(1) $$0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' \to 0$$

A SHORT EXACT SEQUENCE OF CHAIN COMPLEXES AS ABOVE LEADS TO A LONG EXACT SEQENCE

(2) $$\cdots \to H_n C' \xrightarrow{i_*} H_n C \xrightarrow{j_*} H_n C'' \xrightarrow{\partial} H_{n-1} C' \xrightarrow{i_*} H_{n-1} C \to \cdots$$

A BIG DIAGRAM CHOSE HOW TO DEFINE THE CONNECTING HOMOMORPHISM ∂. THE MAPS i_* AND j_* ARE INDUCED BY i AND j. THE COMPOSITE $j_* i_* = (ji)_*$ IS ZERO BECAUSE $ji = 0$. WE NEED TO SHOW THAT (2) IS EXACT AT EACH STAGE. THIS INVOLVES MORE DIAGRAM CHASING. ~ (EXERCISE

QED

THIS IS PART OF
HOMOLOGICAL ALGEBRA

BOOKS BY

EILENBERG - STEENROD

MAC LANE

HILTON - STAMBACH

WEIBEL.

PROOFS ARE EASY, FINDING THE RIGHT STATEMENTS WAS NOT EASY.

DEF TWO CHAIN MAPS

$$C \underset{g}{\overset{f}{\rightrightarrows}} C'$$ ARE CHAIN HOMOTOPIC

IF THERE MAPS $h_n : C_n \to C'_{n+1}$

SUCH THAT

$$\begin{tikzcd}
C_{n-1} \arrow[dr, "h_{n-1}"] & C_n \arrow[l, "d_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dr, "h_n"] & C_{n+1} \arrow[l, "d_{n+1}"'] \\
C'_{n-1} & C'_n \arrow[l, "d'_n"] & C'_{n+1} \arrow[l, "d'_{n+1}"]
\end{tikzcd}$$

$x \mapsto h_n(x)$

$d_n(x) = 0$

$$(3) \quad h_{n-1} d_n + d'_{n+1} h_n = f_n - g_n$$

SUCH A COLLECTION IS A <u>CHAIN HOMOTOPY</u> BETWEEN f AND g. (FOR A GENERALIZATION, SEE MODEL CATEGORIES)

THEOREM. IF f AND g AS ABOVE ARE CHAIN HOMOTOPIC, THEN $H_*(f) = H_*(g)$.

PROOF. IT SUFFICES TO SHOW $H_*(f-g) = 0$. LET $\alpha \in H_n C$ BY REPRESENTED BY $x \in C_n$

$$(f_n - g_n)(x) = (h_{n-1} \underline{d_n} + d'_{n+1} h_n)(x) \text{ BY } (3)$$

$$= d'_{n+1} h_n (x)$$

$$= \text{BOUNDARY}$$

IT REPRESENTS 0 IN $H_n(C')$

QED.

ANALOGOUS STATEMENT IN

TOPOLOGY: IF $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ ARE HOMOTOPIC, THEN $\pi_k(f) = \pi_k(g)$.

LONG TERM STRATEGY

WE WANT TO DEFINE GROUPS $H_n(X)$ AND $H_n(X, A)$ FOR $A \subset X$ THAT SATISFY THE EILENBERG-STEENROD AXIOMS. WILL ASSOCIATE A CHAIN COMPLEX $C(X)$ TO EACH SPACE X. WILL SHOW HOMOTOPIC MAPS $f, g : X \to Y$ INDUCE CHAIN HOMOTOPIC MAPS

$$C(f), C(g) : C(X) \to C(Y).$$

$\rightsquigarrow$ HOMOTOPY AXIOM.

<u>MORE ALGEBRA</u>

WE NEED THE <u>TENSOR PRODUCT</u> $A \otimes B$ OF ABELIAN GROUPS

A AND B.

PROPERTIES:

1) $A \otimes B \cong B \otimes A$

2) $A \otimes (B' \oplus B'') \cong (A \otimes B') \oplus (A \otimes B'')$

3) $\mathbb{Z} \otimes A = A$ ($\mathbb{Z}$ = INTEGERS)

4) $\mathbb{Z}/m \otimes \mathbb{Z}/n \cong \mathbb{Z}/\gcd(m,n)$

5) $\mathbb{Z}/m \otimes \mathbb{Q} \cong 0$

6) $\mathbb{Q} \otimes \mathbb{Q} \cong \mathbb{Q}$

1) – 4) DETERMINE $A \otimes B$ FOR FINITELY GENERATED A AND B.

WANT TO DEFINE A CHAIN COMPLEX $C = C' \otimes C''$ FOR CHAIN COMPLEXES C' AND $''$ WITH $C_n = \bigoplus_{0 \le i \le n} C'_i \otimes C''_{n-i}$

HOW TO DEFINE d IN TERMS OF d' AND d''?

LET $a \in C'_i$ AND $b \in C'_{n-i}$

THEN $d_n(a \otimes b)$

$$\overset{?}{=} d'_i(a) \otimes b - a \otimes d''_{n-i}(b)$$

BUT THEN

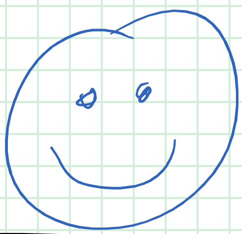
$$d_{n-1}\, d_n (a \otimes b)$$

$$= d_{n-1}\left(d'_i(a) \otimes b + (-1)^i a \otimes d''_{n-i}(b) \right)$$

ESSENTIAL SIGN

$$= \left(\underbrace{d'_{i-1}\, d'_i(a)}_{0} \otimes b + (-1)^{i-1} d'_i(a) \otimes d''_{n-i}(b) \right) + (-1)^i \left(d'_i(a) \otimes d''_{n-i}(b) + (-1)^i a \otimes \underbrace{d''_{n-i-1}\, d''_{n-i}(b)}_{0} \right)$$

$$= 0$$



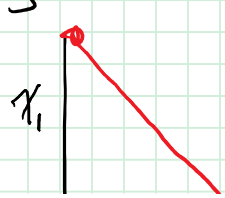
BACK TO TOPOLOGY

DEF THE STANDARD <u>N-SIMPLEX</u>,

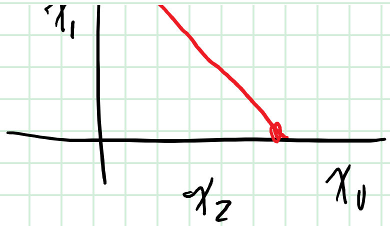
$$\Delta^n := \left\{ (x_0, x_1, \dots x_n) \in \mathbb{R}^{n+1} : \begin{array}{l} x_i \geq 0 \\ \sum x_i = 1 \end{array} \right\}$$

$$\subset [0,1]^{n+1}$$

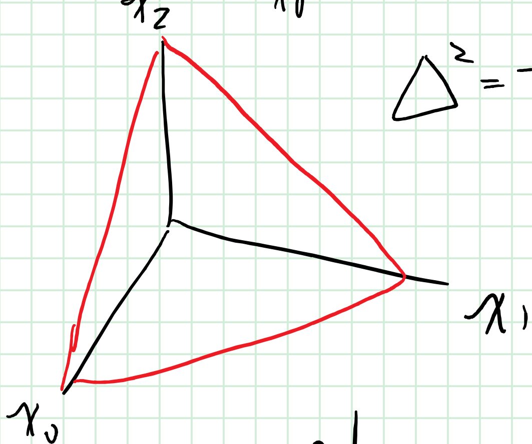
e.g. $n=1$



e.g. $n=1$



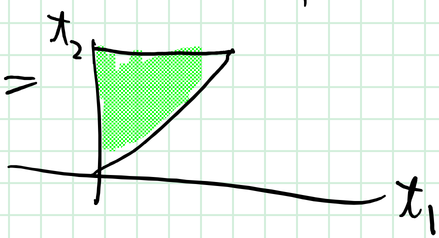
$n=2$



Δ^2 = TRIANGLE

Δ^3 = TETRAHEDRON

$$\boxed{\begin{array}{l}\text{ALTERNATE DEFINITION}\\ \Delta^n = \{(t_1, t_2 \cdots t_n) \in \mathbb{R}^n : 0 \le t_1 \le t_2 \le \cdots \le t_n \le 1\}\\ \Delta^1 = [0,1] \qquad \Delta^2 = \end{array}}$$



FOR $0 \le i \le n$ THERE IS A MAP

$$f_i : \Delta^{n-1} \longrightarrow \Delta^n$$

$$(x_0, x_1 \cdots x_{n-1}) \mapsto (x_0, \cdots x_{i-1}, 0, x_i, \cdots x_{n-1})$$

$$f_0 : (x_0, \cdots x_{n-1}) = (0, x_0, \cdots x_{n-1})$$

$$f_n : (x_0, \cdots x_{n-1}) = (x_0, \cdots x_{n-1}, 0)$$

THESE ARE <u>FACE MAPS</u>

LET X BE A TOP SPACE

WE DEFINE A CHAIN COMPLEX

$C(X)$ BY

$C_n(X)$ = THE FREE ABELIAN GP GENERATED BY THE SET OF CONT. MAP $\sigma : \Delta^n \to X$.

(VERY BIG)

TO DEFINE $d_n : C_n(X) \to C_{n-1}(X)$

$$\begin{tikzcd} \Delta^{n-2} \arrow[r, "f_j"] & \Delta^{n-1} \arrow[r, "f_i"] \arrow[rr, bend right, "\sigma f_i"'] & \Delta^n \arrow[r, "\sigma"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

$0 \le j \le n-1$ $\qquad$ $0 \le i \le n$

$[\sigma f_i] \in C_{n-1}(X)$

LET $[\sigma] \in C_n(X)$ BE THE ELEMENT CORRESPONDING TO $\sigma : \Delta^n \to X$.

THEN $d_n[\sigma] = \sum_{0 \le i \le n} (-1)^i [\sigma f_i] \in C_{n-1}(X)$

NEED TO SHOW $d_{n-1} d_n [\sigma] = 0$

$$d_{n-1} d_n [\sigma] = d_{n-1}\left(\sum_{0 \le i \le n} (-1)^i [\sigma f_i] \right)$$

$$d_{n-1} d_n \sigma = d_{n-1}\Big(\sum_{0 \le i \le n} (-1)^i \sigma f_i\Big)$$

$$= \sum_{0 \le i \le n} (-1)^i \, d_{n-1}[\sigma f_i]$$

$$= \sum_{0 \le i \le n} (-1)^i \boxed{\sum_{0 \le j \le n-1} (-1)^j [\sigma f_i f_j]}$$

EXERCISE: WHAT IS

$$f_i f_j : \Delta^{n-2} \longrightarrow \Delta^n$$

$$(x_0, \dots x_{n-2}) \longmapsto \underbrace{(x_0, \dots\dots x_{n-2})}_{n+1 \text{ COORDINATES}}$$

2 COORDINATES ARE ZERO

THE DOUBLE SUM HAS $n(n+1)$ TERMS. THERE $\binom{n+1}{2} = \frac{(n+1)(n)}{2}$ CHOICES FOR VANISHING COORDINATES. CAN SHOW THE TWO TERMS WITH THE SAME PAIR OF VANISHING COORDS HAVE OPPOSITE SIGNS, SO $d_{n-1} d_n = 0$ AS DESIRED.

$C(X)$ IS THE <u>SINGULAR</u>

CHAIN COMPLEX OF $\underline{X}$.