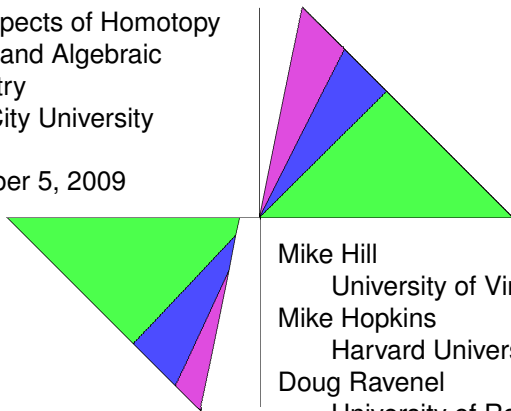


# Lecture 4: The slice spectral sequence and the gap theorem

A solution to the Arf-Kervaire invariant problem

New Aspects of Homotopy  
Theory and Algebraic  
Geometry  
Tokyo City University

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## Our strategy again

Recall our goal is to prove

### Main Theorem

*The Arf-Kervaire elements  $\theta_j \in \pi_{2^{j+1}-2}(S^0)$  do not exist for  $j \geq 7$ .*

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*The Arf-Kervaire elements  $\theta_j \in \pi_{2j+1-2}(S^0)$  do not exist for  $j \geq 7$ .*

Our strategy is to find a map  $S^0 \rightarrow \Omega$  to a nonconnective spectrum  $\Omega$  with the following properties.

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- (i) It has an Adams-Novikov spectral sequence in which the image of each  $\theta_j$  is nontrivial.

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- (i) It has an Adams-Novikov spectral sequence in which the image of each  $\theta_j$  is nontrivial.
- (ii) It is 256-periodic, meaning  $\Sigma^{256}\Omega \cong \Omega$ .

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- (iii)  $\pi_{-2}(\Omega) = 0$ .

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The slice spectral sequence is based on an equivariant analog of the classical Postnikov tower.

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# The classical Postnikov tower

First we need to recall some things about the classical Postnikov tower.

The  $n$ th Postnikov section  $P^n X$  of a space or spectrum  $X$  is obtained by killing all homotopy groups of  $X$  above dimension  $n$  by attaching cells.

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These two functors have some universal properties. Let  $\mathcal{S}$  and  $\mathcal{S}_{>n}$  denote the categories of spectra and  $n$ -connected spectra.

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## The classical Postnikov tower (continued)

Then the functor  $P_{n+1} : \mathcal{S} \rightarrow \mathcal{S}$  satisfies

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- For all  $A \in \mathcal{S}_{>n}$  and  $X \in \mathcal{S}$ , map of function spectra  $\mathcal{S}(A, P_{n+1}X) \rightarrow \mathcal{S}(A, X)$  is a weak equivalence.

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In other words, the map  $P_{n+1}X \rightarrow X$  is universal among maps from  $n$ -connected spectra to  $X$ .

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Similarly the map  $X \rightarrow P^n X$  is universal among maps from  $X$  to spectra which are  $\mathcal{S}_{>n}$ -null in the sense that all maps to them from  $n$ -connected spectra are null. In other words,

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- The spectrum  $P^n X$  is  $\mathcal{S}_{>n}$ -null.
- For any  $\mathcal{S}_{>n}$ -null spectrum  $Z$ , the map  $\mathcal{S}(P^n X, Z) \rightarrow \mathcal{S}(X, Z)$  is an equivalence.

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Since  $\mathcal{S}_{>n} \subset \mathcal{S}_{>n-1}$ , there is a natural transformation  $P^n \rightarrow P^{n-1}$ , whose fiber is denoted by  $P_n^n X$ .

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# An equivariant Postnikov tower

In what follows  $G$  will be an arbitrary finite cyclic 2-group, and  $g = |G|$ .

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In what follows  $G$  will be an arbitrary finite cyclic 2-group, and  $g = |G|$ . The statements made earlier about  $MU_{\mathbf{R}}^{(4)}$  have obvious generalizations to  $MU_{\mathbf{R}}^{(g/2)}$ .

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Recall that  $S_{>n}$  is the category of spectra built up out of spheres of dimension  $> n$  using arbitrary wedges and mapping cones.

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## An equivariant Postnikov tower (continued)

We will replace the set of sphere spectra by

$$\mathcal{A} = \left\{ \widehat{S}(m\rho_h), \Sigma^{-1}\widehat{S}(m\rho_h) : H \subset G, m \in \mathbf{Z}, h = |H| \right\}.$$

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We will refer to the elements in this set as **slice cells** or simply as **cells**.

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In order to define  $\mathcal{S}_{>n}^G$ , we need to assign a dimension to each element in  $\mathcal{A}$ .

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We will replace the set of sphere spectra by

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We will refer to the elements in this set as **slice cells** or simply as **cells**. Note that  $\Sigma^{-2}\widehat{S}(m\rho_H)$  (and larger desuspensions) are **not** cells. A **free cell** is one of the form  $\widehat{S}(m\rho_1)$ , a wedge of  $g$  spheres permuted by  $G$ . Note that  $\Sigma^{-1}\widehat{S}(m\rho_1) = \widehat{S}((m-1)\rho_1)$ . Nonfree cells are said to be **isotropic**.

In order to define  $\mathcal{S}_{>n}^G$ , we need to assign a dimension to each element in  $\mathcal{A}$ . We do this in terms of the underlying wedge summands, namely

$$\dim \widehat{S}(m\rho_H) = mh \quad \text{and} \quad \dim \Sigma^{-1}\widehat{S}(m\rho_H) = mh - 1.$$

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## An equivariant Postnikov tower (continued)

Then  $\mathcal{S}_{>n}^G$  is the category built up out of elements in  $\mathcal{A}$  of dimension  $> n$  using arbitrary wedges, mapping cones and smash products with equivariant suspension spectra.

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With this definition it is possible to construct functors  $P_{n+1}^G$  and  $P_G^n$  with the same formal properties as in the classical case.

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With this definition it is possible to construct functors  $P_{n+1}^G$  and  $P_n^G$  with the same formal properties as in the classical case. Thus we get a tower

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_G^{n+1} X & \longrightarrow & P_G^n X & \longrightarrow & P_G^{n-1} X & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & G P_{n+1}^{n+1} X & & G P_n^n X & & G P_{n-1}^{n-1} X & & \end{array}$$

in which the inverse limit is  $X$  and the direct limit is contractible.

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We call this the **slice tower**.

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We call this the **slice tower**.  ${}^G P_n X$  is the  **$n$ th slice** and the decreasing sequence of subgroups of  $\pi_*(X)$  is the **slice filtration**.

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We call this the **slice tower**.  $G P_n^n X$  is the  **$n$ th slice** and the decreasing sequence of subgroups of  $\pi_*(X)$  is the **slice filtration**. We also get slice filtrations of the  $RO(G)$ -graded homotopy  $\pi_*(X)$  and the homotopy groups of fixed point sets  $\pi_*(X^H)$ .



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There is an important difference between this tower and the classical one.

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There is an important difference between this tower and the classical one. In the classical case the map  $X \rightarrow P^n X$  does not change homotopy groups in dimensions  $\leq n$ .

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There is an important difference between this tower and the classical one. In the classical case the map  $X \rightarrow P^n X$  does not change homotopy groups in dimensions  $\leq n$ . **This is not true in this equivariant case.**

## The slice spectral sequence (continued)

In the classical case,  $P_n^n X$  is an Eilenberg-Mac Lane spectrum whose  $n$ th homotopy group is that of  $X$ . In our case,  $\pi_*({}^G P_n^n X)$  need not be concentrated in dimension  $n$ .

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This means the slice filtration leads to a **slice spectral sequence** converging to  $\pi_*(X)$  and its variants.

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This means the slice filtration leads to a **slice spectral sequence** converging to  $\pi_*(X)$  and its variants.

One variant has the form

$$E_2^{s,t} = \pi_{t-s}^G({}^G P_t^t X) \implies \pi_{t-s}^G(X).$$

Recall that  $\pi_*^G(X)$  is by definition  $\pi_*(X^G)$ , the homotopy of the fixed point set.

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This is the spectral sequence we will use to study  $MU_{\mathbb{R}}^{(4)}$  and its relatives.

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## The slice spectral sequence (continued)

A large portion of our paper is devoted to proving that the slice spectral sequence has the desired properties.

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## The slice spectral sequence (continued)

A large portion of our paper is devoted to proving that the slice spectral sequence has the desired properties. From now on we will drop the symbol  $G$  from the functors  $P^n$ ,  $P_{n+1}$  and  $P_n^n$ .

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### Slice Theorem

*In the slice tower for  $MU_{\mathbf{R}}^{(g/2)}$ , every odd slice is contractible and  $P_{2n}^{2n} = \widehat{W}_n \wedge H\mathbf{Z}$ , where  $\widehat{W}_n$  is the wedge of  $\widehat{S}(m\rho_h)$ s indicated above and  $H\mathbf{Z}$  is the integer Eilenberg-Mac Lane spectrum.*

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## Computing $\pi_*^G(\widehat{S}(m\rho_h) \wedge H\mathbb{Z})$

Thus we need to find the groups

$$\pi_*^G(\widehat{S}(m\rho_h) \wedge H\mathbb{Z}) = \pi_*^H(S^{m\rho_h} \wedge H\mathbb{Z}).$$

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We need this for **all** integers  $m$  because eventually we will invert a certain element in  $\pi_*^G(MU_{\mathbf{R}}^{(g/2)})$ . Here is what we will learn.

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### Vanishing Theorem

- For  $m \geq 0$ ,  $\pi_*^H(S^{m\rho_h} \wedge H\mathbf{Z}) = 0$  for  $k < m$  and for  $k > mh$ .

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### Gap Corollary

For  $h > 1$  and all integers  $m$ ,  $\pi_k^H(S^{m\rho_h} \wedge H\mathbf{Z}) = 0$  for  $-4 < k < 0$ .

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# Computing $\pi_*^G(\widehat{S}(m\rho_h) \wedge H\mathbb{Z})$ (continued)

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This will lead directly to one of the three conditions we are looking for in  $\Omega$ , namely the vanishing of  $\pi_{-2}$ .

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This will lead directly to one of the three conditions we are looking for in  $\Omega$ , namely the vanishing of  $\pi_{-2}$ .

It is our main motivation for using equivariant stable homotopy theory and developing the slice spectral sequence.

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## Computing $\pi_*^G(\widehat{S}(m\rho_h) \wedge H\mathbb{Z})$ (continued)

Here is a picture of some slices  $S^{m\rho_8} \wedge H\mathbb{Z}$ .

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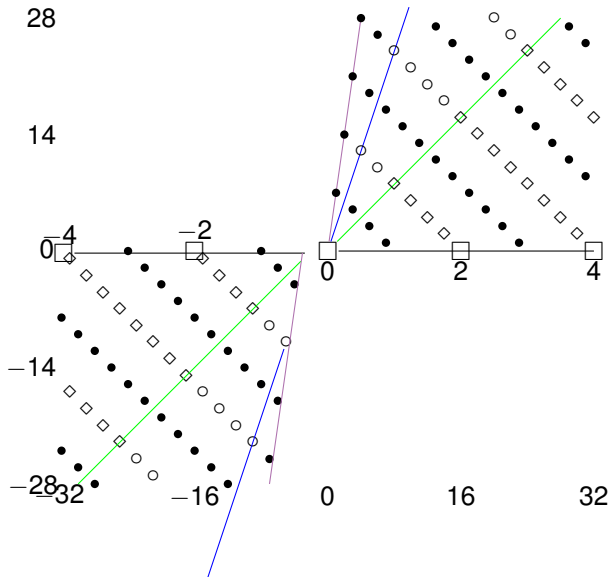
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# Computing $\pi_*^G(\widehat{S}(m\rho_h) \wedge HZ)$ (continued)

- Note that all elements are in the first and third quadrants between certain black lines with slopes 0 and orchid lines with slope 7,

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## Computing $\pi_*^G(\widehat{S}(m\rho_h) \wedge H\mathbb{Z})$ (continued)

- Note that all elements are in the first and third quadrants between certain black lines with slopes 0 and **orchid lines with slope 7**, and are concentrated on diagonals where  $t$  is divisible by 8.

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## Computing $\pi_*^G(\widehat{S}(m\rho_h) \wedge H\mathbb{Z})$ (continued)

- Note that all elements are in the first and third quadrants between certain black lines with slopes 0 and orchid lines with slope 7, and are concentrated on diagonals where  $t$  is divisible by 8.
- Bullets, circles and diamonds indicate cyclic groups of order 2, 4 and 8, and boxes indicate copies of the integers.

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- Bullets, circles and diamonds indicate cyclic groups of order 2, 4 and 8, and boxes indicate copies of the integers.
- A similar picture for  $S^{m\rho_4} \wedge HZ$  would be confined to the regions between the black lines and **blue lines with slope 3**

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- A similar picture for  $S^{m\rho_4} \wedge HZ$  would be confined to the regions between the black lines and **blue lines with slope 3** and concentrated on diagonals where  $t$  is divisible by 4.

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## Computing $\pi_*^G(\widehat{S}(m\rho_h) \wedge H\mathbb{Z})$ (continued)

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- Bullets, circles and diamonds indicate cyclic groups of order 2, 4 and 8, and boxes indicate copies of the integers.
- A similar picture for  $S^{m\rho_4} \wedge H\mathbb{Z}$  would be confined to the regions between the black lines and **blue lines with slope 3** and concentrated on diagonals where  $t$  is divisible by 4.
- A similar picture for  $S^{m\rho_2} \wedge H\mathbb{Z}$  would be confined to the regions between the black lines and **green lines with slope 1**

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# Computing $\pi_*^G(\widehat{S}(m\rho_h) \wedge H\mathbb{Z})$ (continued)

- The slice spectral sequence for  $MU_{\mathbb{R}}^{(4)}$  is concentrated in the first quadrant and confined by the same vanishing lines.

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## Computing $\pi_*^G(\widehat{S}(m\rho_h) \wedge H\mathbb{Z})$ (continued)

- The slice spectral sequence for  $MU_{\mathbf{R}}^{(4)}$  is concentrated in the first quadrant and confined by the same vanishing lines.
- Later we will invert elements in  $\pi_{m\rho_8}(MU_{\mathbf{R}}^{(4)})$ .

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## Computing $\pi_*^G(\widehat{S}(m\rho_h) \wedge HZ)$ (continued)

- The slice spectral sequence for  $MU_{\mathbf{R}}^{(4)}$  is concentrated in the first quadrant and confined by the same vanishing lines.
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$$S^{-\rho_8} \wedge \widehat{S}(m\rho_h) = \widehat{S}((m - 8/h)\rho_h).$$

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## Computing $\pi_*^G(\widehat{S}(m\rho_h) \wedge HZ)$ (continued)

- The slice spectral sequence for  $MU_{\mathbf{R}}^{(4)}$  is concentrated in the first quadrant and confined by the same vanishing lines.
- Later we will invert elements in  $\pi_{m\rho_8}(MU_{\mathbf{R}}^{(4)})$ . The fact that

$$S^{-\rho_8} \wedge \widehat{S}(m\rho_h) = \widehat{S}((m - 8/h)\rho_h).$$

means that the resulting slice spectral sequence is confined to the regions of the first and third quadrants shown in the picture.

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# The proof of the Vanishing Theorem

The proofs of the Vanishing Theorem and Gap Corollary are surprisingly easy.

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# The proof of the Vanishing Theorem

The proofs of the Vanishing Theorem and Gap Corollary are surprisingly easy.

We begin by constructing an equivariant cellular chain complex  $C_*(m\rho_g)$  for  $S^{m\rho_g}$ , where  $m \geq 0$ . In it the cells are permuted by the action of  $G$ .

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$$(S^{m\rho_g})^H = S^{mg/h}$$

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$$(S^{m\rho_g})^H = S^{mg/h}$$

This means there is a  $G$ -CW-complex with one cell in dimension  $m$ , two cells in each dimension from  $m + 1$  to  $2m$ , four cells in each dimension from  $2m + 1$  to  $4m$ , and so on.

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# The proof of the Vanishing Theorem (continued)

In other words,

$$C_k^{m\rho g} = \begin{cases} 0 & \text{for } k < m \\ \mathbf{Z}[G/H] & \text{for } mg/2h < k \leq mg/h \\ 0 & \text{for } k > gm \end{cases}$$

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Each of these is a cyclic  $\mathbf{Z}[G]$ -module.

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## The proof of the Vanishing Theorem (continued)

In other words,

$$C_k^{m\rho_g} = \begin{cases} 0 & \text{for } k < m \\ \mathbf{Z}[G/H] & \text{for } mg/2h < k \leq mg/h \\ 0 & \text{for } k > gm \end{cases}$$

Each of these is a cyclic  $\mathbf{Z}[G]$ -module. The boundary operator is determined by the fact that  $H_*(C(m\rho_g)) = H_*(S^{gm})$ .

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Each of these is a cyclic  $\mathbf{Z}[G]$ -module. The boundary operator is determined by the fact that  $H_*(C(m\rho_g)) = H_*(S^{gm})$ .

Then we have

$$\pi_*^G(S^{m\rho_g} \wedge H\mathbf{Z}) = H_*(\text{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g))).$$

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## The proof of the Vanishing Theorem (continued)

These groups are nontrivial only for  $m \leq k \leq gm$ , which gives the Vanishing Theorem for  $m \geq 0$ .

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## The proof of the Vanishing Theorem (continued)

These groups are nontrivial only for  $m \leq k \leq gm$ , which gives the Vanishing Theorem for  $m \geq 0$ .

We will look at the bottom three groups in the complex  $\text{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C_*^{m\rho g})$ .

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## The proof of the Vanishing Theorem (continued)

These groups are nontrivial only for  $m \leq k \leq gm$ , which gives the Vanishing Theorem for  $m \geq 0$ .

We will look at the bottom three groups in the complex  $\text{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C_*^{m\rho g})$ . Since  $C_k^{m\rho g}$  is a cyclic  $\mathbf{Z}[G]$ -module, the Hom group is always  $\mathbf{Z}$ .

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We will look at the bottom three groups in the complex  $\text{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C_*^{m\rho g})$ . Since  $C_k^{m\rho g}$  is a cyclic  $\mathbf{Z}[G]$ -module, the Hom group is always  $\mathbf{Z}$ .

We have

$$\begin{array}{ccccccc} & C_m(m\rho g) & & C_{m+1}(m\rho g) & & C_{m+2}(m\rho g) & \\ & \parallel & & \parallel & & \parallel & \\ 0 & \longleftarrow \mathbf{Z} & \xleftarrow{\epsilon} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \mathbf{Z}[C_2 \text{ or } C_4] & \xleftarrow{1+\gamma} \dots \end{array}$$

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# The proof of the Vanishing Theorem (continued)

Applying  $\text{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$  to this gives

$$\mathbf{Z} \xleftarrow{2} \mathbf{Z} \xleftarrow{0} \mathbf{Z} \xleftarrow{2} \mathbf{Z} \xleftarrow{0} \mathbf{Z} \xleftarrow{\dots}$$

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$$\mathbf{Z} \xleftarrow{2} \mathbf{Z} \xleftarrow{0} \mathbf{Z} \xleftarrow{2} \mathbf{Z} \xleftarrow{0} \mathbf{Z} \xleftarrow{\dots}$$

so for  $m > 0$ ,

$$\pi_m^G(S^{m\rho_g} \wedge H\mathbf{Z}) = \mathbf{Z}/2$$

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## The proof of the Vanishing Theorem (continued)

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$$\mathbf{Z} \xleftarrow{2} \mathbf{Z} \xleftarrow{0} \mathbf{Z} \xleftarrow{2} \mathbf{Z} \xleftarrow{0} \mathbf{Z} \xleftarrow{\dots}$$

so for  $m > 0$ ,

$$\begin{aligned}\pi_m^G(S^{m\rho_g} \wedge H\mathbf{Z}) &= \mathbf{Z}/2 \\ \pi_{m+1}^G(S^{m\rho_g} \wedge H\mathbf{Z}) &= 0\end{aligned}$$

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## The proof of the Vanishing Theorem (continued)

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Applying  $\text{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$  to this gives

$$\mathbf{Z} \xleftarrow{2} \mathbf{Z} \xleftarrow{0} \mathbf{Z} \xleftarrow{2} \mathbf{Z} \xleftarrow{0} \mathbf{Z} \xleftarrow{\dots}$$

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$$\begin{aligned} \pi_m^G(S^{m\rho_g} \wedge H\mathbf{Z}) &= \mathbf{Z}/2 \\ \pi_{m+1}^G(S^{m\rho_g} \wedge H\mathbf{Z}) &= 0 \\ \pi_{m+2}^G(S^{m\rho_g} \wedge H\mathbf{Z}) &= \begin{cases} 0 & \text{for } m = 1 \text{ and } g = 2 \\ \mathbf{Z} & \text{for } m = 2 \text{ and } g = 2 \\ \mathbf{Z}/2 & \text{otherwise.} \end{cases} \end{aligned}$$

## The proof of the Vanishing Theorem (continued)

For the negative multiples of  $\rho_g$ ,  $S^{-m\rho_g}$  is the equivariant Spanier-Whitehead dual of  $S^{m\rho_g}$ .

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## The proof of the Vanishing Theorem (continued)

For the negative multiples of  $\rho_g$ ,  $S^{-m\rho_g}$  is the equivariant Spanier-Whitehead dual of  $S^{m\rho_g}$ . This means that

$$\pi_*^G(S^{-m\rho_g} \wedge H\mathbf{Z}) = H_*(\mathrm{Hom}_{\mathbf{Z}[G]}(C(m\rho_g), \mathbf{Z})).$$

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$$\pi_*^G(S^{-m\rho_g} \wedge H\mathbf{Z}) = H_*(\mathrm{Hom}_{\mathbf{Z}[G]}(C(m\rho_g), \mathbf{Z})).$$

Applying the functor  $\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$  to our chain complex gives a cochain complex beginning with

$$\mathbf{Z} \xrightarrow{1} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \rightarrow \dots$$

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$$\mathbf{Z} \xrightarrow{1} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \rightarrow \dots$$

The critical fact here is the difference in behavior of the map  $\epsilon : \mathbf{Z}[C_2] \rightarrow \mathbf{Z}$  under the functors  $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$  and  $\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$ .

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Applying the functor  $\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$  to our chain complex gives a cochain complex beginning with

$$\mathbf{Z} \xrightarrow{1} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \rightarrow \dots$$

The critical fact here is the difference in behavior of the map  $\epsilon : \mathbf{Z}[C_2] \rightarrow \mathbf{Z}$  under the functors  $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$  and  $\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$ . They convert it to maps of degrees 2 and 1 respectively.

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# The proof of the Vanishing Theorem (continued)

For  $m < 0$  this gives

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# The proof of the Vanishing Theorem (continued)

For  $m < 0$  this gives

$$\pi_m^G(S^{m\rho_g} \wedge H\mathbf{Z}) = 0$$

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# The proof of the Vanishing Theorem (continued)

For  $m < 0$  this gives

$$\pi_m^G(S^{m\rho_g} \wedge H\mathbf{Z}) = 0$$

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$$\pi_{-2+m}^G(S^{m\rho_g} \wedge H\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{for } (g, m) = (2, -2) \\ 0 & \text{otherwise} \end{cases}$$

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$$\pi_{-3+m}^G(S^{m\rho_g} \wedge H\mathbf{Z}) = \begin{cases} 0 & \text{for } (g, m) = 2, -1 \text{ or } (2, -2) \\ \mathbf{Z}/2 & \text{otherwise} \end{cases}$$

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## The proof of the Vanishing Theorem (continued)

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This gives both the Vanishing Theorem for  $m < 0$  and the Gap Corollary.

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## The element $a_V \in \pi_{-V}(X)$

For future reference we record some elements in the  $RO(G)$ -graded homotopy of a  $G$ -spectrum  $X$ ,  $\pi_{\star}(X)$ .

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For future reference we record some elements in the  $RO(G)$ -graded homotopy of a  $G$ -spectrum  $X$ ,  $\pi_*(X)$ . For any representation  $V$  of  $G$  with  $V^G = 0$ , we have a map  $a_V : S^0 \rightarrow S^V$ .

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Suppose  $X$  is a ring spectrum with unit map  $S^0 \rightarrow X$ .

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Suppose  $X$  is a ring spectrum with unit map  $S^0 \rightarrow X$ . Smashing it with  $a_V$  gives a map  $S^0 \rightarrow \Sigma^V X$  which is adjoint to a map  $S^{-V} \rightarrow X$ .

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It has the multiplicative property  $a_{V+W} = a_V a_W$ .

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It has the multiplicative property  $a_{V+W} = a_V a_W$ .

If  $V$  is a representation of a subgroup  $H \subset G$  with  $V^H = 0$  and  $V'$  is the induced representation of  $G$ , the  $N_H^G(a_V) = a_{V'}$ .

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## The element $u_W \in \pi_{|W|-W}(HZ)$

Let  $W$  be an oriented representation of  $G$ , meaning that it takes values in the **special** orthogonal group.

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Let  $W$  be an oriented representation of  $G$ , meaning that it takes values in the **special** orthogonal group. Then  $\pi_{|W|}(S^W \wedge HZ) = \mathbf{Z}$  and we denote its generator by  $u_W \in \pi_{|W|-W}(HZ)$ .

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We have  $u_{V+W} = u_V u_W$ , and for a trivial representation  $n$ ,  $u_n = 1$ .

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If  $W$  is an oriented representation of a subgroup  $H \subset G$  with induced representation  $W'$  and  $W^H = 0$ , then  $|W|$  is even and the norm functor  $N_H^G$  from  $H$ -spectra to  $G$ -spectra satisfies

$$N_H^G(u_W) u_{2\rho_{G/H}}^{|W|/2} = u_{W'},$$

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$$N_H^G(u_W) u_{2\rho_{G/H}}^{|W|/2} = u_{W'},$$

where  $\rho_{G/H}$  denotes the representation of  $G$  induced up from the degree 1 trivial representation of  $H$ .

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