LIN JINKUN'S WORK ON THE ADAMS SPECTRAL SEQUENCE
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Lin's big theorem

Theorem 1 (Lin [Lin03]). *For each prime $p \geq 5$ and each integer $n \geq 0$, the Hopf invariant one element*

$$h_n \in \mathrm{Ext}_A^{1,p^n q}(H^*(K), \mathbf{Z}/(p))$$

is a permanent cycle and therefore represents a map

$$\omega_n : S^{p^n q-1} \to K.$$

Here $q = 2p-2$ and K is Toda's 4-cell complex $V(1)$.

Background

The Adams spectral sequence [Ada58] is a method for computing the graded group of maps $[X, Y]_*$ for suitable spectra X and Y. Its E_2-term is

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(Y), H^*(X))$$

where A is the mod p Steenrod algebra, and the cohomology groups have coefficients in $\mathbf{Z}/(p)$.This group is contravariant in X and covariant in Y.

We will abbreviate

$$\begin{aligned} \mathrm{Ext}(H^*(Y)) &:= \mathrm{Ext}_A(H^*(Y), \mathbf{Z}/(p)), \\ & \quad \text{the } E_2\text{-term for } \pi_*(Y) \end{aligned}$$

$$\begin{aligned} \text{and} \qquad \mathrm{Ext} &:= \mathrm{Ext}(\mathbf{Z}/(p)), \\ & \quad \text{the } E_2\text{-term for } \pi_*(S^0) \end{aligned}$$

Ext^1 has basis dual to the set of algebra generators for the mod p Steenrod algebra A, the Bockstein operation Δ and the reduced power operations $\mathcal{P}^{p^n}$ for $n \geq 0$, giving elements

$$a_0 \in \mathrm{Ext}^{1,1} \qquad \text{and} \qquad h_n \in \mathrm{Ext}^{1,p^n q}.$$

Ext^2 has basis dual to the set of algebra relations for the mod p Steenrod algebra A. In particular for each $n > 0$, there is an element $b_{n-1} \in \mathrm{Ext}^{2,p^n q}$ corresponding to the relation

$$\left(\mathcal{P}^{p^{n-1}}\right)^p = \dots$$

$$\begin{array}{rclrcl} a_0 & \in & \mathrm{Ext}^{1,1} & h_n & \in & \mathrm{Ext}^{1,p^n q} \\ b_{n-1} & \in & \mathrm{Ext}^{2,p^n q} & & & \end{array}$$

The following facts about these elements are known.
a_0 corresponds to p times the identity map ι and h_0 corresponds to the map $\alpha_1 : S^{q-1} \to S^0$.
Liulevicius [Liu62] (using methods introduced by Adams [Ada60] for $p = 2$) showed that for p odd, h_n for $n > 0$ is not a permanent cycle, but instead there is a nontrivial differential

$$d_2(h_n) = a_0 b_{n-1}.$$

I showed [Rav78] that for $p \geq 5$ and $n \geq 2$,

$$d_{2p-1}(b_{n-1}) = h_0 b_{n-2}^p \neq 0.$$

Ralph Cohen [Coh81] showed that for $p \geq 3$ and $n \geq 1$, $h_0 b_{n-1}$ is a permanent cycle represented by a map

$$\zeta_{n-1} : S^{N-3} \to S^0$$

where $N = (p^n + 1)q$. This is the geometric input for Lin's theorem.

The mod p Moore spectrum M

M is known to be a ring spectrum for $p \geq 3$. There is a cofiber sequence

$$S^0 \xrightarrow{i} M \xrightarrow{j} S^1 \xrightarrow{p\iota} S^1.$$

The element $i_*(a_0)$ vanishes in $\mathrm{Ext}(H^*(M))$, so (2) above is moot here, but we do have

(3′) There is a differential

$$d_{2p-1}(i_*(h_n)) = a_1 i_*(b_{n-2})^p$$

where $a_1 \in \mathrm{Ext}^{1,q+1}(H^*(M))$ corresponds to the composite

$$S^q \xrightarrow{i} \Sigma^q M \xrightarrow{\alpha} M$$

and α is the Adams self-map.

(4′) Cohen [Coh81] showed that for $p \geq 3$ and $n \geq 1$, $i_*(h_0 h_n)$ is a permanent cycle.

Toda's 4-cell complex $K = V(1)$

There is a cofiber sequence

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M$$

and K is known to be a ring spectrum for $p \geq 5$.

Lin's Theorem says that

$$(i'i)_*(h_n) \in \mathrm{Ext}^{1,p^n q}(H^*(K))$$

is a permanent cycle.

Consequences of Lin's theorem

Assuming Lin's theorem is true, let

$$\omega_n : S^{p^n q-1} \to K$$

denote a map detected by $(i'i)_*(h_n)$.

*2 There is a well known map (due to Toda)

$$\beta : \Sigma^{(p+1)q} K \to K$$

whose iterates are all nontrivial.*3 Lin shows that for $n > 2$ and $0 < s < p^{n-2}$, the composite

$$S^{p^n q-1} \xrightarrow{\omega_n} K \xrightarrow{\beta^s} \Sigma^{-s(p+1)q} K \xrightarrow{jj'} S^{q+2-s(p+1)q}$$

is nontrivial and is detected by the element $\gamma_{p^{n-2}/p^{n-2}-s}$ in the Adams-Novikov spectral sequence.

$(3')$ may imply that for $n > 2$,

$$S^{p^n q-1} \xrightarrow{\omega_n} K \xrightarrow{j'} \Sigma^{q+1} M$$

is detected by

$$i_*(b_{n-2}^p) \in \mathrm{Ext}^{2p,p^n q}(H^*(M))$$

and $j'\omega_n$ may lift to S^{q+1}.

For $n = 2$, the composite

$$S^{p^2 q-1} \xrightarrow{\omega_2} K \xrightarrow{jj'} S^{q+2}$$

is $\gamma_1 = \alpha_1 \beta_{p-1} \in \pi_{p^2 q-q-3}(S^0)$.

For $n = 1$, $j'\omega_1 = 0$ and we have

$$\begin{tikzcd}
S^1 & S^{pq-1} \arrow[l, dotted, "\beta_1"'] \arrow[r] \arrow[d, "\omega_1"] \arrow[dl, dotted] & \mathrm{pt.} \arrow[d] \\
M \arrow[u, "j"] \arrow[r, "i'"] & K \arrow[r, "j'"] & \Sigma^{q+1} M
\end{tikzcd}$$

Steps in Lin's argument

Proposition 2 (Toda, [Tod71])**.** *There is a map $\alpha'' : \Sigma^{q-2}K \to K$ with*

$$\begin{tikzcd}
\Sigma^{q-2}M \arrow[r, "i'"] \arrow[d, "j"] & \Sigma^{q-2}K \arrow[r, "\alpha_1\wedge K"] \arrow[ddr, "\alpha''"] & \Sigma^{-1}K \arrow[r, "j'"] & \Sigma^{q}M \arrow[d, "j"] \\
S^{q-1} \arrow[d, "i"] & & & S^{q+1} \arrow[d, "i"] \\
\Sigma^{q-1}M \arrow[r, "i'"] & \Sigma^{q-1}K \arrow[r, "\alpha_1\wedge K"] & K \arrow[r, "j'"] & \Sigma^{q+1}M.
\end{tikzcd}$$

Theorem 3 (Theorem 3.4 of [Lin03])**.** *Let $a'' \in \operatorname{Ext}^{1,q-1}(H^*(K), H^*(K))$ detect α''. Then*

$$(h_0h_n)'' = a''(i'i)_*(h_n) \in \operatorname{Ext}^{2,N-1}(H^*(K), H^*(K))$$

(where $N = (p^n+1)q$) is a permanent cycle detecting a map

$$\eta_n'' : \Sigma^{N-3}K \to K.$$

The proof of this is very difficult.

For the next step we need a minimal Adams resolution

$$\begin{tikzcd}
S^0 \arrow[r, equal] & E_0 \arrow[d, "\overline{b}_0"] & \Sigma^{-1}E_1 \arrow[l, "\overline{a}_0"'] \arrow[d, "\overline{b}_1"] & \Sigma^{-2}E_2 \arrow[l, "\overline{a}_1"'] \arrow[d, "\overline{b}_2"] & \cdots \arrow[l, "\overline{a}_2"'] \\
H/p \arrow[r, equal] & KG_0 & KG_1 & KG_2 &
\end{tikzcd}$$

and the cofiber sequence

$$S^0 \xrightarrow{i'i} K \xrightarrow{r} Y \xrightarrow{\epsilon} S^1.$$

Corollary 4 (Corollary 3.10 of [Lin03])**.** *The map η_n'' lifts to a map $\eta_{n,2}'' : \Sigma^{N-3}K \to \Sigma^{-2}E_2 \wedge K$, and there is a map $(\eta_{n,2}'')_Y$ making the following diagram commute.*

$$\begin{tikzcd}
\Sigma^N K \arrow[r, "r"] \arrow[d, "\eta_{n,2}''"] & \Sigma^N Y \arrow[r, "(\eta_{n,2}'')_Y"] & \Sigma E_2 \wedge K \arrow[d, "\overline{b}_2\wedge K"] \\
\Sigma E_2 \wedge K \arrow[rr, "\overline{b}_2\wedge K"] & & \Sigma KG_2 \wedge K
\end{tikzcd}$$

The proof of this is not difficult.

The next step concerns the cofiber sequence

$$\Sigma^{q-2}K \xrightarrow{\alpha''} K \xrightarrow{w} X \xrightarrow{u} \Sigma^{q-1}K$$

Lemma 5 (Lemma 4.2 of [Lin03]). *Modulo higher Adams filtration, the composite*

$$\Sigma^{N-3}K \xrightarrow{r} \Sigma^{N-3}Y \xrightarrow{(\eta_n'')_Y} K \xrightarrow{w} X$$

is $\lambda' w(\zeta_{n-1} \wedge K)$ *for a nonzero scalar* λ', *where* ζ_{n-1} *is Cohen's map.*

In other words, there is a map f_1'' making the following diagram commute

$$\begin{tikzcd}
\Sigma^N K \arrow[r, "r"] \arrow[dd, "(f_1'', -\zeta_{n-1}\wedge K)"] & \Sigma^N Y \arrow[d, "(\eta_n'')_Y"] \\
 & \Sigma^3 K \arrow[d, "w"] \\
\left(\Sigma^{-1}E_4 \wedge X\right) \vee \Sigma^3 K \arrow[r, "(\overline{a}_{0,3}\wedge X)\vee w"] & \Sigma^3 X
\end{tikzcd}$$

where $\overline{a}_{0,3} = \overline{a}_0\overline{a}_1\overline{a}_2\overline{a}_3$.

The proof of this is difficult, as is the derivation of Theorem 1 from it. The latter involves studying a diagram that uses the cofiber sequences

$$\Sigma^{-1}E_{s+1} \xrightarrow{\overline{a}_s} E_s \xrightarrow{\overline{b}_s} KG_s \xrightarrow{\overline{c}_s} E_{s+1},$$

$$S^0 \xrightarrow{wi'i} X \xrightarrow{w_1} W \xrightarrow{u_1} S^1$$

and

$$\Sigma^{q-2}K \xrightarrow{r\alpha''} Y \xrightarrow{w_2} W \xrightarrow{u_2} \Sigma^{q-1}K.$$

$$
\begin{tikzcd}
 & & & & S^N \arrow[dllll, dotted, "\overline{g}_4"'] \arrow[ddllll, dotted, "\overline{g}_3"'] \arrow[dddllll, dotted, "\overline{g}_2"'] \arrow[ddddllll, dotted, "\overline{g}"'] \arrow[ddll, "i'i"'] \arrow[dddl, dotted, "\overline{f}_2"'] \arrow[ddd, dotted, "\omega_{n,1}"'] \arrow[dddddd, bend left, "\zeta_{n-1,1}"] \\
\Sigma^{-1}KG_3\wedge X \arrow[d, "KG_3\wedge u"'] \arrow[ddrr, "\overline{c}_3\wedge X"] & & & & \\
\Sigma^{q-2}KG_3\wedge K \arrow[dr, "\overline{c}_3\wedge K"] & & \Sigma^N K \arrow[d, "f_1''"'] & & \\
\Sigma^{q-1}KG_2\wedge K \arrow[dr, "\overline{c}_2\wedge K"] & \Sigma^{q-2}E_4\wedge K \arrow[d, "\overline{a}_3\wedge K"] & \Sigma^{-1}E_4\wedge X \arrow[l, "E_4\wedge u"'] \arrow[d, "\overline{a}_3\wedge X"] & \Sigma^{-1}E_4\wedge K \arrow[l, "E_4\wedge w"'] \arrow[d, "\overline{a}_3\wedge K"] & \Sigma^q E_1\wedge K \arrow[dddl, bend left, "E_1\wedge\alpha''"'] \\
\Sigma^q KG_1\wedge K \arrow[dr, "\overline{c}_1\wedge K"] & \Sigma^{q-1}E_3\wedge K \arrow[d, "\overline{a}_2\wedge K"] & E_3\wedge X \arrow[l, "E_3\wedge u"'] \arrow[d, "\overline{a}_2\wedge X"] & E_3\wedge K \arrow[l, "E_3\wedge w"'] \arrow[d, "\overline{a}_2\wedge K"] & \\
 & \Sigma^q E_2\wedge K \arrow[d, "\overline{a}_1\wedge K"] & \Sigma E_2\wedge X \arrow[l, "E_2\wedge u"'] \arrow[d, "\overline{a}_1\wedge X"] & \Sigma E_2\wedge K \arrow[l, "E_2\wedge w"'] \arrow[d, "\overline{a}_1\wedge K"] & \\
 & \Sigma^{q+1}E_1\wedge K & \Sigma^2 E_1\wedge X \arrow[l, "E_1\wedge u"'] \arrow[dr, "E_1\wedge w_1"'] & \Sigma^2 E_1\wedge K \arrow[l, "E_1\wedge w"'] & \Sigma^2 E_1 \arrow[l, "E_1\wedge i'i"] \arrow[d] \\
 & & & \Sigma^2 E_1\wedge W \arrow[ull, bend left, "E_1\wedge u_2"] & \mathrm{pt.} \arrow[l]
\end{tikzcd}
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