Lecture 4: The slice spectral sequence and the gap theorem

A solution to the Arf-Kervaire invariant problem

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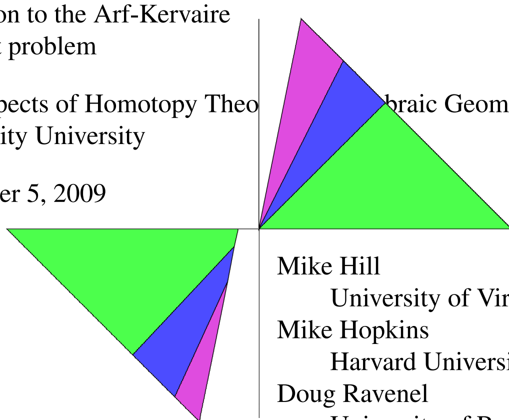

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Our strategy again

Recall our goal is to prove

Main Theorem. *The Arf-Kervaire elements* $\theta_j \in \pi_{2^{j+1}-2}(S^0)$ *do not exist for* $j \geq 7$. 4.2

Our strategy is to find a map $S^0 \to \Omega$ to a nonconnective spectrum Ω with the following properties.

(i) It has an Adams-Novikov spectral sequence in which the image of each θ_j is nontrivial.
(ii) It is 256-periodic, meaning $\Sigma^{256}\Omega \cong \Omega$.
(iii) $\pi_{-2}(\Omega) = 0$.

The slice spectral sequence is based on an equivariant analog of the classical Postnikov tower.

1 Postnikov towers

The classical Postnikov tower

First we need to recall some things about the classical Postnikov tower.

The nth Postnikov section P^nX of a space or spectrum X is obtained by killing all homotopy groups of X above dimension n by attaching cells. The fiber of the map $X \to P^nX$ is $P_{n+1}X$, the n-connected cover of X.

These two functors have some universal properties. Let $\mathscr{S}$ and $\mathscr{S}_{>n}$ denote the categories of spectra and n-connected spectra. 4.3

The classical Postnikov tower (continued)

Then the functor $P_{n+1} : \mathscr{S} \to \mathscr{S}$ satisfies

- For all spectra X, $P_{n+1}X \in \mathscr{S}_{>n}$.
- For all $A \in \mathscr{S}_{>n}$ and $X \in \mathscr{S}$, map of function spectra $\mathscr{S}(A, P_{n+1}X) \to \mathscr{S}(A, X)$ is a weak equivalence.

In other words, the map $P_{n+1}X \to X$ is universal among maps from n-connected spectra to X.

Similarly the map $X \to P^nX$ is universal among maps from X to spectra which are $\mathscr{S}_{>n}$-null in the sense that all maps to them from n-connected spectra are null. In other words,

- The spectrum P^nX is $\mathscr{S}_{>n}$-null.
- For any $\mathscr{S}_{>n}$-null spectrum Z, the map $\mathscr{S}(P^nX, Z) \to \mathscr{S}(X, Z)$ is an equivalence.

Since $\mathscr{S}_{>n} \subset \mathscr{S}_{>n-1}$, there is a natural transformation $P^n \to P^{n-1}$, whose fiber is denoted by P^n_nX. 4.4

2 An equivariant Postnikov tower

An equivariant Postnikov tower

In what follows G will be an arbitrary finite cyclic 2-group, and $g = |G|$. The statements made earlier about $MU_{\mathbf{R}}^{(4)}$ have obvious generalizations to $MU_{\mathbf{R}}^{(g/2)}$.

Let $\mathscr{S}^G$ denote the category of G-equivariant spectra. We need an equivariant analog of $\mathscr{S}_{>n}$. Our choice for this is somewhat novel.

Recall that $\mathscr{S}_{>n}$ is the category of spectra built up out of spheres of dimension $> n$ using arbitrary wedges and mapping cones. 4.5

An equivariant Postnikov tower (continued)

We will replace the set of sphere spectra by

$$\mathscr{A} = \left\{ \widehat{S}(m\rho_h), \Sigma^{-1}\widehat{S}(m\rho_h) \colon H \subset G, m \in \mathbf{Z}, h = |H| \right\}.$$

We will refer to the elements in this set as slice cells or simply as cells. Note that $\Sigma^{-2}\widehat{S}(m\rho_H)$ (and larger desuspensions) are not cells. A free cell is one of the form $\widehat{S}(m\rho_1)$, a wedge of g spheres permuted by G. Note that $\Sigma^{-1}\widehat{S}(m\rho_1) = \widehat{S}((m-1)\rho_1)$. Nonfree cells are said to be isotropic.

In order to define $\mathscr{S}^G_{>n}$, we need to assign a dimension to each element in $\mathscr{A}$. We do this in terms of the underlying wedge summands, namely

$$\dim \widehat{S}(m\rho_H) = mh \quad \text{and} \quad \dim \Sigma^{-1}\widehat{S}(m\rho_H) = mh - 1.$$

4.6

An equivariant Postnikov tower (continued)

Then $\mathscr{S}^G_{>n}$ is the category built up out of elements in $\mathscr{A}$ of dimension $> n$ using arbitrary wedges, mapping cones and smash products with equivariant suspension spectra.

With this definition it is possible to construct functors P^G_{n+1} and P^n_G with the same formal properties as in the classical case. Thus we get a tower

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_G^{n+1}X & \longrightarrow & P_G^nX & \longrightarrow & P_G^{n-1}X & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & {}^GP_{n+1}^{n+1}X & & {}^GP_n^nX & & {}^GP_{n-1}^{n-1}X & & \end{array}$$

in which the inverse limit is X and the direct limit is contractible. 4.7

3 The slice spectral sequence

The slice spectral sequence

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_G^{n+1}X & \longrightarrow & P_G^nX & \longrightarrow & P_G^{n-1}X & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & {}^GP_{n+1}^{n+1}X & & {}^GP_n^nX & & {}^GP_{n-1}^{n-1}X & & \end{array}$$

We call this the slice tower. ${}^GP_n^nX$ is the nth slice and the decreasing sequence of subgroups of $\pi_*(X)$ is the slice filtration. We also get slice filtrations of the $RO(G)$-graded homotopy $\pi_\star(X)$ and the homotopy groups of fixed point sets $\pi_*(X^H)$.

There is an important difference between this tower and the classical one. In the classical case the map $X \to P^nX$ does not change homotopy groups in dimensions $\leq n$. This is not true in this equivariant case. 4.8

The slice spectral sequence (continued)

In the classical case, $P_n^n X$ is an Eilenberg-Mac Lane spectrum whose nth homotopy group is that of X. In our case, $\pi_*({}^G P_n^n X)$ need not be concentrated in dimension n.

This means the slice filtration leads to a slice spectral sequence converging to $\pi_*(X)$ and its variants.

One variant has the form

$$E_2^{s,t} = \pi_{t-s}^G({}^G P_t^t X) \implies \pi_{t-s}^G(X).$$

Recall that $\pi_*^G(X)$ is by definition $\pi_*(X^G)$, the homotopy of the fixed point set.

This is the spectral sequence we will use to study $MU_{\mathbf{R}}^{(4)}$ and its relatives. 4.9

The slice spectral sequence (continued)

A large portion of our paper is devoted to proving that the slice spectral sequence has the desired properties. From now on we will drop the symbol G from the functors P^n, P_{n+1} and P_n^n.

Slice Theorem . *In the slice tower for $MU_{\mathbf{R}}^{(g/2)}$, every odd slice is contractible and $P_{2n}^{2n} = \widehat{W}_n \wedge H\mathbf{Z}$, where $\widehat{W}_n$ is the wedge of $\widehat{S}(m\rho_h)$s indicated above and $H\mathbf{Z}$ is the integer Eilenberg-Mac Lane spectrum. $\widehat{W}_n$ never has any free summands.*

4.10

Computing $\pi_*^G(\widehat{S}(m\rho_h) \wedge H\mathbf{Z})$

Thus we need to find the groups

$$\pi_*^G(\widehat{S}(m\rho_h) \wedge H\mathbf{Z}) = \pi_*^H(S^{m\rho_h} \wedge H\mathbf{Z}).$$

We need this for all integers m because eventually we will invert a certain element in $\pi_*^G(MU_{\mathbf{R}}^{(g/2)})$. Here is what we will learn.

Vanishing Theorem .

- *For $m \geq 0$, $\pi_*^H(S^{m\rho_h} \wedge H\mathbf{Z}) = 0$ for $k < m$ and for $k > mh$.*
- *For $m < 0$ and $h > 1$, $\pi_*^H(S^{m\rho_h} \wedge H\mathbf{Z}) = 0$ for $k < hm$, and for $k > m-3$ except in the case $(h,m) = (2,-2)$ when $\pi_{-4}^H(S^{-2\rho_2} \wedge H\mathbf{Z}) = \mathbf{Z}$.*

4.11

Gap Corollary. *For $h > 1$ and all integers m, $\pi_k^H(S^{m\rho_h} \wedge H\mathbf{Z}) = 0$ for $-4 < k < 0$.*

Computing $\pi_*^G(\widehat{S}(m\rho_h) \wedge H\mathbf{Z})$ (continued)

Gap Corollary. *For $h > 1$ and all integers m, $\pi_k^H(S^{m\rho_h} \wedge H\mathbf{Z}) = 0$ for $-4 < k < 0$.*

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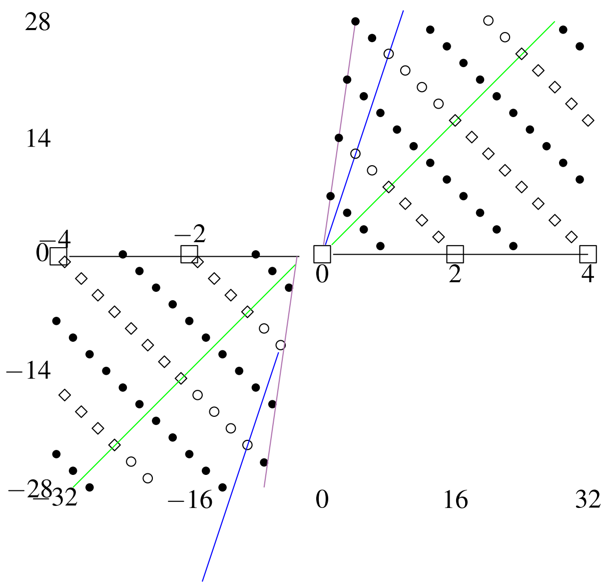
This will lead directly to one of the three conditions we are looking for in Ω, namely the vanishing of π_{-2}.

It is our main motivation for using equivariant stable homotopy theory and developing the slice spectral sequence.

Computing $\pi_*^G(\widehat{S}(m\rho_h)\wedge H\mathbf{Z})$ (continued)

Here is a picture of some slices $S^{m\rho_8}\wedge H\mathbf{Z}$.



4.13

Computing $\pi_*^G(\widehat{S}(m\rho_h)\wedge H\mathbf{Z})$ (continued)

- Note that all elements are in the first and third quadrants between certain black lines with slopes 0 and orchid lines with slope 7, and are concentrated on diagonals where t is divisible by 8.
- Bullets, circles and diamonds indicate cyclic groups of order 2, 4 and 8, and boxes indicate copies of the integers.
- A similar picture for $S^{m\rho_4}\wedge H\mathbf{Z}$ would be confined to the regions between the black lines and blue lines with slope 3 and concentrated on diagonals where t is divisible by 4.
- A similar picture for $S^{m\rho_2}\wedge H\mathbf{Z}$ would be confined to the regions between the black lines and green lines with slope 1 and concentrated on diagonals where t is divisible by 2. 4.14

Computing $\pi_*^G(\widehat{S}(m\rho_h)\wedge H\mathbf{Z})$ (continued)

- The slice spectral sequence for $MU_{\mathbf{R}}^{(4)}$ is concentrated in the first quadrant and confined by the same vanishing lines.
- Later we will invert elements in $\pi_{m\rho_8}(MU_{\mathbf{R}}^{(4)})$. The fact that

  $$S^{-\rho_8}\wedge\widehat{S}(m\rho_h)=\widehat{S}((m-8/h)\rho_h).$$

  means that the resulting slice spectral sequence is confined to the regions of the first and third quadrants shown in the picture. 4.15

4 Proof of Vanishing Theorem

The proof of the Vanishing Theorem

The proofs of the Vanishing Theorem and Gap Corollary are surprisingly easy.

We begin by constructing an equivariant cellular chain complex $C_*(m\rho_g)$ for $S^{m\rho_g}$, where $m\geq 0$. In it the cells are permuted by the action of G. It is a complex of $\mathbf{Z}[G]$-modules and is determined by fixed point data of $S^{m\rho_g}$. For $H\subset G$ we have

$$\left(S^{m\rho_g}\right)^H=S^{mg/h}$$

This means there is a G-CW-complex with one cell in dimension m, two cells in each dimension from $m+1$ to $2m$, four cells in each dimension from $2m+1$ to $4m$, and so on. 4.16

The proof of the Vanishing Theorem (continued)

In other words,

$$C_k^{m\rho_g} = \begin{cases} 0 & \text{for } k < m \\ \mathbf{Z}[G/H] & \text{for } mg/2h < k \leq mg/h \\ 0 & \text{for } k > gm \end{cases}$$

Each of these is a cyclic $\mathbf{Z}[G]$-module. The boundary operator is determined by the fact that $H_*(C(m\rho_g)) = H_*(S^{gm})$.

Then we have

$$\pi_*^G(S^{m\rho_g} \wedge H\mathbf{Z}) = H_*(\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g))).$$

4.17

The proof of the Vanishing Theorem (continued)

These groups are nontrivial only for $m \leq k \leq gm$, which gives the Vanishing Theorem for $m \geq 0$.

We will look at the bottom three groups in the complex $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C_*^{m\rho_g})$. Since $C_k^{m\rho_g}$ is a cyclic $\mathbf{Z}[G]$-module, the Hom group is always $\mathbf{Z}$.

We have

$$\begin{array}{ccccccccc} & & C_m(m\rho_g) & & C_{m+1}(m\rho_g) & & C_{m+2}(m\rho_g) & & \\ & & \| & & \| & & \| & & \\ 0 & \longleftarrow & \mathbf{Z} & \xleftarrow{\varepsilon} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \mathbf{Z}[C_2 \text{ or } C_4] & \xleftarrow{1+\gamma} & \cdots \end{array}$$

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The proof of the Vanishing Theorem (continued)

Applying $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$ to this gives

$$\mathbf{Z} \xleftarrow{2} \mathbf{Z} \xleftarrow{0} \mathbf{Z} \xleftarrow{2} \mathbf{Z} \xleftarrow{0} \mathbf{Z} \longleftarrow \cdots$$

so for $m > 0$,

$$\begin{aligned} \pi_m^G(S^{m\rho_g} \wedge H\mathbf{Z}) &= \mathbf{Z}/2 \\ \pi_{m+1}^G(S^{m\rho_g} \wedge H\mathbf{Z}) &= 0 \\ \pi_{m+2}^G(S^{m\rho_g} \wedge H\mathbf{Z}) &= \begin{cases} 0 & \text{for } m = 1 \text{ and } g = 2 \\ \mathbf{Z} & \text{for } m = 2 \text{ and } g = 2 \\ \mathbf{Z}/2 & \text{otherwise.} \end{cases} \end{aligned}$$

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The proof of the Vanishing Theorem (continued)

For the negative multiples of ρ_g, $S^{-m\rho_g}$ is the equivariant Spanier-Whitehead dual of $S^{m\rho_g}$.This means that

$$\pi_*^G(S^{-m\rho_g} \wedge H\mathbf{Z}) = H_*(\mathrm{Hom}_{\mathbf{Z}[G]}(C(m\rho_g), \mathbf{Z})).$$

Applying the functor $\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$ to our chain complex gives a cochain complex beginning with

$$\mathbf{Z} \xrightarrow{1} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \longrightarrow \cdots$$

The critical fact here is the difference in behavior of the map $\varepsilon : \mathbf{Z}[C_2] \to \mathbf{Z}$ under the functors $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$ and $\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$. They convert it to maps of degrees 2 and 1 respectively. 4.20

The proof of the Vanishing Theorem (continued)

For $m<0$ this gives

$$\begin{array}{rcl}
\pi^G_m(S^{m\rho_g}\wedge H\mathbf{Z}) &=& 0\\
\pi^G_{-1+m}(S^{m\rho_g}\wedge H\mathbf{Z}) &=& 0\\
\pi^G_{-2+m}(S^{m\rho_g}\wedge H\mathbf{Z}) &=& \left\{\begin{array}{ll}\mathbf{Z} & \text{for } (g,m)=(2,-2)\\ 0 & \text{otherwise}\end{array}\right.\\
\pi^G_{-3+m}(S^{m\rho_g}\wedge H\mathbf{Z}) &=& \left\{\begin{array}{ll}0 & \text{for } (g,m)=2,-1 \text{ or } (2,-2)\\ \mathbf{Z}/2 & \text{otherwise}\end{array}\right.
\end{array}$$

This gives both the Vanishing Theorem for $m<0$ and the Gap Corollary. 4.21

5 $RO(G)$-graded homotopy

5.1 a_V

The element $a_V\in\pi_{-V}(X)$

For future reference we record some elements in the $RO(G)$-graded homotopy of a G-spectrum X, $\pi_\star(X)$. For any representation V of G with $V^G=0$, we have a map $a_V: S^0\to S^V$.

Suppose X is a ring spectrum with unit map $S^0\to X$. Smashing it with a_V gives a map $S^0\to\Sigma^V X$ which is adjoint to a map $S^{-V}\to X$. We also denote this by $a_V\in\pi_{-V}(X)$.

It has the multiplicative property $a_{V+W}=a_Va_W$.

If V is a representation of a subgroup $H\subset G$ with $V^H=0$ and V' is the induced representation of G, the $N_H^G(a_V)=a_{V'}$. 4.22

5.2 u_W

The element $u_W\in\pi_{|W|-W}(H\mathbf{Z})$

Let W be an oriented representation of G, meaning that it takes values in the special orthogonal group. Then $\pi_{|W|}(S^W\wedge H\mathbf{Z})=\mathbf{Z}$ and we denote its generator by $u_W\in\pi_{|W|-W}(H\mathbf{Z})$.

We have $u_{V+W}=u_Vu_W$, and for a trivial representation n, $u_n=1$.

If W is an oriented representation of a subgroup $H\subset G$ with induced representation W' and $W^H=0$, then $|W|$ is even and the norm functor N_H^G from H-spectra to G-spectra satisfies

$$N_H^G(u_W)u_{2\rho_{G/H}}^{|W|/2}=u_{W'},$$

where $\rho_{G/H}$ denotes the representation of G induced up from the degree 1 trivial representation of H.

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