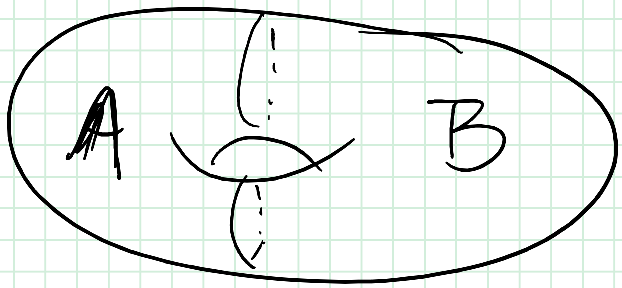
$$(a, b) \longmapsto j_1(a) - j_2(b)$$

$$\begin{tikzcd} & A \arrow[dr, "j_1"] & \\ A \cap B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] \arrow[rr, phantom, "\text{COMMUTES}"] & & X \\ & B \arrow[ur, "j_2"'] & \end{tikzcd}$$

$$j_1 i_1 = j_2 i_2$$

PROOF LATER

EXAMPLE $X =$ TORUS



$A \cap B = S^1 \amalg S^1$

$A, B \cong S^1 \times I \simeq S^1$

$H_i(A) \cong H_i(B) \cong H_i(S^1) = \begin{cases} \mathbb{Z} & \text{FOR } i=0,1 \\ 0 & \text{ELSE} \end{cases}$

$$\underbrace{H_2(A) \oplus H_2(B)}_{0} \longrightarrow H_2(X) = \mathbb{Z} \longrightarrow$$

$$H_1(A \cap B) \xrightarrow{f_1} H_1(A) \oplus H_1(B) \longrightarrow H_1(X) = \mathbb{Z} \oplus \mathbb{Z} \longrightarrow$$

$$H_0(A \cap B) \xrightarrow{f_0} H_0(A) \oplus H_0(B) \longrightarrow H_0(X) = \mathbb{Z} \longrightarrow 0$$

$H_1(A \cap B) = \mathbb{Z} \oplus \mathbb{Z} = H_1(A) \oplus H_1(B)$, $H_0(A \cap B) = \mathbb{Z} \oplus \mathbb{Z} = H_0(A) \oplus H_0(B)$

CAN ANALYZE f_0 AND f_1 AND FIND

① $H_2 X = \ker f_1 \cong \mathbb{Z}$

② $\operatorname{im} f_1 = \mathbb{Z} =$ DIRECT SUM OF $H_1 A \oplus H_1(B)$

SO $\operatorname{coker} f_1 \cong \mathbb{Z}$

SIMILARLY FOR f_0 WITH $\operatorname{coker} f_0 = H_0 X = \mathbb{Z}$

FOR $H_1(X)$ WE HAVE A SHORT EXACT SEQUENCE

$$0 \to \underset{\mathbb{Z}}{\underset{\parallel}{\operatorname{coker} f_1}} \to \underset{\mathbb{Z} \oplus \mathbb{Z}}{\underset{\parallel}{H_1(X)}} \longrightarrow \underset{\mathbb{Z}}{\underset{\parallel}{\ker f_0}} \to 0$$

NOTE: IN THE SHORT EXACT SEQUENCE

$\mathbb{Z} \oplus \mathbb{Z} \qquad \mathbb{Z}$

NOTE: IN THE SHORT EXACT SEQUENCE

$$0 \to \mathbb{Z}/2 \longrightarrow \,?\, \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

$\mathbb{Z}/2 \oplus \mathbb{Z}/2$

OR $\mathbb{Z}/4$