A solution to the Arf-Kervaire invariant problem

New Aspects of Homotopy Theo braic Geometry
Tokyo City University

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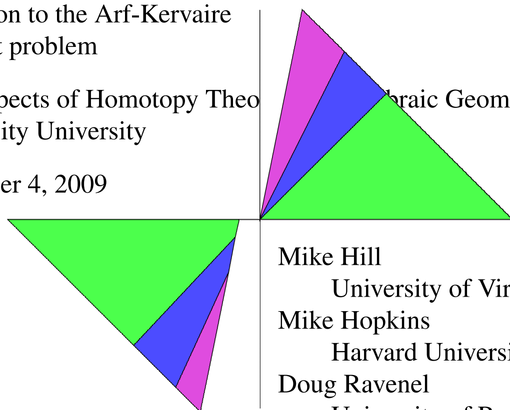

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2.1

1 The spectrum Ω

The spectrum Ω

Our goal is to prove

Main Theorem. *The Arf-Kervaire elements $\theta_j \in \pi_{2^{j+1}-2}(S^0)$ do not exist for $j \geq 7$.* 2.2

Our strategy is to find a map $S^0 \to M$ to a nonconnective spectrum Ω with the following properties.

(i) It has an Adams-Novikov spectral sequence in which the image of each θ_j is nontrivial.
(ii) It is 256-periodic, meaning $\Sigma^{256}M \cong M$.
(iii) $\pi_{-2}(M) = 0$.

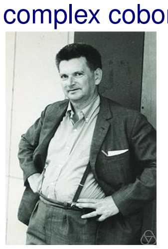
The spectrum Ω (continued)

We will construct an equivariant C_8-spectrum $\tilde{\Omega}$ and show that its homotopy fixed point set $\tilde{\Omega}^{hC_*}$ (to be defined below) and its actual fixed point set $\tilde{\Omega}^{C_8}$ are equivalent.

- The homotopy of $\tilde{\Omega}^{hC_*}$ can be computed using a spectral sequence similar to that of Hopkins-Miller. Twenty year old algebraic methods can be used to show that it detects the θ_js.
- In order to establish (ii) and (iii), we will use equivariant methods to construct a new spectral sequence (the slice spectral sequence) converging to the homotopy of the actual fixed point set $\tilde{\Omega}^{C_8}$. 2.3

2 *MU*

The complex cobordism spectrum



MU is the Thom spectrum for the universal complex vector bundle, which is defined over the classifying space of the stable unitary group, *BU*.

- *MU* has an action of the group C_2 via complex conjugation. The fixed point set is *MO*, the Thom spectrum for the universal real vector bundle.
- $H_*(MU;\mathbf{Z}) = \mathbf{Z}[b_i : i > 0]$ where $|b_i| = 2i$.
- $H_*(MO;\mathbf{Z}/2) = \mathbf{Z}/2[a_i : i > 0]$ where $|a_i| = i$.
- $\pi_*(MU) = \mathbf{Z}[x_i : i > 0]$ where $|x_i| = 2i$. This is the complex cobordism ring.
- $\pi_*(MO) = \mathbf{Z}/2[y_i : i > 0, i \neq 2^k - 1]$ where $|y_i| = i$. This is the unoriented cobordism ring. 2.4

3 Formal group laws

Formal group laws

The following algebraic structure plays a central role in complex cobordism theory.

A (1-dimensional commutative) formal group law over a ring R is a power series

$$F(x,y) = \sum_{i,j \geq 0} a_{i,j} x^i y^j \in R[[x,y]]$$

satisfying

(i) (Commutativity) $F(y,x) = F(x,y)$. This implies $a_{j,i} = a_{i,j}$.

(ii) (Identity element) $F(x,0) = F(0,x) = x$. This implies $a_{1,0} = a_{0,1} = 1$ and $a_{i,0} = a_{0,i} = 0$ for $i \neq 1$.

(iii) (Associativity) $F(x, F(y,z)) = F(F(x,y),z)$. This implies more complicated relations among the $a_{i,j}$. 2.5

Examples of formal group laws

- $x+y$, the additive formal group law.
- $x+y+xy$, the multiplicative formal group law. Note here that $1+F(x,y) = (1+x)(1+y)$.
- $(x+y)/(1-xy)$, the addition formula for the tangent function. 2.6

Another example of a formal group law

$$\frac{x\sqrt{1-y^4}+y\sqrt{1-x^4}}{1+x^2y^2},$$

This formal group law is defined over $\mathbf{Z}[1/2]$. It is the addition formula for the elliptic integral

$$\int_0^x \frac{dt}{\sqrt{1-t^4}}.$$

It is originally due to Euler, see De integratione aequationis differentialis $(m\mathrm{d}x)/\sqrt{1-x^4} = (n\mathrm{d}y)/\sqrt{1-x^4}$, 1753.

2.7

The Lazard ring and the universal formal group law

Let

$$L = \mathbf{Z}[a_{i,j}]/(\text{relations})$$

where the relations are those implied by the definition of a formal group law. We give this ring a grading by $|a_{i,j}| = 2(i+j-1)$.

There is formal group law G over L given by the formula in the definition. It is universal in the following sense.

Given any formal group law F over any ring R, there is a unique ring homomorphism $\lambda : L \to R$ such that

$$F(x,y) = \lambda(G(x,y)),$$

where $\lambda(G(x,y))$ is the formal group law over R obtained from G by applying λ to each of the $a_{i,j}$. 2.8

Quillen's theorem

Lazard showed that L and $\pi_*(MU)$ are isomorphic as graded rings. Quillen showed that this is not an accident. The isomorphism is defined by a formal group law over $\pi_*(MU)$ defined as follows.

There is a cohomology theory associated with MU under which

$$\begin{array}{rrcl} & MU^*(\mathbf{C}P^\infty) & = & \pi_*(MU)[[x]] \\ \text{and} & MU^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) & = & \pi_*(MU)[[x \otimes 1, 1 \otimes x]]. \end{array}$$

The map $\mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$ (corresponding to tensor product of complex line bundles) induces a homomorphism

$$MU^*(\mathbf{C}P^\infty) \to MU^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty)$$

that sends x to a power series in $x \otimes 1$ and $1 \otimes x$ which is a formal group law over $\pi_*(MU)$. 2.9

Quillen's theorem (continued)

Quillen's Theorem (1969). *The homomorphism $\theta : L \to \pi_*(MU)$ induced by the formal group law over $\pi_*(MU)$ defined above is an isomorphism.* 2.10

This means that the internal structure of MU, and the associated homology and cohomology theories, is intimately related to the structure of formal group laws.

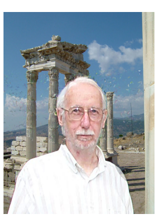
4 Some relatives of MU

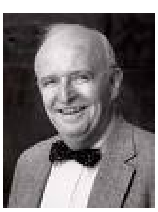
Some relatives of MU

Here is an example of this connection.

After localizing at a prime p, MU splits into a wedge of suspensions of smaller spectra (Brown-Peterson) BP with

$$\pi_*(BP) = \mathbf{Z}_{(p)}[v_n : n > 0] \qquad \text{where } |v_n| = 2p^n - 2.$$





Brown and Peterson originally constructed it (in 1967) via its Postnikov tower. 2.11

More relatives of MU

Quillen's 1969 paper gave a more elegant construction in terms of p-typical formal group laws. A theorem of Cartier says that any formal group law over a $\mathbf{Z}_{(p)}$-algebra is canonically isomorphic to one with certain special properties.

The Brown-Peterson splitting is the topological analog of Cartier's theorem. 2.12

More relatives of MU

The Morava spectrum E_n (for a positive integer n) is an E_∞-ring spectrum such that $\pi_*(E_n)$ obtained from $\pi_*(BP)$ as follows:

(i) Invert v_n and kill the higher generators.

(ii) Complete with respect to the ideal $I_n = (p, v_1, \ldots, v_{n-1})$.

(iii) Tensor over $\mathbf{Z}_p$ (the p-adic integers) with the Witt ring $W(\mathbf{F}_{p^n})$; this is equivalent to adjoining (p^n-1)th roots of unity.

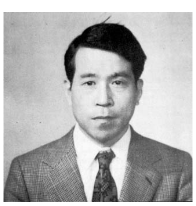
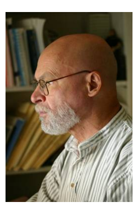
The ring $\pi_*(E_n)$ was studied by Lubin-Tate. They showed that it classifies liftings (to Artinian rings) of a certain formal group law F_n over $\mathbf{F}_{p^n}$, the Honda formal group law. 2.13

5 The Hopkins-Miller theorem

The Morava stabilizer group S_n

S_n is the automorphism group of the Honda formal group law F_n. It a crucial ingredient in chromatic stable homotopy theory.

Its action on F_n lifts to an action on $\pi_*(E_n)$, the Lubin-Tate ring. This action is defined by certain formulas but is mysterious in practice.

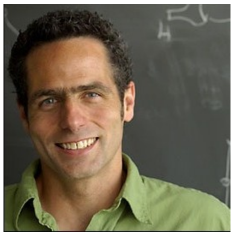

It is a pro-p-group isomorphic to a group of units in a certain division algebra D_n of rank n^2 over the p-adic numbers $\mathbf{Q}_p$.

D_n contains each degree n field extension of $\mathbf{Q}_p$, including the cyclotomic ones.

We will be interested in some finite subgroups of S_n. 2.14

The Hopkins-Miller theorem

The algebraically defined action of S_n on $\pi_*(E_n)$ leads to action on E_n itself, but it is defined only up to homotopy.



In the early 90s Hopkins and Miller showed that the action can be rigidified enough to construct homotopy fixed points sets E_n^{hG} for finite subgroups G.

$E_n^{hS_n}$ is $L_{K(n)}S^0$, the localization of the sphere spectrum with respect to the nth Morava K-theory.

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The Hopkins-Miller theorem (continued)

Hopkins-Miller Theorem (1992?). *For each closed subgroup $G \subset S_n$ there is a homotopy fixed point set E_n^{hG} and a spectral sequence*

$$H^*(G;\pi_*(E_n)) \implies \pi_*(E_n^{hG}).$$

It coincides with the Adams-Novikov spectral sequence for E_n^{hG}.

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Finite subgroups of S_n

The finite subgroups of S_n have been completely classified by Hewett, but only three of them concern us here. The prime is always 2.

- $C_2 = \{\pm 1\} \subset S_1$, which is $\mathbf{Z}_2^\times$, the units in the 2-adic integers.
- $C_4 \subset S_2$. The group S_2 is in the division algebra D_2 which contains each quadratic extension of the 2-adic numbers. Hence it contains fourth roots of unity.
- $C_8 \subset S_4$. The division algebra D_4 contains eighth roots of unity for similar reasons.

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6 Our first guess at Ω

A first attempt to define the magic spectrum Ω

- The spectrum $E_4^{hC_8}$ can be shown to satisfy the first condition required of Ω, namely its Adams-Novikov spectral sequence detects all of the θ_js. $E_1^{hC_2}$ and $E_2^{hC_4}$ do not have this property.
- The Hopkins-Miller spectral sequence for $E_1^{hC_2}$ is very simple and we will describe it at the end of the third lecture.
- The one for $E_2^{hC_4}$ is very rich and is similar to the one for tmf (topological modular forms), whose $K(2)$-localization is the homotopy fixed point set for a certain subgroup of order 24.
- The one for $E_4^{hC_8}$ is too complicated for us to use it to prove that $\pi_{-2} = 0$.

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A C_8-equivariant substitute for E_4

A G-equivariant spectrum is more than a spectrum with an action of G. We will give the precise definitions in the next lecture.

After describing a C_8-equivariant substitute for E_4, we will present a new spectral sequence, the slice spectral sequence, for computing the homotopy of its fixed point set.

A convenient property of the slice spectral sequence is that π_{-2} vanishes at the E_2-level, making property (iii) immediate.

Property (ii) (periodicity) involves some differentials in the slice spectral sequence.

Property (i) (detection) requires some algebra that has been knonw for over 20 years. It will be the subject of the last lecture. 2.19

7 Two spectral sequences for KO

The Hopkins-Miller spectral sequence for KO

The simplest case of a finite subgroup of S_n acting on E_n is that of C_2 acting on E_1 for $p = 2$. It has been known since the 70s. E_1 is 2-adic complex K-theory and the group action is complex conjugation. The homotopy fixed point set is 2-adic real K-theory.

It has a slice spectral sequence that was the subject of Dan Dugger's thesis.

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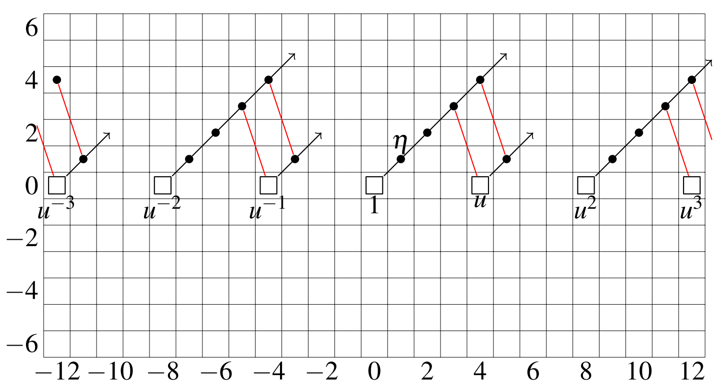
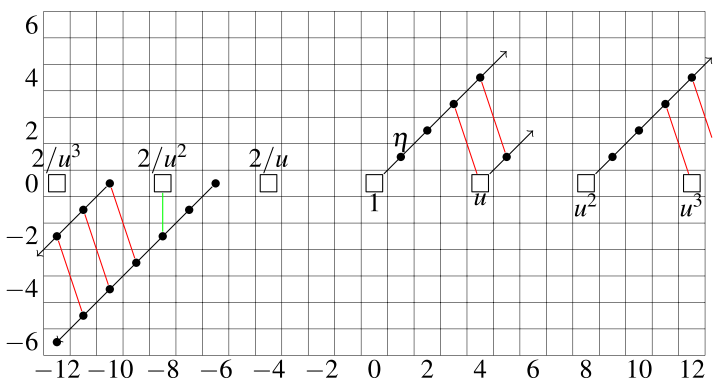
The Hopkins-Miller spectral sequence for KO (continued)



Here is the Hopkins-Miller spectral sequence it. 2.21

The slice spectral sequence for KO



Here is the slice spectral sequence for the actual fixed point set. 2.22

Actual fixed points and homotopy fixed points

These two spectral sequences are computing different things.

- The Hopkins-Miller spectral sequence converges to $\pi_*(E_1^{hC_2})$, the homotopy of the homotopy fixed point set, $F(EC_2, E_1)^{C_2}$, the spectrum of equivariant maps from a contractible free C_2-spectrum EC_2 to E_1.
- The slice spectral sequence converges to $\pi_*(E_1^{C_2})$, the homotopy groups of the actual fixed point set.

In general the homotopy and actual fixed point sets need not be equivalent, but in this case they are.

With this in mind, comparing the two E_2-terms enables us to determine the complete behavior of each SS. 2.23