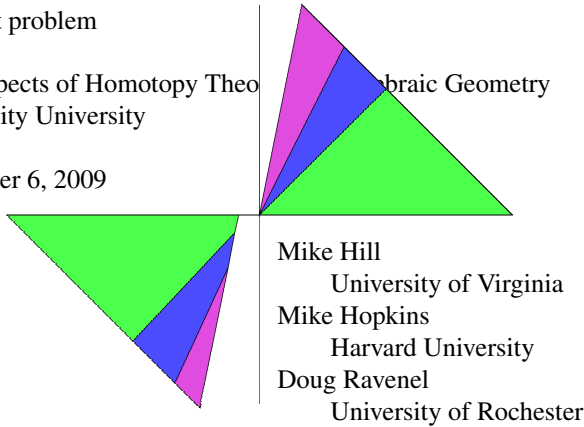


Lecture 6: The detection theorem

A solution to the Arf-Kervaire invariant problem

New Aspects of Homotopy Theory and Algebraic Geometry
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6.1

1 The Detection Theorem

1.1 θ_j in the Adams-Novikov spectral sequence

θ_j in the Adams-Novikov spectral sequence

Browder's Theorem says that θ_j is detected in the classical Adams spectral sequence by

$$h_j^2 \in \text{Ext}_A^{2, 2^{j+1}}(\mathbf{Z}/2, \mathbf{Z}/2).$$

This element is known to be the only one in its bidegree.

It is more convenient for us to work with the Adams-Novikov spectral sequence, which maps to the Adams spectral sequence. It has a family of elements in filtration 2, namely

$$\beta_{i/j} \in \text{Ext}_{MU_*(MU)}^{2, 6i-2j}(MU_*, MU_*)$$

for certain values of i and j . When $j = 1$, it is customary to omit it from the notation. The definition of these elements can be found in Chapter 5 of the third author's book [Complex Cobordism and Stable Homotopy Groups of Spheres](#).

6.2

θ_j in the Adams-Novikov spectral sequence (continued)

Here are the first few of these in the relevant bidegrees.

$$\begin{aligned} \theta_5 &: \beta_{8/8} \text{ and } \beta_{6/2} \\ \theta_6 &: \beta_{16/16}, \beta_{12/4} \text{ and } \beta_{11} \\ \theta_7 &: \beta_{32/32}, \beta_{24/8} \text{ and } \beta_{22/2} \\ \theta_8 &: \beta_{64/64}, \beta_{48/16}, \beta_{44/4} \text{ and } \beta_{43} \end{aligned}$$

and so on. In the bidegree of θ_j , only $\beta_{2^{j-1}/2^{j-1}}$ has a nontrivial image (namely h_j^2) in the Adams spectral sequence. There is an additional element in this bidegree, namely $\alpha_1 \alpha_{2^{j-1}}$.

We need to show that any element mapping to h_j^2 in the classical Adams spectral sequence has nontrivial image in the Adams-Novikov spectral sequence for Ω .

6.3

θ_j in the Adams-Novikov spectral sequence (continued)

Detection Theorem. Let $x \in \text{Ext}_{MU_*(MU)}^{2,2^{j+1}}(MU_*, MU_*)$ be any element whose image in $\text{Ext}_A^{2,2^{j+1}}(\mathbf{Z}/2, \mathbf{Z}/2)$ is h_j^2 with $j \geq 6$. (Here A denotes the mod 2 Steenrod algebra.) Then the image of x in $H^{2,2^{j+1}}(C_8; \pi_*(\Omega))$ is nonzero.

6.4

We will prove this by showing the same is true after we map the latter to a simpler object involving another algebraic tool, [the theory of formal \$A\$ -modules](#), where A is the ring of integers in a suitable field.

1.2 Formal A -modules

Formal A -modules

Recall the a formal group law over a ring R is a power series

$$F(x, y) = x + y + \sum_{i, j > 0} a_{i, j} x^i y^j \in R[[x, y]]$$

with certain properties.

For positive integers m one has power series $[m](x) \in R[[x]]$ defined recursively by $[1](x) = x$ and

$$[m](x) = F(x, [m-1](x)).$$

These satisfy

$$[m+n](x) = F([m](x), [n](x)) \text{ and } [m]([n](x)) = [mn](x).$$

With these properties we can define $[m](x)$ uniquely for all integers m , and we get a homomorphism τ from \mathbf{Z} to $\text{End}(F)$, the endomorphism ring of F .

6.5

Formal A -modules (continued)

If the ground ring R is an algebra over the p -local integers $\mathbf{Z}_{(p)}$ or the p -adic integers \mathbf{Z}_p , then we can make sense of $[m](x)$ for m in $\mathbf{Z}_{(p)}$ or \mathbf{Z}_p .

Now suppose R is an algebra over a larger ring A , such as the ring of integers in a number field or a finite extension of the p -adic numbers. We say that the formal group law F is a [formal \$A\$ -module](#) if the homomorphism τ extends to A in such a way that

$$[a](x) \equiv ax \pmod{x^2} \text{ for } a \in A.$$

The theory of formal A -modules is well developed. Lubin-Tate used them to do local class field theory.

6.6

Formal A -modules (continued)

The example of interest to us is $A = \mathbf{Z}_2[\zeta_8]$, where ζ_8 is a primitive 8th root of unity. The maximal ideal of A is generated by $\pi = \zeta_8 - 1$, and π^4 is a unit multiple of 2. There is a formal A -module F over $R_* = A[w^{\pm 1}]$ (with $|w| = 2$) satisfying

$$\log_F(F(x, y)) = \log_F(x) + \log_F(y)$$

where

$$\log_F(x) = \sum_{n \geq 0} \frac{w^{2^n - 1} x^{2^n}}{\pi^n}.$$

The classifying map $\lambda : MU_* \rightarrow R_*$ for F factors through BP_* , where the logarithm is

$$\log(x) = \sum_{n \geq 0} \ell_n x^{2^n}.$$

6.7

Formal A -modules (continued)

Recall that $BP_* = \mathbf{Z}_{(2)}[v_1, v_2, \dots]$ with $|v_n| = 2(2^n - 1)$. The v_n and the ℓ_n are related by Hazewinkel's formula,

$$\begin{aligned}\ell_1 &= \frac{v_1}{2} \\ \ell_2 &= \frac{v_2}{2} + \frac{v_1^3}{4} \\ \ell_3 &= \frac{v_3}{2} + \frac{v_1 v_2^2 + v_2 v_1^4}{4} + \frac{v_1^7}{8} \\ \ell_4 &= \frac{v_4}{2} + \frac{v_1 v_3^2 + v_2^5 + v_3 v_1^8}{4} + \frac{v_1^3 v_2^4 + v_1^9 v_2^2 + v_2 v_1^{12}}{8} + \frac{v_1^{15}}{16} \\ &\vdots\end{aligned}$$

6.8

1.3 $\pi_*(MU_{\mathbf{R}}^{(4)})$ and R_*

The relation between $MU^{(4)}$ and formal A -modules

What does this have to do with our spectrum $\Omega = D^{-1}MU_{\mathbf{R}}^{(4)}$? Recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$. We saw earlier that inverting a product of this sort is needed to get the Periodicity Theorem, but we did not explain the choice of subscripts of $\overline{\Delta}$. They are the smallest ones that satisfy the second part of the following.

Lemma. *The classifying homomorphism $\lambda : \pi_*(MU) \rightarrow R_*$ for F factors through $\pi_*(MU_{\mathbf{R}}^{(4)})$ in such a way that*

- the homomorphism $\lambda^{(4)} : \pi_*(MU_{\mathbf{R}}^{(4)}) \rightarrow R_*$ is equivariant, where C_8 acts on $\pi_*(MU_{\mathbf{R}}^{(4)})$ as before, it acts trivially on A and $\gamma w = \zeta_8 w$ for a generator γ of C_8 .
- The element $D \in \pi_*(MU_{\mathbf{R}}^{(4)})$ that we invert to get Ω goes to a unit in R_* .

We will prove this later.

6.9

1.4 The proof of the Detection Theorem

The proof of the Detection Theorem

It follows that we have a map

$$H^*(C_8; \pi_*(D^{-1}MU_{\mathbf{R}}^{(4)})) = H^*(C_8; \pi_*(\Omega)) \rightarrow H^*(C_8; R_*).$$

The source here is the E_2 -term of the homotopy fixed point spectral sequence for Ω , and the target is easy to calculate. We will use it to prove the Detection Theorem, namely

Detection Theorem. *Let $x \in \text{Ext}_{MU_*(MU)}^{2,2^{j+1}}(MU_*, MU_*)$ be any element whose image in $\text{Ext}_A^{2,2^{j+1}}(\mathbf{Z}/2, \mathbf{Z}/2)$ is h_j^2 with $j \geq 6$. (Here A denotes the mod 2 Steenrod algebra.) Then the image of x in $H^{2,2^{j+1}}(C_8; \pi_*(\Omega))$ is nonzero.*

6.10

We will prove this by showing that the image of x in $H^{2,2^{j+1}}(C_8; R_*)$ is nonzero.

The proof of the Detection Theorem (continued)

We will calculate with BP -theory. Recall that

$$BP_*(BP) = BP_*[t_1, t_2, \dots] \quad \text{where } |t_n| = 2(2^n - 1).$$

We will abbreviate $\text{Ext}_{BP_*(BP)}^{s,t}(BP_*, BP_*)$ by $\text{Ext}^{s,t}$.

There is a map from this Hopf algebraoid to one associated with $H^*(C_8; R_*)$ in which t_n maps to an R_* -valued function on C_8 (regarded as the group of 8th roots of unity) determined by

$$[\zeta](x) = \sum_{n \geq 0} \langle t_n, \zeta \rangle x^{2^n}.$$

An easy calculation shows that the function t_1 sends a primitive root in C_8 to a unit in R_* .

6.11

The proof of the Detection Theorem (continued)

Let

$$b_{1,j-1} = \frac{1}{2} \sum_{0 < i < 2^j} \binom{2^j}{i} \left[t_1^i | t_1^{2^j-i} \right] \in \text{Ext}^{2,2^{j+1}}$$

It is known to be cohomologous to $\beta_{2^{j-1}/2^{j-1}}$ and to have order 2. We will show that its image in $H^{2,2^{j+1}}(C_8; R_*)$ is nontrivial for $j \geq 2$.

$H^*(C_8; R_*)$ is the cohomology of the cochain complex

$$R_*[C_8] \xrightarrow{\gamma-1} R_*[C_8] \xrightarrow{\text{Trace}} R_*[C_8] \xrightarrow{\gamma-1} \dots$$

where Trace is multiplication by $1 + \gamma + \dots + \gamma^7$.

6.12

The proof of the Detection Theorem (continued)

The cohomology groups $H^s(C_8; R_*)$ for $s > 0$ are periodic in s with period 2. We have

$$\begin{aligned} H^1(C_8; R_{2^m}) &= \ker(1 + \zeta_8^m + \dots + \zeta_8^{7m}) / \text{im}(\zeta_8^m - 1) \\ &= \begin{cases} w^m A / (\pi) & \text{for } m \text{ odd} \\ w^m A / (\pi^2) & \text{for } m \equiv 2 \pmod{4} \\ w^m A / (2) & \text{for } m \equiv 4 \pmod{8} \\ 0 & \text{for } m \equiv 0 \pmod{8} \end{cases} \\ H^2(C_8; R_{2^m}) &= \ker(\zeta_8^m - 1) / \text{im}(1 + \zeta_8^m + \dots + \zeta_8^{7m}) \\ &= \begin{cases} w^m A / (8) & \text{for } m \equiv 0 \pmod{8} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

An easy calculation shows that $b_{1,j-1}$ maps to $4w^{2^j}$, which is the element of order 2 in $H^2(C_8; R_{2^{j+1}})$.

6.13

The proof of the Detection Theorem (continued)

To finish the proof we need to show that the other β s in the same bidegree map to zero. We will do this for $j \geq 6$. The set of these is

$$\left\{ \beta_{c(j,k)/2^{j-1-2k}} : 0 \leq k < j/2 \right\}$$

where $c(j,k) = 2^{j-1-2k}(1 + 2^{2k+1})/3$. Note that $\beta_{c(j,0)/2^{j-1}} = \beta_{2^{j-1}/2^{j-1}}$, so we need to show that the elements with $k > 0$ map to zero.

We will see in the proof of the Lemma below that v_1 and v_2 map to unit multiples of $\pi^3 w$ and $\pi^2 w^3$ respectively. This means we can define a valuation on BP_* compatible with the one on A in which $\|2\| = 1$, $\|\pi\| = 1/4$, $\|v_1\| = 3/4$ and $\|v_2\| = 1/2$. We extend the valuation on A to R_* by setting $\|w\| = 0$.

6.14

The proof of the Detection Theorem (continued)

Hence for $k \geq 1$ and $j \geq 6$ we have

$$\begin{aligned}
 \|\beta_{c(j,k)/2^{j-1-2k}}\| &= \left\| \frac{v_2^{c(j,k)}}{2v_1^{2^{j-1-2k}}} \right\| \\
 &= \frac{c(j,k)}{2} - \frac{3 \cdot 2^{j-1-2k}}{4} - 1 \\
 &= \frac{2^j + 2^{j-1-2k}}{6} - \frac{3 \cdot 2^{j-1-2k}}{4} - 1 \\
 &= (2^{j-1} - 7 \cdot 2^{j-3-2k})/3 - 1 \\
 &\geq 5.
 \end{aligned}$$

This means $\beta_{c(j,k)/2^{j-1-2k}}$ maps to an element that is divisible by 8 and therefore zero. 6.15

The proof of the Detection Theorem (continued)

We have to make a similar computation with the element $\alpha_1 \alpha_{2^{j-1}}$. We have

$$\begin{aligned}
 \|\alpha_{2^{j-1}}\| &= \left\| \frac{v_1^{2^j-1}}{2} \right\| \\
 &= \frac{3(2^j-1)}{4} - 1 \\
 &\geq \frac{21}{4} - 1 \geq 4 \quad \text{for } j \geq 3.
 \end{aligned}$$

This completes the proof of the Detection Theorem modulo the Lemma. 6.16

1.5 The proof of the Lemma

The proof of the Lemma

Here it is again.

Lemma. *The classifying homomorphism $\lambda : \pi_*(MU) \rightarrow R_*$ for F factors through $\pi_*(MU_{\mathbf{R}}^{(4)})$ in such a way that*

- *the homomorphism $\lambda^{(4)} : \pi_*(MU_{\mathbf{R}}^{(4)}) \rightarrow R_*$ is equivariant, where C_8 acts on $\pi_*(MU_{\mathbf{R}}^{(4)})$ as before, it acts trivially on A and $\gamma w = \zeta_8 w$ for a generator γ of C_8 .*
- *The element $D \in \pi_*(MU_{\mathbf{R}}^{(4)})$ that we invert to get Ω goes to a unit in R_* .*

6.17

The proof of the Lemma (continued)

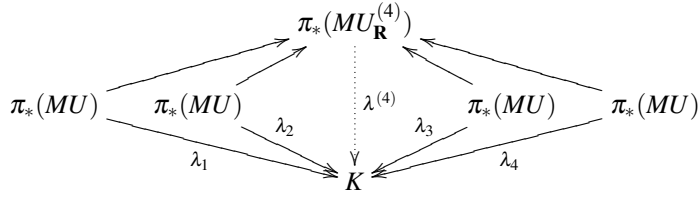
To prove the first part, consider the following diagram for an arbitrary ring K .

$$\begin{array}{ccccc}
 & & MU_*(MU) & & \\
 & \nearrow \eta_L & \parallel & \nwarrow \eta_R & \\
 \pi_*(MU) & & \pi_*(MU_{\mathbf{R}}^{(2)}) & & \pi_*(MU) \\
 & \searrow \lambda_1 & \downarrow \lambda^{(2)} & \swarrow \lambda_2 & \\
 & & K & &
 \end{array}$$

The maps λ_1 and λ_2 classify two formal group laws F_1 and F_2 over K . The Hopf algebra $MU_*(MU)$ represents strict isomorphisms between formal group laws. Hence the existence of $\lambda^{(2)}$ is equivalent to that of a compatible strict isomorphism between F_1 and F_2 . 6.18

The proof of the Lemma (continued)

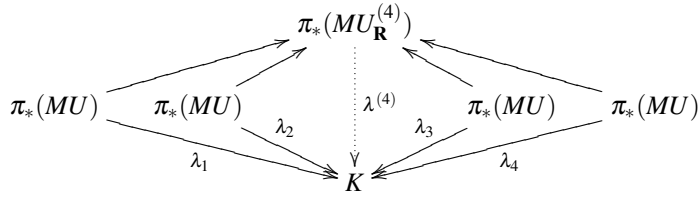
Similarly consider the diagram



The existence of $\lambda^{(4)}$ is equivalent to that of compatible strict isomorphisms between the formal group laws F_j classified by the λ_j .

6.19

The proof of the Lemma (continued)



Now suppose that K has a C_8 -action and that $\lambda^{(4)}$ is equivariant with respect to the previously defined C_8 -action on $MU_{\mathbf{R}}^{(4)}$. Then the isomorphism induced by the fourth power of a generator $\gamma \in C_8$ is the isomorphism sending x to its formal inverse on each of the F_j .

This means that the existence of an equivariant $\lambda^{(4)}$ is equivalent to that of a formal $\mathbf{Z}[\zeta_8]$ -module structure on each of the F_j , which are all isomorphic. This proves the first part of the Lemma.

6.20

The proof of the Lemma (continued)

For the second part, recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$, where

$$\overline{\Delta}_k^{(g)} = \begin{cases} x_{2^k-1} & \text{for } g = 2 \\ N_4^g(r_{2^k-1}) & \text{otherwise.} \end{cases}$$

Since our formal A -module is 2-typical we can do the calculations using BP in place of MU . Hence we can replace x_{2^k-1} by v_k and r_{2^k-1} by t_k . We have $\overline{\Delta}_k^{(2)} = v_k$.

6.21

The proof of the Lemma (continued)

Using Hazewinkel's formula we find that

$$\begin{aligned} v_1 &\mapsto (-\pi^3 - 4\pi^2 - 6\pi - 4)w = \pi^3 \cdot \text{unit} \cdot w \\ v_2 &\mapsto (4\pi^3 + 11\pi^2 + 6\pi - 6)w^3 = \pi^2 \cdot \text{unit} \cdot w^3 \\ v_3 &\mapsto (40\pi^3 + 166\pi^2 + 237\pi + 100)w^7 \\ &= \pi \cdot \text{unit} \cdot w^7 \\ v_4 &\mapsto (-15754\pi^3 - 56631\pi^2 - 63495\pi - 9707)w^{15} \\ &= \text{unit} \cdot w^7. \end{aligned}$$

where each unit is in A . It follows that v_4 (but not v_n for $n < 4$) and therefore $N_2^8(\overline{\Delta}_4^{(2)})$ maps to a unit.

6.22

The proof of the Lemma (continued)

We have $\bar{\Delta}_k^{(2)} = t_k$. We consider the equivariant composite

$$BP_*^{(2)} \rightarrow BP_*^{(4)} \rightarrow R_*$$

under which

$$\eta_R(\ell_n) \mapsto \frac{\zeta_8^2 w^{2^n-1}}{\pi^n}.$$

Using the right unit formula we find that

$$\begin{aligned} t_1 &\mapsto (\pi + 2)w = \pi \cdot \text{unit} \cdot w \\ t_2 &\mapsto (\pi^3 + 5\pi^2 + 9\pi + 5)w^3 = \text{unit} \cdot w^3. \end{aligned}$$

This means t_2 (but not t_1) and therefore $N_4^8(\bar{\Delta}_2^{(4)})$ maps to a unit. 6.23

The proof of the Lemma (continued)

Finally, we have $\bar{\Delta}_n^{(8)} = t_n(1) \in BP_*^{(4)}$, where $t_n(1)$ is the analog of $r_{2^n-1}(1)$. Then we find

$$\begin{aligned} \ell_n(1) &\mapsto \frac{w^{2^n-1}}{\pi^n} \\ \ell_n(2) &\mapsto \frac{(\zeta_8 w)^{2^n-1}}{\pi^n}. \end{aligned}$$

This implies

$$\bar{\Delta}_1^{(8)} = \ell_1(2) - \ell_1(1) \mapsto w.$$

Thus we have shown that each factor of

$$D = \bar{\Delta}_1^{(8)} N_4^8(\bar{\Delta}_2^{(4)}) N_2^8(\bar{\Delta}_4^{(2)})$$

and hence D itself maps to a unit in R_* , thus proving the lemma. 6.24