THE THOMIFIED EILENBERG-MOORE SPECTRAL SEQUENCE

MARK MAHOWALD, DOUGLAS C. RAVENEL AND PAUL SHICK

December 8, 1999

1. Introduction

In this paper we will construct a generalization of the Eilenberg-Moore spectral sequence, which in some interesting cases turns out to be a form of the Adams spectral sequence. We recall the construction of both of these in general terms. Suppose we have a diagram of spectra of the form

$$\begin{array}{ccccccc} X_0 & \longleftarrow & X_1 & \longleftarrow & X_2 & \longleftarrow & \cdots \\ \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \\ K_0 & & K_1 & & K_2 & & \end{array} \tag{1.1}$$

where X_{s+1} is the fiber of g_s. We get an exact couple of homotopy groups and a spectral sequence with

$$E_1^{s,t} = \pi_{t-s}(K_s) \quad \text{and} \quad d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

This spectral sequence converges to $\pi_*(X)$ (where $X = X_0$) if the homotopy inverse limit $\lim_{\leftarrow} X_s$ is contractible and certain $\lim^1$ groups vanish. When X is connective, it is a first quadrant spectral sequence. For more background, see [Rav86].

In the case of the classical Adams spectral sequence, we have some additional conditions on on (1.1), namely

- Each spectrum K_s is a generalized mod p Eilenberg-Mac Lane spectrum, and
- each map g_s induces a monomorphism in mod p homology

These conditions enable us to identify the E_2-term as an Ext group over the Steenrod algebra, and to prove convergence when X is connective and p-adically complete.

The second author acknowledges support from NSF grant DMS-9802516 and the Centre de Recerca Matematica in Barcelona.

For the Eilenberg-Moore spectral sequence, let

$$X \xrightarrow{i} E \xrightarrow{h} B \tag{1.2}$$

be a fiber sequence with simply connected base space B. Then one uses this (in a manner to be described below) to produce a diagram of the form (1.1) where X_0 is the suspension spectrum of X. This will yield a spectral sequence converging to the stable homotopy of X, but in practice it is not very useful. However if we smash everything in sight with the mod p Eilenberg-Mac Lane spectrum H/p, we get the Eilenberg-Moore spectral sequence converging to $H_*(X)$, where E_2 is a certain Cotor group over $H_*(B)$.

In this paper we will explain a way to twist this construction using a p-local spherical fibration over the total space E. The entire construction can be Thomified to yield a spectral sequence converging to the homotopy of the Thom spectrum for the induced bundle over X. In §2 we recall a geometric construction of the Eilenberg-Moore spectral sequence, and in §3 we explain how it can be Thomified. In §4 we identify the E_2-term under certain circumstances as an Ext group over the Massey-Peterson algebra of the base space of the fibration in question, and in §5 we show that in some other cases we get a BP-theoretic analog of this result. In §6, we show that a special case of the $\mathbf{Z}/(p)$-equivariant Adams spectral sequence of Greenlees can be constructed using the Thomified Eilenberg-Moore spectral sequence.

The authors wish to thank Bill Dwyer, John Greenlees and Brooke Shipley for helpful conversations and correspondence.

2. A geometric construction of the Eilenberg-Moore spectral sequence

We begin by recalling the stable cosimplicial construction associated with the Eilenberg-Moore spectral sequence, due to Larry Smith [Smi69] and Rector [Rec70]. Given the fibration (1.2), for $s \geq 0$ let

$$G_s = E \times \overbrace{B \times \cdots \times B}^{s \text{ factors}}.$$

Define maps $h_t : G_{s-1} \to G_s$ for $0 \leq t \leq s$ by

$$h_t(e, b_1, b_2, \ldots, b_{s-1}) = \begin{cases} (e, b_1, b_2, \ldots, b_{s-1}, *) & \text{if } t = 0 \\ (e, b_1, b_2, \ldots, b_t, b_t, b_{t+1}, \ldots, b_{s-1}) & \text{if } 1 \leq t \leq s-1 \\ (e, h(e), b_1, b_2, \ldots, b_{s-1}) & \text{if } t = s. \end{cases}$$

Let $E_0 = E$, $X_0 = X$, $X_1 = E/\mathrm{Im}\, i$, and for $s \geq 1$ we define spectra

$$\begin{aligned} \Sigma^s E_s &= G_s/\mathrm{Im}\, h_0 \cup \cdots \cup \mathrm{Im}\, h_{s-1} \\ \Sigma^{s+1} X_{s+1} &= G_s/\mathrm{Im}\, h_0 \cup \cdots \cup \mathrm{Im}\, h_s \end{aligned}$$

i.e., the spectra X_s and E_s are desuspensions of suspension spectra of the indicated spaces. Then for $s \geq 0$, h_s induces a map $X_s \to E_s$ giving a cofiber sequence

$$X_s \xrightarrow{\quad h_s \quad} E_s \xrightarrow{\quad \partial_s \quad} \Sigma X_{s+1}, \tag{2.1}$$

where ∂_s is projection from the topological quotient of G_s by one subspace to the quotient by a bigger subspace.

For $s \geq 0$ there is a homology isomorphism

$$H_*(E_s) = \Sigma^{-s} H_*(E) \otimes \overline{H}_*(B^{(s)}),$$

where $\overline{H}$ denotes reduced homology. Since B is simply connected, the connectivity of E_s increases without bound with s. Note also that

$$H_*(X_s) = \Sigma^{-s} H_*(X) \otimes \overline{H}_*(B^{(s)}),$$

for $s > 0$, so the homotopy inverse limit of the X_s is contractible. The homology exact couple associated with the cofiber sequences (2.1) leads to the Eilenberg-Moore spectral sequence for the fibration (1.2). The Eilenberg-Moore spectral sequence also converges for non-simply-connected B, also. Dwyer has proved ([D74]) that the Eilenberg-Moore spectral sequence for the fibration

$$X \to E \to B$$

converges strongly to $H_*(X)$ if and only if $\pi_1(B)$ acts nilpotently on $H_i(E)$ for all $i \geq 0$.

3. The Thomified Eilenberg-Moore spectral sequence

Now suppose that in addition to the fibration (1.2) we also have a p-local stable spherical fibration ξ over E which is oriented with respect to mod p homology. Projection onto the first coordinate gives compatible maps of the G_s to E, and hence a stable spherical fibration over each of them. This means that we can Thomify the entire construction. To each of the quotients X_s and E_s we associate a *reduced Thom spectrum*, which is defined as follows. Given a space A with a spherical fibration and a subspace $B \subset A$, the reduced Thom space for A/B is the space $D_A/(S_A \cup D_B)$ where D_X and S_X denote disk and sphere bundles over the space X. Thus we can associate reduced Thom spectra to the topological quotients E_s and X_{s+1} of G_s.

Let Y, K, Y_s and K_s be the Thomifications of X, E, X_s and E_s. Then the cofiber sequence of (2.1) Thomifies to

$$Y_s \longrightarrow K_s \longrightarrow \Sigma Y_{s+1}. \tag{3.1}$$

and we have

$$H_*(K_s) = \Sigma^{-s} H_*(K) \otimes \overline{H}_*(B^{(s)}).$$

The exact couple of homotopy groups for (3.1) leads to a spectral sequence converging to $\pi_*(Y)$. There is an associated diagram

$$\begin{array}{ccccccc} Y = & Y_0 & \longleftarrow & Y_1 & \longleftarrow & Y_2 & \longleftarrow \cdots \\ & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & \\ & K_0 & & K_1 & & K_2 & \end{array} \tag{3.2}$$

where Y_{s+1} is the fiber of g_s. This is similar to the Adams diagram of (1.1), but $H_*(g_s)$ need not be a monomorphism in general. We will call this the *Thomified Eilenberg-Moore spectral sequence*. We will use the indexing conventions of Adams rather than Eilenberg-Moore, namely

$$E_1^{s,t} = \pi_{t-s}(K_s) \qquad \text{with} \qquad E_r^{s,t} \xrightarrow{d_r} E_r^{s+r,t+r-1}.$$

This puts our spectral sequence in the first rather than the second quadrant.

We will see below (Theorem 4.4(ii) and Corollary 4.5) that under suitable hypotheses (including that the map i of (1.2) induces a monomorphism in homology), the Thomified Eilenberg-Moore spectral sequence coincides with the usual Adams spectral sequence for $\pi_*(Y)$.

The following lemma will be useful.

Lemma 3.3. *For each prime p there is a p-local spherical fibration over $\Omega^2 S^3$ whose Thom spectrum is the mod p Eilenberg-Mac Lane spectrum H/p.*

Proof. For $p = 2$ we can use an ordinary vector bundle. We extend the nontrivial map $S^1 \to BO$ to $\Omega^2 S^3$ using the double loop space structure on BO. It was shown in [Mah79] that the resulting Thom spectrum is $H/2$.

The following argument for odd primes is due to Mike Hopkins. Let $BF(n)_{(p)}$ denote the classifying space for the monoid of homotopy equivalences of the p-local n-sphere. Its fundamental group is $\mathbf{Z}_{(p)}^\times$. A p-local n-dimensional spherical fibration of a space X, i.e., a fibration with fiber $S^n_{(p)}$, is classified by a map $X \to BF(n)_{(p)}$. Its Thom space is the cofiber of the projection map to X. Such fibrations and Thom spectra can be stabilized in the usual way. We denote the direct limit of the $BF(n)_{(p)}$ by $BF_{(p)}$.

Now consider a p-local spherical fibration over S^1 corresponding to an element $u \in \mathbf{Z}_{(p)}^{\times}$. It Thomifies to the Moore spectrum $S^0 \cup_{1-u} e^1$. If we set $u = 1 - p$ (which is a p-local unit) we get the mod p Moore spectrum $V(0)$.

As in the case $p = 2$, we can extend this map $S^1 \to BF_{(p)}$ to $\Omega^2 S^3$ using the double loop space structure on $BF_{(p)}$, and similar arguments to those of [Mah79] identify the resulting Thom spectrum as H/p. □

4. Identifying the E_2-term

Observe that $H_*(K)$ is simultaneously a comodule over A_* and (via the Thom isomorphism and the map h_*) $H_*(B)$, which is itself a comodule over A_*. Following Massey-Peterson [MP67], we combine these two structures by defining the *Massey-Peterson coalgebra* (they called the dual object the semitensor product)

$$R_* = H_*(B) \otimes A_* \tag{4.1}$$

in which the coproduct is the composite

$$\begin{array}{c} H_*(B) \otimes A_* \\ \Big\downarrow {\scriptstyle \Delta_B \otimes \Delta_A} \\ H_*(B) \otimes H_*(B) \otimes A_* \otimes A_* \\ \Big\downarrow {\scriptstyle H_*(B) \otimes \psi_B \otimes A_* \otimes A_*} \\ H_*(B) \otimes A_* \otimes H_*(B) \otimes A_* \otimes A_* \\ \Big\downarrow {\scriptstyle H_*(B) \otimes A_* \otimes T \otimes A_*} \\ H_*(B) \otimes A_* \otimes A_* \otimes H_*(B) \otimes A_* \\ \Big\downarrow {\scriptstyle H_*(B) \otimes m_A \otimes H_*(B) \otimes A_*} \\ (H_*(B) \otimes A_*) \otimes (H_*(B) \otimes A_*), \end{array} \tag{4.2}$$

where Δ_A and Δ_B are the coproducts on A_* and $H_*(B)$, T is the switching map, $\psi_B : H_*(B) \to A_* \otimes H_*(B)$ is the comodule structure map, and m_A is the multiplication in A_*.

Massey-Peterson gave this definition in cohomological terms. They denoted the semitensor algebra R by $H^*(B) \odot A$, which is additively isomorphic to $H^*(B) \otimes A$ with multiplication given by

$$(x_1 \otimes a_1)(x_2 \otimes a_2) = x_1 a_1'(x_2) \otimes a_1'' a_2,$$

where $x_i \in H^*(B)$, $a_i \in A$, and $a_1' \otimes a_1''$ denotes the coproduct expansion of a_1 given by the Cartan formula. Our definition is the homological reformulation of theirs.

Note that given a map $f : V \to B$ and a subspace $U \subset V$, $\bar{H}^*(V/U) = H^*(V, U)$ is an R-module since it is an $H^*(V)$-module via relative cup products, even if the map f does not extend to the quotient V/U. In our case we have maps $G_s \to B$ for all $s \geq 0$ given by

$$(e, b_1, \ldots, b_s) \mapsto h_e.$$

These are compatible with all of the maps h_t, so $H_*(Y_s)$ and $H_*(K_s)$ are R_*-comodules, and the maps between them respect this structure.

We will see in the next theorem that under suitable hypotheses, the E_2-term of the Thomified Eilenberg-Moore spectral sequence is $\mathrm{Ext}_{R_*}(\mathbf{Z}/(p), H_*(K))$. When B is an H-space we have a Hopf algebra extension (see [Rav86, A1.1.15] for a definition)

$$A_* \longrightarrow R_* \longrightarrow H_*(B).$$

This gives us a Cartan-Eilenberg spectral sequence ([CE56, page 349] or [Rav86, A1.3.14]) converging to this Ext group with

$$E_2 = \mathrm{Ext}_{A_*}(\mathbf{Z}/(p), \mathrm{Ext}_{H_*(B)}(\mathbf{Z}/(p), H_*(K))). \tag{4.3}$$

Note that the inner Ext group above is the same as $\mathrm{Ext}_{H_*(B)}(\mathbf{Z}/(p), H_*(E))$, the E_2-term of the classical Eilenberg-Moore spectral sequence converging to $H_*(X)$. If the latter collapses from E_2, then the Ext group of (4.3) can be thought of as

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(Y)),$$

where $H_*(Y)$ is equipped with the Eilenberg-Moore bigrading. This is the usual Adams E_2-term for Y when $H_*(Y)$ is concentrated in Eilenberg-Moore degree 0, but the Ext group of (4.3) is graded differently in general.

Theorem 4.4. (i) *Suppose that B is simply connected. Then the Thomified Eilenberg-Moore spectral sequence associated with the homotopy of (3.2) converges to $\pi_*(Y)$. If, in addition, $H^*(K)$ is a free A-module, then*

$$E_2 = \mathrm{Ext}_{R_*}(\mathbf{Z}/(p), H_*(K)),$$

where R_* is the Massey-Peterson coalgebra of (4.1).

(ii) *If, in addition, the map $i : X \to E$ induces a monomorphism in mod p homology, then the Thomified Eilenberg-Moore spectral sequence coincides with the classical Adams spectral sequence for Y.*

The hypotheses on $H_*(K)$ may be unnecessary, but they are adequate for our purposes. The result may not be new, but we know of no published proof. Before proving the theorem we give a corollary that indicates that the hypotheses are not as restrictive as they may appear.

Corollary 4.5. *Given a fibration*

$$X \longrightarrow E \longrightarrow B$$

with X p-adically complete, a p-local spherical fibration over E, and B simply connected, there is a spectral sequence converging to $\pi_*(Y)$ (where Y is the Thomification of X) with

$$E_2 = \mathrm{Ext}_{H_*(B)\otimes A_*}(\mathbf{Z}/(p), H_*(K)),$$

where K as usual is the Thomification of E.

Proof. We can apply 4.4 to the product of the given fibration with pt. $\to \Omega^2 S^3 \to \Omega^2 S^3$, where $\Omega^2 S^3$ is equipped with the p-local spherical fibration of Lemma 3.3. Then the Thomified total space is $K \wedge H/p$, so its cohomology is a free A-module. Thus the E_2-term is

$$\mathrm{Ext}_{H_*(B\wedge H/p)\otimes A_*}(\mathbf{Z}/(p), H_*(K \wedge H/p)) = \mathrm{Ext}_{H_*(B)\otimes A_*}(\mathbf{Z}/(p), H_*(K)).$$

□

Proof of Theorem 4.4 (i) The freeness of $H_*(K)$ over A_* does not make (3.2) an Adams resolution because $H_*(g_s)$ need not be a monomorphism and the cofiber sequence

$$\Sigma^s Y_s \xrightarrow{g_s} \Sigma^s K_s \longrightarrow \Sigma^{s+1} Y_{s+1}$$

need not induce a short exact sequence in homology.

We will finesse this problem by producing a commutative diagram

$$\text{(4.6)}\qquad \begin{array}{ccccc} \Sigma^s Y_s & \xrightarrow{g_s} & \Sigma^s K_s & \longrightarrow & \Sigma^{s+1} Y_{s+1} \\ \downarrow{\scriptstyle g_s} & & \downarrow & & \downarrow{\scriptstyle h_{s+1}} \\ \Sigma^s K_s & \longrightarrow & \Sigma^s W_s & \longrightarrow & \Sigma^{s+1} K_{s+1} \end{array} \qquad \text{for } s \geq 0.$$

in which the cofiber sequence in the bottom row does induce a short exact sequence in homology with

$$H_*(W_s) = H_*(K_s) \otimes H_*(B). \tag{4.7}$$

By the change-of-rings isomorphism of Milnor-Moore [MM65], this implies that

$$\mathrm{Ext}_{R_*}(\mathbf{Z}/(p), H_*(W_s)) = \mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(K_s)). \tag{4.8}$$

Splicing the short exact sequences in homology from the bottom row of (4.6) gives a long exact sequence

$$0 \longrightarrow H_*(K) \longrightarrow H_*(W_0) \longrightarrow H_*(\Sigma W_1) \longrightarrow \cdots,$$

which gives an algebraic spectral sequence (see [Rav86, A1.3.2]) converging to $\mathrm{Ext}_{R_*}(\mathbf{Z}/(p), H_*(K))$ with

$$E_1 = \mathrm{Ext}_{R_*}(\mathbf{Z}/(p), H_*(W_s)),$$

suitably indexed.

The freeness hypothesis on $H_*(K)$ implies (via (4.7)) that $H_*(W_s)$ is free over R_*, so the algebraic spectral sequence collapses from E_2, i.e., $\mathrm{Ext}_{R_*}(\mathbf{Z}/(p), H_*(K))$ is the cohomology of the cochain complex

$$\mathrm{Ext}^0_{R_*}(\mathbf{Z}/(p), H_*(W_0)) \longrightarrow \mathrm{Ext}^0_{R_*}(\mathbf{Z}/(p), H_*(\Sigma W_1)) \longrightarrow \cdots$$

By (4.8) this is the same as

$$\mathrm{Ext}^0_{A_*}(\mathbf{Z}/(p), H_*(K_0)) \longrightarrow \mathrm{Ext}^0_{A_*}(\mathbf{Z}/(p), H_*(\Sigma K_1)) \longrightarrow \cdots$$

and our freeness hypothesis along with (4.6) allows us to identify this cochain complex with the E_1-term of the Thomified Eilenberg-Moore spectral sequence.

Thus the Thomified Eilenberg-Moore spectral sequence has the desired E_2-term if we can produce the diagram (4.6) satisfying (4.7). We shall do this now by geometric construction.

We define the following subspaces of G_s for $s \geq 1$:

$$\begin{aligned} A_s &= \mathrm{Im}\, h_0 \cup \mathrm{Im}\, h_2 \cup \ldots \cup \mathrm{Im}\, h_{s-1}, \\ B_s &= A_s \cup \mathrm{Im}\, h_1 \\ \text{and} \quad C_s &= B_s \cup \mathrm{Im}\, h_s. \end{aligned}$$

Then it follows that h_s sends C_{s-1} to B_s and B_{s-1} to A_s, $B_s/A_s = G_{s-1}/B_{s-1}$ and $C_s/B_s = G_{s-1}/C_{s-1}$. Thus for $s \geq 0$ we get the following pointwise commutative diagram in which each row is a cofiber sequence.

$$\begin{array}{ccccc} X & \xrightarrow{i} & E & \xrightarrow{\partial_0} & E/i(X) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_1} \\ E & \xrightarrow{h_1} & G_1/A_1 & \xrightarrow{\partial_1} & G_1/B_1 \end{array} \qquad \text{for } s = 0$$

$$\text{and} \qquad \begin{array}{ccccc} G_{s-1}/C_{s-1} & \xrightarrow{h_s} & G_s/B_s & \xrightarrow{\partial_s} & G_s/C_s \\ \downarrow{\scriptstyle h_s} & & \downarrow{\scriptstyle h_{s+1}} & & \downarrow{\scriptstyle h_{s+1}} \\ G_s/B_s & \xrightarrow{h_{s+1}} & G_{s+1}/A_{s+1} & \xrightarrow{\partial_{s+1}} & G_{s+1}/B_{s+1} \end{array} \qquad \text{for } s \geq 1.$$

We define $\Sigma^{s-1}W_{s-1}$ to be the Thomification of G_s/A_s, and we have previously defined $\Sigma^s K_s$ and $\Sigma^{s+1}X_{s+1}$ to be the Thomifications of G_s/B_s and G_s/C_s, so Thomification converts the diagrams above to (4.6).

Let $p_s : G_{s+1} \to G_s \times B$ be the homeomorphism given by

$$p_s(e, b_1, \ldots, b_{s+1}) = ((e, b_2, \ldots, b_{s+1}), b_1).$$

Then we have

$$\begin{aligned} p_s h_0 &= (h_0 \times B)p_{s-1} \\ \text{and} \qquad p_s h_t &= (h_{t-1} \times B)p_{s-1} \qquad \text{for } 2 \leq t \leq s. \end{aligned}$$

It follows that

$$G_{s+1}/A_{s+1} = (G_s \times B)/(B_s \times B) = (G_s/B_s) \times B$$

and (4.7) follows.

(ii) If the $H_*(i)$ is monomorphic and $H^*(K_s)$ is a free A-module, then the diagram (3.2) is an Adams resolution for Y. Thus, the identity map on the resolution provides a comparison map from the Thomified Eilenberg-Moore spectral sequence to the Adams spectral sequence. We can identify the inner Ext group of (4.3) with $H_*(Y)$ concentrated in degree 0, the Cartan-Eilenberg spectral sequence collapses and our E_2-term is the usual

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(Y)).$$

So the comparison map induces an isomorphism on the E_2 term of the spectral sequences, completing the proof of the theorem.

□

5. An Adams-Novikov analog

We now describe a case of the Thomified Eilenberg-Moore spectral sequence leading to variants of the Adams-Novikov spectral sequence. Suppose that in the fibration of (1.2), the spherical fibration over E is a complex vector bundle and that $MU_*(K)$ is free as a comodule over $MU_*(MU)$. If in addition $MU_*(i)$ is a monomorphism, then we get the usual Adams-Novikov spectral sequence converging to $\pi_*(Y)$.

We want an analog of 4.4 in the p-local case identifying the E_2-term for more general i. For this we need a BP-theoretic analog of the Massey-Peterson algebra R_* of (4.1), additively isomorphic to

$$\Gamma(B) = BP_*(B) \otimes_{BP_*} \Gamma, \tag{5.1}$$

where $\Gamma = BP_*(BP)$. In order to define a coproduct on this as in (4.2), we need a coalgebra structure on $BP_*(B)$. This does not exist in general, but it does when $H_*(B)$ is torsion free and $BP_*(B)$ is therefore a free BP_*-module. If B is also an H-space, then $BP_*(B)$ is a Hopf algebra over BP_* and $(BP_*, \Gamma(B))$ is a Hopf algbebroid (defined in [Rav86, A1.1.1])

$$(BP_*, \Gamma) \longrightarrow (BP_*, \Gamma(B)) \longrightarrow (BP_*, BP_*(B))$$

is a Hopf algebroid extension as defined in [Rav86, A1.1.15]. This means there is a Cartan-Eilenberg spectral sequence (see [CE56, page 349] or [Rav86, A1.3.14]) converging to $\mathrm{Ext}_{\Gamma(B)}(BP_*, BP_*(K))$ with

$$E_2 = \mathrm{Ext}_{\Gamma}(BP_*, \mathrm{Ext}_{BP_*(B)}(BP_*, BP_*(K))). \tag{5.2}$$

Then we get the following analog of Theorem 4.4, which can be proved in the same way.

Theorem 5.3. (i) *Suppose that $BP_*(K)$ is free as a $BP_*(BP)$-comodule and B is simply connected with torsion free homology. Then the Thomified Eilenberg-Moore spectral sequence associated with the homotopy of (3.2) converges to $\pi_*(Y)$ with*

$$E_2 = \mathrm{Ext}_{\Gamma(B)}(BP_*, BP_*(K)),$$

where $\Gamma(B)$ is the Massey-Peterson coalgebra of (5.1).

(ii) *If in addition the map $i : X \to E$ induces a monomorphism in BP-homology, then the Thomified Eilenberg-Moore spectral sequence coincides with the Adams-Novikov spectral sequence for Y.*

There is an analog of 4.5 in which we retain the hypothesis on B while dropping the one on K.

Corollary 5.4. *Given a fibration*

$$X \longrightarrow E \longrightarrow B$$

with X p-local, a a complex vector bundle over E, and B simply connected with torsion free homology, there is a spectral sequence converging to $\pi_*(Y)$ (where Y is the Thomification of X) with

$$E_2 = \mathrm{Ext}_{\Gamma(B)}(BP_*, BP_*(K)),$$

where K as usual is the Thomification of E.

This can be proved by applying 5.3 to the product of the given fibration with

$$\mathrm{pt.} \longrightarrow BU \longrightarrow BU$$

with the universal complex vector bundle over BU.

6. A construction of the equivariant Adams spectral sequence

In this section we provide an alternative construction of a special case of the equivariant Adams spectral sequence, due to Greenlees ([G88] and [G90].) We first recall Greenlees' approach.

Let G be a finite p-group. (Later, we will restrict our attention to the case where G is elementary abelian.) We work in the equivariant stable homotopy category of [LMS86], with all spaces pointed and all homology groups reduced. In this setting, G-free means that the action of G is free away from the base point. Greenlees' version of the equivariant Adams spectral sequence is based on mod p Borel cohomology, defined for a based G-spectrum X as

$$b_G^*(X) = H^*(EG_+ \wedge_G X; \mathbf{Z}/(p)),$$

where, as above, the $\mathbf{Z}/(p)$ coefficient groups will hereafter be suppressed. This is an $RO(G)$-graded cohomology theory, defined as follows for α any virtual real representation of G:

$$b_G^\alpha(X) = H^{|\alpha|}(EG_+ \wedge_G X).$$

Since G is a p-group, all representations are orientable, and the suspension isomorphisms in b_G^* are given by the Thom maps. This cohomology theory b is representable in the equivariant stable category. Adams and Greenlees identify the algebra $b_G^* b$ of natural cohomology operations as

$$b_G^* b \cong H^*(BG_+) \tilde{\otimes} A.$$

Greenlees actually defines the spectral sequence in terms of a variant of Borel cohomology, namely f- or coBorel-cohomology, represented by

$$c = b \wedge EG_+.$$

Greenlees shows in [G88] that $c^*_G c \cong b^*_G b$.

Greenlees' main result is the following cohomology version of the spectral sequence.

Theorem 6.1. *([G88]) For G a finite p-group, X and Y any G-spectra, with Y p-complete, bounded below, G-free and homologically locally finite, there is a convergent Adams spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{c^*_G c}(c^*_G Y, c^*_G X) \Longrightarrow [X, Y]^G_*,$$

natural in both variables.

One can define a similar spectral sequence based on $b^*_G(\cdot)$, but this requires the additional hypothesis that X is G-free to guarantee proper convergence. A homology version of the spectral sequence can be written using the homology theory represented by the G-spectrum b, ([G90]) which does calculate $[X, Y]^G_*$ when X or Y is not G-free, provided we take G to be elementary abelian. The hypotheses on Y can just be checked nonequivariantly, if Y is G-free, by looking at the nonequivariant spectrum $EG_+ \wedge_G Y$.

Greenlees' construction involves building a resolution of $b^*_G Y$ by free $b^*_G b$-modules,

$$0 \longleftarrow b^* Y \xleftarrow{\epsilon} P_0 \xleftarrow{\delta_0} P_1 \xleftarrow{\delta_1} P_1 \longleftarrow \cdots$$

and realizing this resolution geometrically. Apply the functor $[X, -]^G$ to this geometric resolution, obtaining a spectral sequence with

$$E_1 = [\Sigma^{t-s} X, Q_s]^G \Longrightarrow [\Sigma^{t-s} X, Y/ \operatorname*{holim}_s Y_s]^G,$$

where Q_s is a locally finite wedge of copies of the spectrum representing b made free (i.e. a wedge of copies of $c = b \wedge EG_+$,) with $P_s = b^*_G \Sigma^s Q_s$. One identifies the E_2 term in the usual manner, and proves convergence by comparing c^*- (or b^*-) connectivity with H^*-connectivity to show that $\mathrm{holim}_s Y_s \simeq *$.

We now show how to identify this equivariant Adams spectral sequence as a case of the Thomified Eilenberg-Moore spectral sequence. From here onward we take G to be $\mathbf{Z}/(p)$, and we'll work with the spectrum X G-fixed (so that we'll use the $c^*_G c$-based spectral sequence, rather than it $b^*_G b$-based analog.) Let Z be a p-complete free G-spectrum with a spherical G-fibration $F \to E(\xi) \xrightarrow{p} Z$. Consider the

Borel fibration

$$Z \to EG_+ \wedge_G Z \to BG.$$

Then $EG_+ \wedge_G E(\xi) \to EG_+ \wedge_G Z$ is also a G-fibration, with fiber $EG_+ \wedge_G F$. We smash this fibration with

$$\text{pt.} \to \Omega^2 S^3 \to \Omega^2 S^3$$

(with the trivial G-action) and apply the Thomified Eilenberg-Moore spectral sequence construction to the resulting fibration. The resulting resolution is $EG_+ \wedge_G H\mathbf{Z}/(p)$-free. Now for a G-fixed spectrum W (like $H\mathbf{Z}/(p)$ here,) the Borel construction is very simple: $EG_+ \wedge_G W \simeq BG_+ \wedge W$, So the Thomified Eilenberg-Moore spectral sequence resolution is free over $BG_+ \wedge H\mathbf{Z}/(p)$. Let $T(W)$ denote the Thom spectrum of the bundle over W. Then the resulting Thomified Eilenberg-Moore spectral sequence has

$$\begin{aligned} E_2 &= \operatorname{Ext}_{H_*(BG_+)\tilde{\otimes}A_*}(H_*BG_+, H_*(T(EG_+ \wedge_G Z))) \\ &= \operatorname{Ext}_{H_*(BG_+)\tilde{\otimes}A_*}(H_*BG_+, H_*(EG_+ \wedge_G T(Z))) \\ &= \operatorname{Ext}_{b_G^* b}(b_G^*(T(Z)), b_G^*(S^0)), \end{aligned}$$

by $\mathbf{F}_p$-duality, so that the Thomified Eilenberg-Moore spectral sequence E_2 agrees with the equivariant Adams spectral sequence E_2 term. This special case of the Thomified Eilenberg-Moore spectral sequence converges if $[S^0, T(Z)]^G_*$ is isomorphic to $[EG_+, T(Z)]^G_* = [S^0, F(EG_+, T(Z))]^G_*$ via the comparison map $T(Z) \to F(EG_+, T(Z))$, which is indeed an equivalence when $T(Z)$ is finite, confirmed by the Segal Conjecture ([Car84].) Thus, when Z is finite and G-free, the Thomified Eilenberg-Moore spectral sequence converges to

$$\pi_*(T(EG_+ \wedge_G Z)) \cong \pi_*(EG_+ \wedge_G T(Z)) \cong \pi_*(T(Z)/G) \cong \pi_*(T(Z)^G),$$

as we wish, we think of $\pi_*(T(Z))^G$ as $[S^0, T(Z)]^G$ with the sphere G-fixed. This completes the proof of the following:

Theorem 6.2. *Let Z be a p-complete finite based free $\mathbf{Z}/(p)$-spectrum, with a spherical $\mathbf{Z}/(p)$-fibration over Z. The Thomified Eilenberg-Moore spectral sequence for the smash product of the fibrations*

$$pt. \to \Omega^2 S^3 \to \Omega^2 S^3$$

and

$$Z \to E\mathbf{Z}/(p)_+ \wedge_{\mathbf{Z}/(p)} Z \to B\mathbf{Z}/(p)$$

is the equivariant Adams spectral sequence converging to $\pi_*(T(Z))^{\mathbf{Z}/(p)}$.

One might ask why the G-spectrum Z in Theorem 6.2 must be finite, while the Thomified Eilenberg-Moore spectral sequence has no such requirement, in general. We recall Dwyer's result ([D74]) that the Eilenberg-Moore spectral sequence for the fibration

$$X \to E \to B$$

converges strongly to $H_*(X)$ if and only if $\pi_1(B)$ acts nilpotently on $H_i(E)$ for all $i \geq 0$. In our situation, we're looking at whether $\pi_1(\Omega^2 S^3 \wedge B\mathbf{Z}/(p)_+)$ acts nilpotently on $H_*(\Omega^2 S^3 \wedge E\mathbf{Z}/(p)_+ \wedge_{Z/p} T(Z))$. If Z is not finite, this action can be nonnilpotent (as it is in the case of $T(Z) = E\mathbf{Z}/(p)_+ \wedge H\mathbf{Z}/(p)$, as pointed out to us by the referee.)

References

[CE56] H. Cartan and S. Eilenberg. *Homological Algebra.* Princeton University Press, Princeton, 1956.

[Car84] G. Carlsson, Equivariant stable homotopy theory and Segal's Burnside ring conjecture, *Annals of Mathematics* 120:189-224, 1984.

[D74] W. G. Dwyer, Strong convergence of the Eilenberg-Moore spectral sequence, *Topology* 13:255-265, 1974.

[G88] J.P.C. Greenlees, Stable maps into free G-spaces, *Trans. A.M.S.* 310:199-215, 1988.

[G90] J.P.C. Greenlees, The power of mod p Borel homology, *Homotopy Theory and Related Topics,* ed. by M. Mimura, Lecture Notes Math. 1418:140-151, 1990.

[LMS86] L.G. Lewis, J.P. May, and M. Steinberger (with contributions by J.E. McClure), *Equivariant Stable Homotopy Theory,* Lecture Notes in Math. 1213, Springer-Verlag 1986.

[Mah79] M. E. Mahowald. Ring spectra which are Thom complexes. *Duke Mathematical Journal*, 46:549–559, 1979.

[MM65] J. W. Milnor and J. C. Moore. On the structure of Hopf algebras. *Annals of Mathematics*, 81(2):211–264, 1965.

[MP67] W. S. Massey and F. P. Peterson. *The* mod *2 cohomology structure of certain fibre spaces*, volume 74 of *Memoirs of the American Mathematical Society.* American Mathematical Society, Providence, Rhode Island, 1967.

[Rav86] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres.* Academic Press, New York, 1986.

[Rec70] D. L. Rector. Steenrod operations in the Eilenberg-Moore spectral sequence. *Commentarii Mathematici Helvetici*, 45:540–552, 1970.

[Smi69] L. Smith. On the construction of the Eilenberg-Moore spectral sequence. *Bulletin of the American Mathematical Society*, 75:873–878, 1969.

Northwestern University, Evanston IL 60208, University of Rochester, Rochester NY 14627, John Carroll University, University Hts OH 44118