Lin Jinkun's work on the Adams spectral sequence

UR Topology Seminar
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Lin's big theorem

Theorem 1 (Lin [Lin03]) *For each prime $p \geq 5$ and each integer $n \geq 0$, the Hopf invariant one element*

$$h_n \in \mathrm{Ext}_A^{1,p^n q}(H^*(K), \mathbf{Z}/(p))$$

is a permanent cycle and therefore represents a map

$$\omega_n : S^{p^n q - 1} \to K.$$

Here $q = 2p - 2$ and K is Toda's 4-cell complex $V(1)$.

Background

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where A is the mod p Steenrod algebra, and the cohomology groups have coefficients in $\mathbf{Z}/(p)$.
This group is contravariant in X and covariant in Y.

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$$\begin{aligned} \mathrm{Ext} &:= \mathrm{Ext}(\mathbf{Z}/(p)), \\ & \quad \text{the } E_2\text{-term for } \pi_*(S^0) \end{aligned}$$

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Ext^1 has basis dual to the set of algebra generators for the mod p Steenrod algebra A, the Bockstein operation Δ and the reduced power operations $\mathcal{P}^{p^n}$ for $n \geq 0$, giving elements

$$a_0 \in \mathrm{Ext}^{1,1} \qquad \text{and} \qquad h_n \in \mathrm{Ext}^{1,p^n q}.$$

Background

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$$\mathrm{Ext}^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbf{Z}/(p), \mathbf{Z}/(p))$$

Ext^2 has basis dual to the set of algebra relations for the mod p Steenrod algebra A. In particular for each $n > 0$, there is an element $b_{n-1} \in \mathrm{Ext}^{2,p^n q}$ corresponding to the relation

$$\left(\mathcal{P}^{p^{n-1}}\right)^p = \ldots$$

Background

$$\begin{aligned} a_0 &\in \text{Ext}^{1,1} \qquad h_n \in \text{Ext}^{1,p^n q} \\ b_{n-1} &\in \text{Ext}^{2,p^n q} \end{aligned}$$

The following facts about these elements are known.

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The following facts about these elements are known.

(1) a_0 corresponds to p times the identity map ι and h_0 corresponds to the map $\alpha_1 : S^{q-1} \longrightarrow S^0$.

Background

$$a_0 \in \mathrm{Ext}^{1,1} \qquad h_n \in \mathrm{Ext}^{1,p^n q}$$
$$b_{n-1} \in \mathrm{Ext}^{2,p^n q}$$

The following facts about these elements are known.

(2) Liulevicius [Liu62] (using methods introduced by Adams [Ada60] for $p = 2$) showed that for p odd, h_n for $n > 0$ is not a permanent cycle, but instead there is a nontrivial differential

$$d_2(h_n) = a_0 b_{n-1}.$$

Background

$$\begin{array}{rclcrcl} a_0 & \in & \mathrm{Ext}^{1,1} & \quad & h_n & \in & \mathrm{Ext}^{1,p^n q} \\ b_{n-1} & \in & \mathrm{Ext}^{2,p^n q} & & & & \end{array}$$

The following facts about these elements are known.

(3) I showed [Rav78] that for $p \geq 5$ and $n \geq 2$,

$$d_{2p-1}(b_{n-1}) = h_0 b_{n-2}^p \neq 0.$$

Background

$$\begin{aligned} a_0 &\in \mathrm{Ext}^{1,1} \qquad h_n \in \mathrm{Ext}^{1,p^n q} \\ b_{n-1} &\in \mathrm{Ext}^{2,p^n q} \end{aligned}$$

The following facts about these elements are known.

(4) Ralph Cohen [Coh81] showed that for $p \geq 3$ and $n \geq 1$, $h_0 b_{n-1}$ is a permanent cycle represented by a map

$$\zeta_{n-1} : S^{N-3} \longrightarrow S^0$$

where $N = (p^n + 1)q$. This is the geometric input for Lin's theorem.

The mod p Moore spectrum M

M is known to be a ring spectrum for $p \geq 3$.

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There is a cofiber sequence

$$S^0 \xrightarrow{i} M \xrightarrow{j} S^1 \xrightarrow{p\iota} S^1.$$

The mod p Moore spectrum M

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$(3')$ There is a differential

$$d_{2p-1}(i_*(h_n)) = a_1 i_*(b_{n-2})^p$$

where $a_1 \in \mathrm{Ext}^{1,q+1}(H^*(M))$ corresponds to the composite

$$S^q \xrightarrow{i} \Sigma^q M \xrightarrow{\alpha} M$$

and α is the Adams self-map.

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$(4')$ Cohen [Coh81] showed that for $p \geq 3$ and $n \geq 1$, $i_*(h_0 h_n)$ is a permanent cycle.

Toda's 4-cell complex $K = V(1)$

There is a cofiber sequence

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M$$

and K is known to be a ring spectrum for $p \geq 5$. Lin's Theorem says that

$$(i'i)_*(h_n) \in \mathrm{Ext}^{1,p^n q}(H^*(K))$$

is a permanent cycle.

Consequences of Lin's theorem

Assuming Lin's theorem is true, let

$$\omega_n : S^{p^n q-1} \longrightarrow K$$

denote a map detected by $(i'i)_*(h_n)$.

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There is a well known map (due to Toda)

$$\beta : \Sigma^{(p+1)q} K \longrightarrow K$$

whose iterates are all nontrivial.

Consequences of Lin's theorem

Assuming Lin's theorem is true, let

$$\omega_n : S^{p^n q-1} \longrightarrow K$$

denote a map detected by $(i'i)_*(h_n)$.
Lin shows that for $n > 2$ and $0 < s < p^{n-2}$, the composite

$$S^{p^n q-1} \xrightarrow{\omega_n} K \xrightarrow{\beta^s} \Sigma^{-s(p+1)q} K \xrightarrow{jj'} S^{q+2-s(p+1)q}$$

is nontrivial and is detected by the element $\gamma_{p^{n-2}/p^{n-2}-s}$ in the Adams-Novikov spectral sequence.

Consequences of Lin's theorem

Assuming Lin's theorem is true, let

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denote a map detected by $(i'i)_*(h_n)$.
$(3')$ may imply that for $n > 2$

$$S^{p^n q-1} \xrightarrow{\omega_n} K \xrightarrow{j'} \Sigma^{q+1} M$$

is detected by

$$i_*(b^p_{n-2}) \in \mathrm{Ext}^{2p,p^n q}(H^*(M))$$

and $j'\omega_n$ may lift to S^{q+1}.

Consequences of Lin's theorem

Assuming Lin's theorem is true, let

$$\omega_n : S^{p^n q-1} \to K$$

denote a map detected by $(i'i)_*(h_n)$.
For $n = 2$, the composite

$$S^{p^2 q-1} \xrightarrow{\omega_2} K \xrightarrow{jj'} S^{q+2}$$

is $\gamma_1 = \alpha_1 \beta_{p-1} \in \pi_{p^2 q-q-3}(S^0)$.

Consequences of Lin’s theorem

Assuming Lin’s theorem is true, let

$$\omega_n : S^{p^n q-1} \longrightarrow K$$

denote a map detected by $(i'i)_*(h_n)$.
For $n = 1$, $j'\omega_1 = 0$ and we have

$$\begin{array}{ccccc}
S^1 & \overset{\beta_1}{\longleftarrow} & S^{pq-1} & \longrightarrow & \text{pt.} \\
{\scriptstyle j}\uparrow & \swarrow & \downarrow{\scriptstyle \omega_1} & & \downarrow \\
M & \overset{i'}{\longrightarrow} & K & \overset{j'}{\longrightarrow} & \Sigma^{q+1}M
\end{array}$$

Steps in Lin's argument

Proposition 2 (Toda, [Tod71]) *There is a map* $\alpha'' : \Sigma^{q-2}K \to K$ *with*

$$
\begin{array}{ccccccc}
\Sigma^{q-2}M & \xrightarrow{i'} & \Sigma^{q-2}K & \xrightarrow{\alpha_1 \wedge K} & \Sigma^{-1}K & \xrightarrow{j'} & \Sigma^{q}M \\
\downarrow{\scriptstyle j} & & & \searrow{\scriptstyle \alpha''} & & & \downarrow{\scriptstyle j} \\
S^{q-1} & & & & & & S^{q+1} \\
\downarrow{\scriptstyle i} & & & & & & \downarrow{\scriptstyle i} \\
\Sigma^{q-1}M & \xrightarrow{i'} & \Sigma^{q-1}K & \xrightarrow{\alpha_1 \wedge K} & K & \xrightarrow{j'} & \Sigma^{q+1}M.
\end{array}
$$

Steps in Lin's argument

Theorem 3 (Theorem 3.4 of [Lin03]) *Let $a'' \in \mathrm{Ext}^{1,q-1}(H^*(K), H^*(K))$ detect α''. Then*

$$(h_0 h_n)'' = a''(i'i)_*(h_n) \in \mathrm{Ext}^{2,N-1}(H^*(K), H^*(K))$$

(where $N = (p^n + 1)q$) is a permanent cycle detecting a map

$$\eta''_n : \Sigma^{N-3} K \to K.$$

Steps in Lin's argument

Theorem 3 (Theorem 3.4 of [Lin03]) *Let* $a'' \in \mathrm{Ext}^{1,q-1}(H^*(K), H^*(K))$ *detect* α''. *Then*

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(where $N = (p^n + 1)q$*) is a permanent cycle detecting a map*

$$\eta_n'' : \Sigma^{N-3} K \to K.$$

The proof of this is very difficult.

Steps in Lin's argument

For the next step we need a minimal Adams resolution

$$\begin{array}{ccccccc} S^0 & = & E_0 & \xleftarrow{\overline{a}_0} & \Sigma^{-1}E_1 & \xleftarrow{\overline{a}_1} & \Sigma^{-2}E_2 & \xleftarrow{\overline{a}_2} & \cdots \\ & & \downarrow{\scriptstyle \overline{b}_0} & & \downarrow{\scriptstyle \overline{b}_1} & & \downarrow{\scriptstyle \overline{b}_2} & & \\ H/p & = & KG_0 & & KG_1 & & KG_2 & & \end{array}$$

and the cofiber sequence

$$S^0 \xrightarrow{i'i} K \xrightarrow{r} Y \xrightarrow{\epsilon} S^1.$$

Steps in Lin's argument

Corollary 4 (Corollary 3.10 of [Lin03]) *The map* η''_n *lifts to a map* $\eta''_{n,2} : \Sigma^{N-3}K \to \Sigma^{-2}E_2 \wedge K$, *and there is a map* $(\eta''_{n,2})_Y$ *making the following diagram commute.*

$$\begin{array}{ccccc}
\Sigma^N K & \xrightarrow{r} & \Sigma^N Y & \xrightarrow{(\eta''_{n,2})_Y} & \Sigma E_2 \wedge K \\
\downarrow{\scriptstyle \eta''_{n,2}} & & & & \downarrow{\scriptstyle \overline{b}_2 \wedge K} \\
\Sigma E_2 \wedge K & & \xrightarrow{\overline{b}_2 \wedge K} & & \Sigma K G_2 \wedge K
\end{array}$$

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Corollary 4 (Corollary 3.10 of [Lin03]) *The map* η''_n *lifts to a map* $\eta''_{n,2} : \Sigma^{N-3}K \to \Sigma^{-2}E_2 \wedge K$*, and there is a map* $(\eta''_{n,2})_Y$ *making the following diagram commute.*

$$\begin{array}{ccccc} \Sigma^N K & \xrightarrow{r} & \Sigma^N Y & \xrightarrow{(\eta''_{n,2})_Y} & \Sigma E_2 \wedge K \\ \downarrow \eta''_{n,2} & & & & \downarrow \bar{b}_2 \wedge K \\ \Sigma E_2 \wedge K & & \xrightarrow{\bar{b}_2 \wedge K} & & \Sigma K G_2 \wedge K \end{array}$$

The proof of this is not difficult.

Steps in Lin's argument

The next step concerns the cofiber sequence

$$\Sigma^{q-2}K \xrightarrow{\alpha''} K \xrightarrow{w} X \xrightarrow{u} \Sigma^{q-1}K$$

Lemma 5 (Lemma 4.2 of [Lin03]) *Modulo higher Adams filtration, the composite*

$$\Sigma^{N-3}K \xrightarrow{r} \Sigma^{N-3}Y \xrightarrow{(\eta''_n)_Y} K \xrightarrow{w} X$$

is $\lambda' w(\zeta_{n-1} \wedge K)$ *for a nonzero scalar* λ'*, where* ζ_{n-1} *is Cohen's map.*

Steps in Lin’s argument

In other words, there is a map f_1'' making the following diagram commute

$$\begin{array}{ccc}
\Sigma^N K & \xrightarrow{\ r\ } & \Sigma^N Y \\
\Big\downarrow{\scriptstyle (f_1'', -\zeta_{n-1}\wedge K)} & & \Big\downarrow{\scriptstyle (\eta_n'')_Y} \\
 & & \Sigma^3 K \\
 & & \Big\downarrow{\scriptstyle w} \\
\left(\Sigma^{-1}E_4 \wedge X\right) \vee \Sigma^3 K & \xrightarrow{(\overline{a}_{0,3}\wedge X)\vee w} & \Sigma^3 X
\end{array}$$

where $\overline{a}_{0,3} = \overline{a}_0\overline{a}_1\overline{a}_2\overline{a}_3$.

Steps in Lin's argument

The proof of this is difficult, as is the derivation of Theorem 1 from it. The latter involves studying a diagram that uses the cofiber sequences

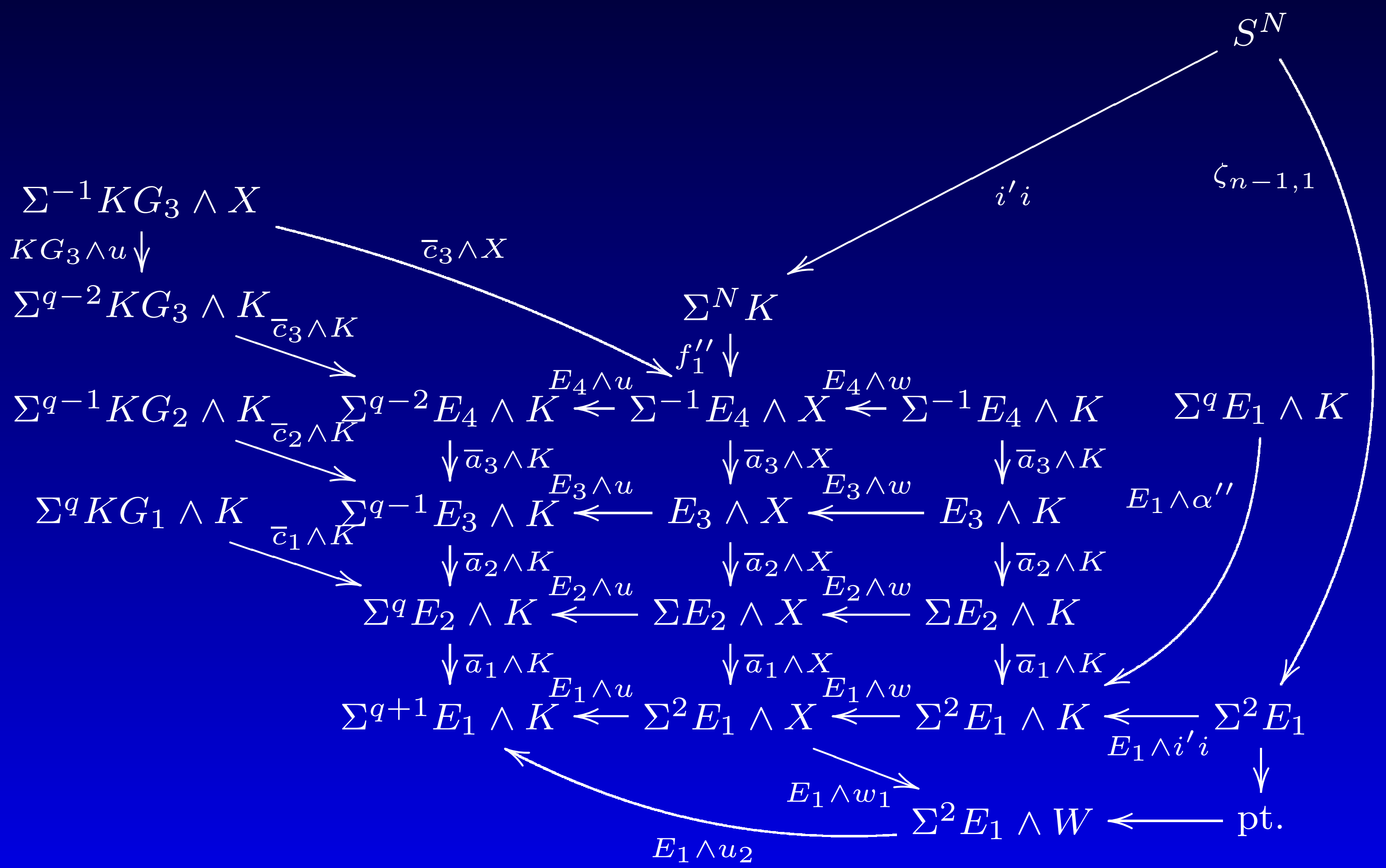
$$\Sigma^{-1}E_{s+1} \xrightarrow{\overline{a}_s} E_s \xrightarrow{\overline{b}_s} KG_s \xrightarrow{\overline{c}_s} E_{s+1},$$

$$S^0 \xrightarrow{wi'i} X \xrightarrow{w_1} W \xrightarrow{u_1} S^1$$

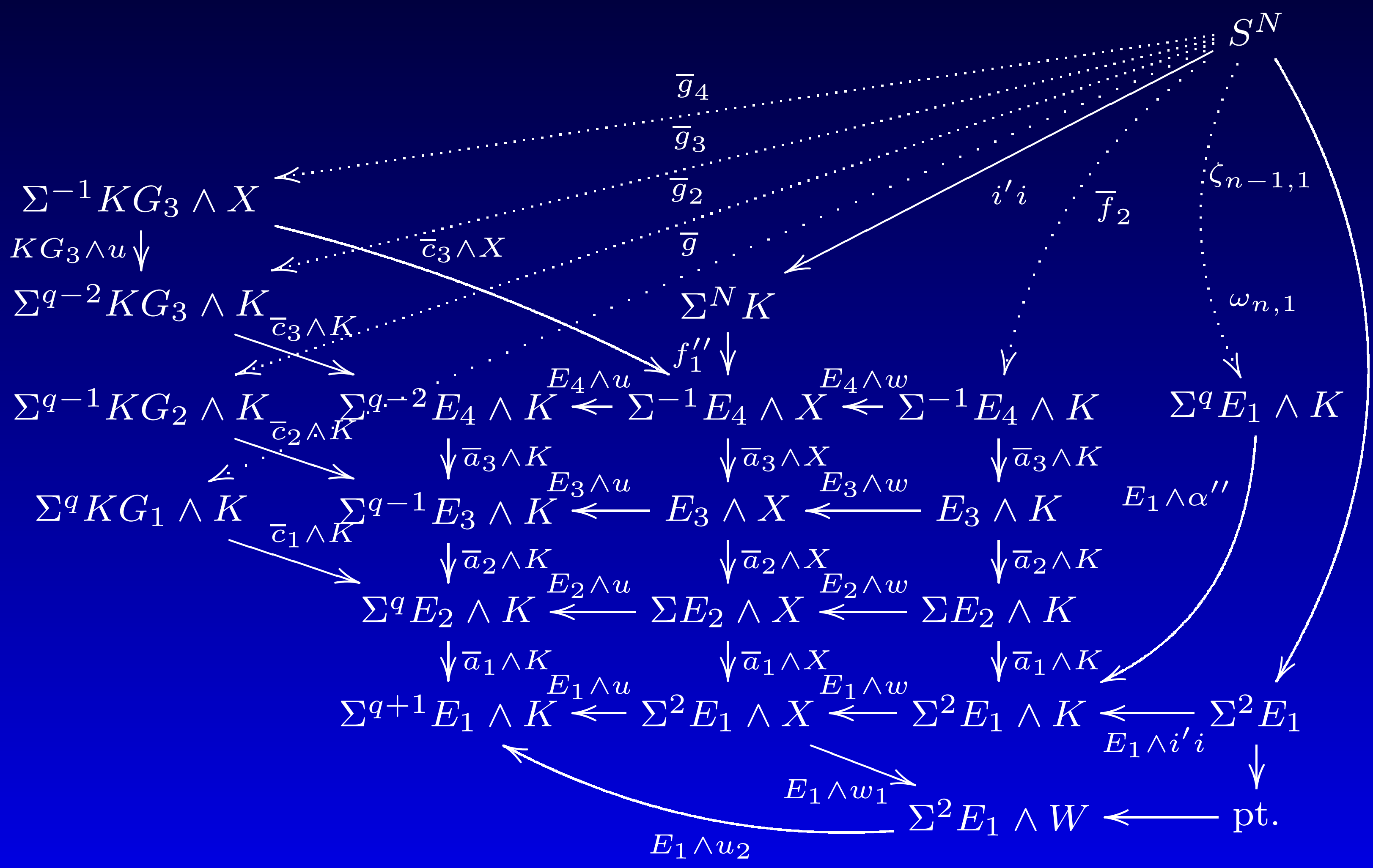
and

$$\Sigma^{q-2}K \xrightarrow{r\alpha''} Y \xrightarrow{w_2} W \xrightarrow{u_2} \Sigma^{q-1}K.$$

Steps in Lin's argument



Steps in Lin's argument



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