Math 443 **Final exam**

December 16, 2017

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ______________________________

Be sure to write your name on your bluebook.
INSERT THIS PAGE WITH THE HONOR PLEDGE SIGNED INTO YOUR BLUEBOOK.
Use a separate page (or pages) for each problem. Show all of your work.

1. (30 POINTS) **Three dimensional grid question.** Let $X_0 \subset \mathbf{R}^3$ be the set of points (x, y, z) in which two of the three coordinates are integers. It is an infinite union of lines parallel to the coordinate axes, and each point with three integer coordinates is the intersection of three such lines.

   Choose a number ϵ with $0 < \epsilon < 1/2$ and let $X_1 \subset \mathbf{R}^3$ be the set of points whose distance from X_0 is $\leq \epsilon$. It is a 3-manifold whose boundary X_2 is a noncompact surface.

   Let $G = \mathbf{Z}^3 \subset \mathbf{R}^3$. It acts freely on all three spaces by translation, and each orbit space is compact.

   (a) Decscribe the graph X_0/G and compute its Euler characteristic and fundamental group.

   (b) Find the genus of the surface X_2/G.

   HINT: Consider the intersection of each X_i with the cube $[-1/2,\, 1/2]^3$. Then $\mathbf{R}^3/G$ is the 3-torus obtained by making appropriate identifications on this cube. The orbit spaces X_0/G and X_2/G can be studied in a similar way.

   Solution:

   (a) X_0/G has one vertex, the image of $(0, 0, 0)$, and three edges, the images of the three coordinate axes. Hence its Euler characteristic is -2 and its fundamental group is free on 3 generators.

   (b) X_1/G is the 3-manifold obtained by attaching three handles to D^3 and its boundary X_2/G has genus 3.

2. **Projective plane question.** Let $X = \mathbf{R}P^2$ and let X^k denote the k-fold Cartesian product of X.

   (a) (5 POINTS) Find $H_*(X^2; \mathbf{Z}/2)$. Recall that homology with field coefficients converts Cartesian products to tensor products.

Solution:

$$H_*(X^2;\mathbf{Z}/2) = H_*(X;\mathbf{Z}/2)\otimes H_*(X;\mathbf{Z}/2)$$

$$H_n(X^2;\mathbf{Z}/2) = \bigoplus_{i+j=n} H_i(X;\mathbf{Z}/2)\otimes H_j(X;\mathbf{Z}/2)$$

$$= \begin{cases} \mathbf{Z}/2 & \text{for } n=0 \\ (\mathbf{Z}/2)^2 & \text{for } n=1 \\ (\mathbf{Z}/2)^3 & \text{for } n=2 \\ (\mathbf{Z}/2)^2 & \text{for } n=3 \\ \mathbf{Z}/2 & \text{for } n=4 \\ 0 & \text{otherwise} \end{cases}$$

(b) (5 POINTS) Find $H_*(X^2;\mathbf{Z})$.

Solution: Here we have to use the Künneth formula, which says that

$$H_n(X^2;\mathbf{Z}) = \bigoplus_{i+j=n} H_i(X;\mathbf{Z})\otimes H_j(X;\mathbf{Z}) \oplus \bigoplus_{i+j=n-1} (H_i(X;\mathbf{Z}), H_j(X;\mathbf{Z}))$$

Then we have

$$\bigoplus_{i+j=n} H_i(X;\mathbf{Z})\otimes H_j(X;\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{for } n=0 \\ \mathbf{Z}/2\oplus\mathbf{Z}/2 & \text{for } n=1 \\ \mathbf{Z}/2 & \text{for } n=2 \\ 0 & \text{otherwise} \end{cases}$$

$$\bigoplus_{i+j=n-1} (H_i(X;\mathbf{Z}), H_j(X;\mathbf{Z})) = \begin{cases} \mathbf{Z}/2 & \text{for } n=3 \\ 0 & \text{otherwise} \end{cases}$$

so

$$H_n(X^2;\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{for } n=0 \\ \mathbf{Z}/2\oplus\mathbf{Z}/2 & \text{for } n=1 \\ \mathbf{Z}/2 & \text{for } n=2 \\ \mathbf{Z}/2 & \text{for } n=3 \\ 0 & \text{otherwise} \end{cases}$$

(c) (5 POINTS) Find $H_*(X^3;\mathbf{Z}/2)$.

Solution: A similar calculation to (a) gives

$$H_n(X^3;\mathbf{Z}/2) = \bigoplus_{i+j=n} H_i(X;\mathbf{Z}/2)\otimes H_j(X^2;\mathbf{Z}/2)$$
$$= \begin{cases} \mathbf{Z}/2 & \text{for } n=0 \\ (\mathbf{Z}/2)^3 & \text{for } n=1 \\ (\mathbf{Z}/2)^6 & \text{for } n=2 \\ (\mathbf{Z}/2)^7 & \text{for } n=3 \\ (\mathbf{Z}/2)^6 & \text{for } n=4 \\ (\mathbf{Z}/2)^3 & \text{for } n=5 \\ \mathbf{Z}/2 & \text{for } n=6 \\ 0 & \text{otherwise} \end{cases}$$

(d) (5 POINTS) For a space Y define the mod 2 to be

$$g_{\mathbf{Z}/2}(Y)(t) = \sum_{n\geq 0} \operatorname{rank}(H_n(Y;\mathbf{Z}/2))t^n.$$

Find $g_{\mathbf{Z}/2}(X^k)(t)$ for $k \geq 0$.

Solution: Since mod 2 homology converts Cartesian products to tensor products,

$$g_{\mathbf{Z}/2}(A\times B)(t) = g_{\mathbf{Z}/2}(A)(t)g_{\mathbf{Z}/2}(B)(t).$$

Since $g_{\mathbf{Z}/2}(X)(t) = 1+t+t^2$,

$$g_{\mathbf{Z}/2}(X^k)(t) = (1+t+t^2)^k.$$

3. (20 POINTS) The 5-lemma says that given a commutative diagram of abelian groups with exact rows,

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "\ell"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\ A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "\ell'"] & E', \end{tikzcd}$$

if α, β, δ and ϵ are isomorphisms, then so is γ. Show by counterexample that the triviality of α, β, δ and ϵ does *not* imply the triviality of γ.

Solution: Let p by a prime. One counterexample is

$$\begin{tikzcd} 0 \arrow[r, "i"] \arrow[d, "0"] & \mathbf{Z}/p \arrow[r, "j"] \arrow[d, "0"] & \mathbf{Z}/p^2 \arrow[r, "k"] \arrow[d, "p"] & \mathbf{Z}/p \arrow[r, "\ell"] \arrow[d, "0"] & 0 \arrow[d, "0"] \\ 0 \arrow[r, "i'"] & \mathbf{Z}/p \arrow[r, "j'"] & \mathbf{Z}/p^2 \arrow[r, "k'"] & \mathbf{Z}/p \arrow[r, "\ell'"] & 0. \end{tikzcd}$$

Another one is

$$\begin{tikzcd} 0 \arrow[r, "i"] \arrow[d, "0"] & \mathbf{Z} \arrow[r, "j"] \arrow[d, "0"] & \mathbf{Z} \arrow[r, "k"] \arrow[d, "1"] & 0 \arrow[r, "\ell"] \arrow[d, "0"] & 0 \arrow[d, "0"] \\ 0 \arrow[r, "i'"] & 0 \arrow[r, "j'"] & \mathbf{Z}/p^2 \arrow[r, "k'"] & \mathbf{Z} \arrow[r, "\ell'"] & 0. \end{tikzcd}$$

4. (20 POINTS) Let X be a finite CW-complex which is the union of two sub-CW-complexes A and B such that the intersection $A \cap B$ is also a sub-CW-complex. Show that the Euler characteristic $\chi(X)$ satisfies the formula

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

Solution: Let x_i, a_i, b_i and c_i denote the number of i-cells in X, A, B and $A \cap B$ respectively. Each cell of X lies in either A or B or possibly both. This means that $x_i = a_i + b_i - c_i$; we subtract c_i so as not to count cells in the intersection twice. It follows that

$$\begin{aligned} \chi(X) &= \sum_{i \geq 0} (-1)^i x_i \\ &= \sum_{i \geq 0} (-1)^i (a_i + b_i - c_i) \\ &= \sum_{i \geq 0} (-1)^i a_i + \sum_{i \geq 0} (-1)^i b_i - \sum_{i \geq 0} (-1)^i c_i \\ &= \chi(A) + \chi(B) - \chi(A \cap B). \end{aligned}$$

5. (20 POINTS) Prove or disprove the Borsuk-Ulam theorem for the torus, which says the following. For every map $f : S^1 \times S^1 \to \mathbf{R}^2$, there is a point (x, y) in $S^1 \times S^1$ such that $f(-x, -y) = f(x, y)$. Here we regard S^1 as the unit circle in the complex numbers in order to define $-z$ for $z \in S^1$.

Solution: The theorem is false. Let f be the composite of projection p_1 onto the first coordinate followed by an embedding i of S^1 into the plane. Then we have

$$f(-x, -y) = i(-x) \neq i(x) = f(x, y).$$

6. (20 POINTS) Prove the 2-dimensional case of the Brouwer Fixed Point Theorem, i.e., that any continuous map of the 2-dimensional disk D^2 to itself has a fixed point. You may assume $\pi_1 S^1 = \mathbf{Z}$.

Solution: See page 32 of Hatcher.