The Homology and Morava K–theory of $\Omega^2 SU(n)$

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The object of this paper is to compute $H_*(\Omega^2 SU(n); \mathbf{Z}/(p))$ and the Morava K–theory $K(m)_*(\Omega^2 SU(n))$. $SU(n)$ as usual denotes the group of $n \times n$ unitary complex matrices with determinant one. The methods we describe will also work for the spaces $\Omega^2 SU(n+1)/SU(k)$ for any $k \leq n$, but we will leave these questions for the interested reader.

We will now state our main results. First we recover a theorem due to Waggoner [Wag85] and Yamaguchi [Yam86].

Theorem A *(a) For each odd prime p, $H_*(\Omega^2 SU(n+1); \mathbf{Z}/(p))$ is*

$$E(x_{2p^kj-1} : 0 < j \leq n,\ k \geq 0,\ p \not| j) \otimes P(y_{n,p^kj-n} : 0 < j \leq n,\ p^kj \geq n,\ p \not| j).$$

where $x_{2i-1} \in H_{2i-1}$ and $y_{n,i-n} \in H_{2i}$. $E(x)$ denotes the exterior algebra on x and $P(y)$ denotes the polynomial algebra on y.

(b) For $p = 2$, $H_*(\Omega^2 SU(n+1); \mathbf{Z}/(2))$ is

$$\begin{gathered} E(x_{2^{k+1}\ell-1} : 0 < \ell < n/2,\ k \geq 0,\ \ell \text{ odd}) \otimes \\ P(y_{n,2^{k+1}\ell-n-1} : 0 < \ell < n/2,\ 2^{k+1}\ell \geq n,\ \ell \text{ odd}) \otimes \\ P(x_{2^{k+1}\ell-1} : n/2 \leq \ell \leq n,\ k \geq 0,\ \ell \text{ odd}) \end{gathered}$$

where for $n/2 \leq \ell \leq n$, $k \geq 0$ and ℓ odd, $x^2_{2^{k+1}\ell-1} = y_{n,2^{k+1}\ell-n-1}$.

These are proved in Section 1 as 1.12 and 1.14. The elements $y_{n,i-n}$ and x_{2i-1} arise in the following way. Consider the maps

$$\Omega^3 SU/SU(n+1) \longrightarrow \Omega^2 SU(n+1) \longrightarrow \Omega^2 SU = U.$$

We will see below (1.2) that

$$H_*(\Omega^3 SU/SU(n+1); \mathbf{Z}/(p)) = P(y_{n,i} : i \geq 0)$$

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for all primes p. The element $y_{n,i-n} \in H_{2i}(\Omega^2 SU(n+1))$ is by definition the image of $y_{n,i-n} \in H_{2i}(\Omega^3 SU/SU(n+1))$. *The main idea of this paper is to exploit the nice structure of $H_*(\Omega^3 SU/SU(n+1))$.*

It is well known that

$$H_*(U; \mathbf{Z}/(p)) = E(x_{2i-1} : i \geq 0).$$

The element $x_{2i-1} \in H_{2i-1}(\Omega^2 SU(n+1))$ maps to $x_{2i-1} \in H_{2i-1}(U)$.

Similar statements can be made about the Morava K–theories of these spaces. Our second main result is

Theorem B *(a) The structure of $K(m)_*(\Omega^2 SU(n+1))$ for odd primes p is*

$$\bigotimes_{\substack{0<j\leq n \\ p\nmid j}} (E(x_{2jp^k+1} : 0 \leq k < t(m+1)) \otimes T_{tm}(y_{n,jp^k-n-1} : k \geq t))$$

where $T_h(y)$ denotes the truncated polynomial algebra on y of height p^h, and t is the smallest integer such that $p^t j > n$.

(b) The structure of $K(m)_*(\Omega^2 SU(n+1))$ for $p = 2$ is

$$\bigotimes_{\substack{0<j\leq n/2 \\ j \text{ odd}}} (T_{tm}(y_{n,2^k j-n-1} : k \geq t) \otimes E(x_{2^{k+1}j-1} : 0 \leq k < t(m+1)))$$

$$\bigotimes$$

$$\bigotimes_{\substack{n/2<j\leq n \\ j \text{ odd}}} (T_{m+1}(x_{2^{k+1}j-1} : 0 \leq k \leq m) \otimes T_m(y_{n,2^{k+1}j-n-1} : k \geq m+1))$$

where $x^2_{2^{k+1}j-1} = y_{n,2^{k+1}j-1-n}$ for $n/2 < j \leq n$.

This is proved in Section 2 as 2.4 and 2.6.

The proofs of these results requires some knowledge of $BP_*\Omega^2 S^{2n+1}$, which is studied in Section 3. Our main result there (proved as 3.3) is

Theorem C *For each prime p and each integer $n > 0$,*

$$BP_*(\Omega^2 S^{2n+1}) = E(x_{(0)}) \otimes P(y_{(i)} : i > 0)/(\rho_1, \rho_2, \cdots)$$

where $x_{(0)} \in BP_{2n-1}(\Omega^2 S^{2n+1})$, $y_{(i)} \in BP_{2p^i n-2}(\Omega^2 S^{2n+1})$ and

$$\rho_i \equiv \sum_{0\leq j\leq i} v_j y_{(i-j)}^{p^j} \bmod I^2,$$

the v_j are the usual generators of BP_* (with $v_0 = p$) and $I = (p, v_1, v_2, \cdots)$.

We do not have a precise formula for the ρ_i. A conjecture about them is given in 3.4.

We also determine $K(m)_*(\Omega^2 S^{2n+1})$ (3.7, which is originally due to Yamaguchi [Yam88]) and $K(m)^*(\Omega^2 S^{2n+1})$ (3.8).

Our main tool will be a generalization of the Serre spectral sequence due to Dold [Dol62] (see Dyer's book [Dye69] for a detailed proof). Given a fibre sequence

$$F \longrightarrow E \longrightarrow B$$

satisfying suitable hypotheses (e.g. either B is simply connected or both maps are loop maps), and a generalized homology theory h_*, there is a spectral sequence converging to $h_*(E)$ with

$$E_2 = H_*(B; h_*(F)).$$

If h_* is ordinary homology then this is the classical Serre spectral sequence. If F is a point this is the Atiyah–Hirzebruch spectral sequence for $h_*(B)$. If h_* is the Morava K–theory $K(m)_*$ then the E_2–term is simply

$$H_*(B; \mathbf{Z}/(p)) \otimes K(m)_*(F)).$$

Morava K–theory satisfies a Künneth isomorphism, i.e.

$$K(m)_*(X) \otimes K(m)_*(Y) = K(m)_*(X \times Y).$$

Therefore one hopes for an Eilenberg–Moore spectral sequence similar to the one for ordinary homology with coefficients in a field. In other words there ought to be a spectral sequence converging to $K(m)_*(F)$ with

$$E_2 = \mathrm{Cotor}_R(K(m)_*(E), K(m)_*(pt.)),$$

where $R = K(m)_*(B)$. Unfortunately such a spectral sequence has no chance of converging in general since there are spaces X such that

$$K(m)_*(X) = K(m)_*(\mathrm{pt.})$$

but

$$K(m)_*(\Omega X) \neq K(m)_*(\mathrm{pt.}).$$

This happens when X is the Eilenberg–Mac Lane space $K(G, m+1)$ for a finite abelian p–group G (see [RW80]). Conceivably such a spectral sequence would converge under the additional hypothesis that E and B are $K(m)_*$–local, which would imply that F is also $K(m)_*$–local.

I became interested in this problem in 1985 through conversations with two graduate students, Dan Wagonner and Atsuchi Yamaguchi. The former computed the homology of $\Omega^2 SU(n+1)$ in his thesis [Wag85]. He obtained his results by studying the Serre spectral sequence for the fibration

$$\Omega^2 SU(n) \longrightarrow \Omega^2 SU(n+1) \longrightarrow \Omega^2 S^{2n+1}.$$

Yamaguchi latter extended this calculation to double loop spaces of the complex Stiefel manifolds $SU(n+1)/SU(k)$ in [Yam86] by studying the Eilenberg–Moore spectral sequence on the path fibration for the single loop space. He also computed the Morava K–theory of $\Omega^2 S^{2n+1}$ in [Yam88] by essentially the same method used in this paper.

A draft of this paper was written and circulated in 1985. It was the preprint referred to (under a different title) in [Yam88].

1 The homology of $\Omega^2 SU(n)$

Now we will turn to the specifics. Consider the diagram

$$\begin{array}{ccccc} \mathbf{Z}\times BU=\Omega^3 SU & \longrightarrow & \text{pt.} & \longrightarrow & \Omega^2 SU = U \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ \Omega^3 SU/SU(n+1) & \longrightarrow & \Omega^2 SU(n+1) & \longrightarrow & \Omega^2 SU \end{array} \tag{1.1}$$

where each row is a fibre sequence. The behavior of the Serre spectral sequence for the upper fibration is well known. We will use the Serre spectral sequence of the lower fibration to compute $H_*(\Omega^2 SU(n+1);\mathbf{Z}/(p))$ and $K(m)_*(\Omega^2 SU(n+1))$. The following Lemma will imply that the groups are determined by the map f_* in homology.

Lemma 1.2 *Let h_* be a generalized multiplicative homology theory with a complex orientation. Then*

$$h_*(\Omega^3 SU/SU(n+1)) = P(y_{n,i} : i \geq 0)$$

where $|y_{n,i}| = 2(n+i)$.

Proof. We will give the argument in terms of ordinary homology theory H_* and use the known structure of $H_*(SU(n))$ and $H_*(BU)$. The analogous facts for an arbitrary theory h_* with complex orientation are proved in [Ada74].

We know

$$H_*(SU/SU(n+1)) = E(x_{2n+3}, x_{2n+5}, \cdots)$$

where $|x_i| = i$. It follows (using the Serre spectral sequence) that

$$H_*(\Omega SU/SU(n+1)) = P(b_{n+1}, b_{n+2}, \cdots)$$

where $b_i \in H_{2i}(BU) = H_{2i}(\Omega SU)$ is the standard polynomial generator.

This polynomial algebra is *not* primitively generated. On the contrary its coproduct is such that $H^*(\Omega SU/SU(n+1))$ is polynomial. This follows from the fact that $H_*(BU)$ maps surjectively onto $H_*(\Omega SU/SU(n+1))$, so $H^*(\Omega SU/SU(n+1))$ maps injectively into $H^*(BU)$. The map is one of Hopf algebras and any sub–Hopf algebra of a polynomial Hopf algebra is also polynomial. Since $H^*(\Omega SU/SU(n+1))$ is polynomial on even dimensional generators, it follows that both the homology and cohomology of $\Omega^2 SU/SU(n+1)$ are exterior algebras on odd dimensional generators. The latter implies the statement of the Lemma. ■

Now go back to the Serre spectral sequence for the diagram (1.1). For the upper fibration we have

$$H_*(U) = E(x_1, x_3, x_5, \cdots)$$

and $\tau(x_{2i+1}) = b_i \in H_{2i}(BU \times \mathbf{Z})$ where τ is the transgression. Therefore by naturality we have $\tau(x_{2i+1}) = f_*(b_i)$ in the Serre spectral sequence for the lower fibration. Thus we need to study the map f_*. This is best done in stages. Consider the fibration

$$\Omega^3 SU/SU(n) \xrightarrow{f_n} \Omega^3 SU/SU(n+1) \longrightarrow \Omega^2 S^{2n+1} \tag{1.3}$$

The map f of (1.1) is the composite $f_n f_{n-1} \cdots f_1$. We have knowledge of $BP_*(\Omega^2 S^{2n+1})$ (as will be explained in Section 3) that can be used to get enough information about f for our purposes. The result we obtain is the following.

Lemma 1.4 *Let*

$$a_{(i)} \in BP_{2(np^i-1)}(\Omega^3 SU/SU(n))$$

and

$$c_{(i)} \in BP_{2(np^i-1)}(\Omega^3 SU/SU(n+1))$$

be polynomial generators. They can be chosen such that

$$f_{n*}(a_{(i)}) \equiv \sum_{j\geq 0} v_j c_{(i-j)}^{p^j} \bmod I^2$$

where $I = (p, v_1, v_2, \cdots)$, $v_0 = p$, *and* v_i *is the* i^{th} *polynomial generator of* $\pi_*(BP)$.

Moreover the polynomial generators in other dimensions can be chosen so that $f_{n*}(y_{n-1,0}) = 0$ *and* $f_{n*}(y_{n-1,k}) = (y_{n,k-1})$ *for* $k > 0$.

When we specialize this to $K(m)_*$ for $m > 0$, we get

$$f_{n*}(a_{(i+m)}) = v_m c_{(i)}^{p^m} \bmod I^2.$$

If we knew that the error term here was a $(p^m)^{\text{th}}$, we could easily deduce the following.

Lemma 1.5 *Let* $a_{(i)}$ *and* $c_{(i)}$ *denote the corresponding polynomial generators of* $H_*(X; Z_{(p)})$ *and* $K(m)_*(X)$ *for* X *as above. In the case of* H_*, $f_{n*}(a_{(i)}) = pc_{(i)}$. *In the case of* $K(m)_*$, $f_{n*}(a_{(i+m)}) = v_m c_{(i)}^{p^m}$.

Proof. The indeterminacy in 1.4 forces us to go further afield for a proof. We can deduce this description of $K(m)_*(f_n)$ from Yamaguchi's computation of $K(m)_*(\Omega^2 S^{2n+1})$, given below as Theorem 3.7. If $f_{n*}(a_{(i+m)})$ was not a $(p^m)^{\text{th}}$ power, then $K(m)_*(\Omega^2 S^{2n+1})$ would have a different ring structure. ∎

With this information in hand it is possible to read off $H_*(f)$ and $K(m)_*(f)$ and to compute these theories on $\Omega^2 SU(n)$.

Example 1.6 *We will do the computation at* $p = 2$ *for* $\Omega^2 SU(4)$. *The map* f *is the composite* $f_3 f_2 f_1$. *The following table shows the affect of these three maps in integer homology as given by Corollary 1.5 in dimensions up to 22.*

i	$f_{1*}(b_i)$	$f_{2*}(y_{1,i-1})$	$f_{3*}(y_{2,i-2})$
1	$2y_{1,0}$	0	
2	$y_{1,1}$	$y_{2,0}$	0
3	$2y_{1,2}$	$2y_{2,1}$	$y_{3,0}$
4	$2y_{1,3}$	$2y_{2,2}$	$y_{3,1}$
5	$y_{1,4}$	$y_{2,3}$	$2y_{3,2}$
6	$y_{1,5}$	$y_{2,4}$	$y_{3,3}$
7	$2y_{1,6}$	$2y_{2,5}$	$y_{3,4}$
8	$y_{1,7}$	$y_{2,6}$	$y_{3,5}$
9	$y_{1,8}$	$y_{2,7}$	$y_{3,6}$
10	$y_{1,9}$	$y_{2,8}$	$y_{3,7}$
11	$y_{1,10}$	$y_{2,9}$	$2y_{3,8}$

In other words the behavior of f_* in $H_*(;Z_{(2)})$ is as follows.

$$f_*(b_i) = \begin{cases} 0 & \textit{if } i = 1 \textit{ or } 2; \\ 2y_{3,i-3} & \textit{if } i = 3{\cdot}2^k - 1 \textit{ for } k \geq 1; \\ 4y_{3,i-3} & \textit{if } i = 2^k - 1 \textit{ for } k \geq 2 \qquad \textit{and} \\ y_{3,i-3} & \textit{otherwise.} \end{cases} \tag{1.7}$$

Thus in the mod 2 Serre spectral sequence for the fibration

$$\Omega^3 SU/SU(4) \longrightarrow \Omega^2 SU(4) \longrightarrow \Omega^2 SU = U$$

we have

$$\tau(x_{2i+1}) = \begin{cases} 0 & \textit{if } i = 3\cdot 2^k - 1 \textit{ or } 2^{k+1} - 1 \textit{ for } k \geq 0 \qquad \textit{and} \\ y_{3,i-3} & \textit{otherwise.} \end{cases} \tag{1.8}$$

It follows that the Serre E_∞–term has the form

$$E(x_1) \otimes E(x_{2^{k+1}-1},\, x_{3\cdot 2^k - 1} : k \geq 1) \otimes P(y_{3,2^{k+1}-4},\, y_{3,3\cdot 2^k - 4} : k \geq 1) \tag{1.9}$$

There are some nontrivial multiplicative extensions, namely $x^2_{3\cdot 2^k - 1} = y_{3,3\cdot 2^k - 4}$ for $k \geq 1$. This is a special case of Theorem 1.13 below.

It follows that

$$H_*(\Omega^2 SU(4); Z/(2)) = E(x_{2^{k+1}-1} : k \geq 0) \otimes P(y_{3,2^{k+2}-4},\, x_{3\cdot 2^{k+1}-1} : k \geq 0) \tag{1.10}$$

where

$$x^2_{3\cdot 2^{k+1}-1} = y_{3,3\cdot 2^{k+1}-4}.$$

Now we want to generalize this discussion to $\Omega^2 SU(n+1)$. The following result is straightforward.

Theorem 1.11 *(a) For a given prime p the polynomial generators $y_{n,i-n} \in H_{2i}(\Omega^3 SU/SU(n+1))$ can be chosen so that*

$$f_*(b_i) = \begin{cases} 0 & \text{if } i < n; \\ p^{\varphi(n,i)} y_{n,i-n} & \text{otherwise.} \end{cases}$$

where $\varphi(n,i)$ is the number of ways to write $i = jp^k - 1$ for $j \leq n$ and $k \geq 0$. (Notice than $\varphi(n,i)$ is zero for most values of i and never more than $1 + \log_p(n)$. It is also bounded above by the p–adic valuation of $1+i$.)

(b) The transgression in mod p homology for the fibration

$$\Omega^3 SU/SU(n+1) \longrightarrow \Omega^2 SU(n+1) \longrightarrow \Omega^2 SU = U$$

is given by

$$\tau(x_{2i+1}) = \begin{cases} 0 & \text{if } i = p^k j - 1 \text{ with} \\ & \qquad 0 < j \leq n, \quad k \geq 0 \text{ and } p \not| j \\ y_{n,i-n} & \text{otherwise.} \end{cases}$$

(c) The Serre E_∞–term for the fibration is

$$E(x_{2p^k j-1} : 0 < j \leq n, k \geq 0) \otimes P(y_{n,p^k j-n} : 0 < j \leq n, p^k j \geq n).$$

where p does not divide j.

For odd primes there are no multiplicative extensions in the spectral sequence, so we have

Corollary 1.12 *With notation as in 1.11, for each odd prime p, the structure of $H_*(\Omega^2 SU(n+1); \mathbf{Z}/(p))$ is*

$$E(x_{2p^k j-1} : 0 < j \leq n,\ k \geq 0,\ p \not| j) \otimes P(y_{n,p^k j-n} : 0 < j \leq n,\ p^k j \geq n,\ p \not| j).$$

where $x_{2i-1} \in H_{2i-1}$ and $y_{n,i-n} \in H_{2i}$.

This is the first part of Theorem A.

Before we can describe the mod 2 case, we must determine the multiplicative extensions.

Theorem 1.13 *For $p = 2$, let $x_{2i+1} \in H_*(\Omega^2 SU(n+1))$ be one of the odd–dimensional generators given by 1.11(c) above. Then $x_{2i+1}^2 = y_{n,2i+1-n}$ if $i = 2^k j - 1$ with $k \geq 0$, j odd and $j > n/2$; $x_{2i+1}^2 = 0$ for all other values of i.*

Proof. Recall that in $H_*(\Omega^2 S^{2\ell+1})$, all of the odd–dimensional generators have nontrivial squares; these generators occur in dimensions $2^{k+1}\ell - 1$ for $k \geq 0$. Now consider the map $\Omega^2 SU(\ell+1) \to \Omega^2 S^{2\ell+1}$.

If ℓ is odd we find (by induction on ℓ) that $y_{\ell-1,0} \in H_{2\ell-2}(\Omega^2 SU(\ell))$ is trivial. Consider the Serre spectral sequence for the fibration

$$\Omega^2 SU(\ell) \longrightarrow \Omega^2 SU(\ell+1) \longrightarrow \Omega^2 S^{2\ell+1}.$$

The fundamental class $u_{2\ell-1} \in H_{2\ell-1}(\Omega^2 S^{2\ell+1})$ survives to E_∞ because there is no primitive element in $H_{2\ell-2}(\Omega^2 SU(\ell))$ for it to transgress to. All of the other generators $u_{2^{k+1}\ell-1}$ of $H_*(\Omega^2 S^{2\ell+1})$ can be expressed in terms of Dyer–Lashof operations acting on $u_{2\ell-1}$. Since the fibration is one of double loop spaces, each of these generators must also survive to E_∞ and the Serre spectral sequence collapses. Thus we have

$$u_{2^{k+1}\ell-1} = x_{2^{k+1}\ell-1}$$

and

$$x^2_{2^{k+1}\ell-1} = y_{\ell,2^{k+1}\ell-\ell-1}$$

for all $k \geq 0$. Furthermore, 1.5 tells us that for $n > \ell$, the image of $y_{\ell,2^{k+1}\ell-\ell-1}$ is $y_{n,2^{k+1}\ell-n-1}$ for $n < 2\ell$ and zero for $n \geq 2\ell$.

On the other hand if ℓ is even, we find that $y_{\ell-1,0}$ is nontrivial and that $u_{2\ell-1}$ transgresses to it. It follows that $u_{2^{k+1}\ell-1}$ transgresses to $y_{\ell-1,2^k\ell-\ell}$. 1.5 tells us that each of these has trivial image in $H_*(\Omega^2 SU(\ell+1))$. Moreover $x_{2\ell-1}$ is present in $H_{2\ell-1}(\Omega^2 SU(\ell/2+1))$ where its square is a multiple (nonzero if and only if $\ell/2$ is odd) of $y_{\ell/2,3\ell/2-1}$. By 1.5 this element has trivial image in $H_*\Omega^2 SU(\ell+1)$.

The result follows. ■

Corollary 1.14 *With notation as in 1.11, $H_*(\Omega^2 SU(n+1); \mathbf{Z}/(2))$ is*

$$\begin{array}{c} E(x_{2^{k+1}\ell-1} : 0 < \ell < n/2,\ k \geq 0,\ \ell \text{ odd}) \quad \otimes \\ P(y_{n,2^{k+1}\ell-n-1} : 0 < \ell < n/2,\ 2^{k+1}\ell \geq n,\ \ell \text{ odd}) \quad \otimes \\ P(x_{2^{k+1}\ell-1} : n/2 \leq \ell \leq n,\ k \geq 0,\ \ell \text{ odd}) \end{array}$$

where for $n/2 \leq \ell \leq n$, $k \geq 0$ and ℓ odd, $x^2_{2^{k+1}\ell-1} = y_{n,2^{k+1}\ell-n-1}$.

This is the second part of Theorem A.

2 The Morava K–theory of $\Omega^2 SU(n)$

Now we turn to Morava K–theory $K(m)_*$. We can use 1.5 to determine the behavior of $K(m)_*(f)$.

Example 2.1 *We will do the computation at $p = 2$ for $\Omega^2 SU(4)$. We will illustrate the behavior of $K(m)_*(f)$ for $m = 1$ with a table similar to that of 1.6. In this case some polynomial generators are sent to squares of same by f_{i*}.*

i	$f_{1*}(b_i)$	$f_{2*}(y_{1,i-1})$	$f_{3*}(y_{2,i-2})$
1	0	0	
2	$y_{1,1}$	$y_{2,0}$	0
3	$v_1 y_{1,0}^2$	0	$y_{3,0}$
4	$y_{1,3}$	$y_{2,2}$	$y_{3,1}$
5	$y_{1,4}$	$y_{2,3}$	0
6	$y_{1,5}$	$y_{2,4}$	$y_{3,3}$
7	$v_1 y_{1,2}^2$	$v_1 y_{2,1}^2$	$y_{3,4}$
8	$y_{1,7}$	$y_{2,6}$	$y_{3,5}$
9	$y_{1,8}$	$y_{2,7}$	$y_{3,6}$
10	$y_{1,9}$	$y_{2,8}$	$y_{3,7}$
11	$y_{1,10}$	$y_{2,9}$	$v_1 y_{3,2}^2$

More generally Corollary 1.5 gives

$$f_*(b_i) = \begin{cases} 0 & \textit{if } i = 2^k - 1 \textit{ and } 0 < k < 2m+2; \\ 0 & \textit{if } i = 3 \cdot 2^k - 1 \textit{ and } 0 < k < m+1; \\ v_m^{2^m+1} y_{3,2^{k-2m}-4}^{2^{2m}} & \textit{if } i = 2^k - 1 \textit{ and } k \geq 2m+2; \\ v_m y_{3,3\cdot 2^{k-m}-4}^{2^m} & \textit{if } i = 3 \cdot 2^k - 1 \textit{ and } k \geq m+1 \textit{ and} \\ y_{3,i-3} & \textit{otherwise.} \end{cases}$$

Then the E_∞–term of the Serre spectral sequence is

$$\begin{gathered} E(x_1, x_3, x_7, \ldots x_{2^{2m+2}-1}) \otimes E(x_5, x_{11}, \ldots x_{3\cdot 2^{m+1}-1}) \otimes \\ T_{2m}(y_{3,2^{k+2}-4} : k \geq 0) \otimes T_m(y_{3,3\cdot 2^{k+1}-4} : k \geq 0) \end{gathered}$$

where $T_i(x)$ denotes the truncated polynomial algebra of height 2^i.

As in the case of ordinary homology, there are some nontrivial multiplicative extensions (that will be explained in Theorem 2.5), namely

$$x_{3\cdot 2^k-1}^2 = y_{3,3\cdot 2^{k+1}-4} \quad \textit{for } 1 \leq k \leq m+1 \qquad \textit{and}$$

so $K(m)_*(\Omega^2 SU(4))$ is

$$\begin{gathered} E(x_1, x_3, \ldots x_{2^{2m+2}-1}) \otimes T_{2m}(y_{3,2^{k+2}-4} : k \geq 0) \otimes \\ T_{m+1}(x_5, x_{11}, \ldots x_{3\cdot 2^{m+1}-1}) \otimes T_m(y_{3,3\cdot 2^{k+m+2}-4} : k \geq 0). \end{gathered}$$

Now we will generalize this to $\Omega^2 SU(n+1)$. To describe the map f_*, first observe that if i is not congruent to -1 mod p, then

$$f_*(b_i) = \begin{cases} 0 & \text{if } i < n \\ y_{n,i-n} & \text{otherwise.} \end{cases}$$

If i is one less than a multiple of p, we can write

$$i = jp^{k+1} - 1 \qquad \text{with } k \geq 0 \text{ and } p \not| j.$$

Let t be the smallest integer such that $n < p^t j$. If $t = 0$, i.e. if $j > n$, then

$$f_*(b_{jp^{k+1}-1}) = y_{n,jp^{k+1}-1-n}.$$

For $t > 0$ we have

$$
\begin{aligned}
f_*(b_{jp^{k+1}-1}) &= v_m y^{p^m}_{j,jp^{k+1-m}-1-j} \\
&= v_m^{1+p^m} y^{p^{2m}}_{pj,jp^{k+1-2m}-1-pj} \\
&\cdots \\
&= v_m^{(p^{tm}-1)/(p^m-1)} y^{p^{tm}}_{p^{t-1}j,jp^{k+1-tm}-1-p^{t-1}j} \\
&= v_m^{(p^{tm}-1)/(p^m-1)} y^{p^{tm}}_{n,jp^{k+1-tm}-1-n}
\end{aligned}
$$

where we are identifying

$$y_{\ell,s} \in K(m)_*(\Omega^3 SU/SU(\ell+1)) \qquad \text{for } \ell < n$$

with its image in $K(m)_*(\Omega^3 SU/SU(n+1))$, and it is understood that $y_{n,s} = 0$ if s is not a nonnegative integer. In $y_{n,jp^{k+1-tm}-1-n}$ the second subscript is a nonnegative integer precisely when $k+1-tm \geq t$. In other words we have

$$
f_*(b_{jp^{k+1}-1}) = \begin{cases} y_{n,jp^{k+1}-1-n} & \text{if } j > n \\ 0 & \text{if } k < tm+t-1 \\ v_m^{(p^{tm}-1)/(p^m-1)} y^{p^{tm}}_{n,jp^{k+1-tm}-1-n} & \text{otherwise} \end{cases}
$$

Combining these gives the following.

Theorem 2.2 *The map*

$$f_* : K(m)_*(\mathbf{Z} \times BU) \longrightarrow K(m)_*(\Omega^3 SU/SU(n+1))$$

behaves as follows. Let

$$i = jp^k - 1 \qquad \text{with } p \not| j$$

and let t be the smallest integer such that $p^t j > n$. (In particular, $t = 0$ if $j > n$.) Then

$$
f_*(b_i) = \begin{cases} 0 & \text{if } 0 \leq k < t(m+1) \\ v_m^{(p^{tm}-1)/(p^m-1)} y^{p^{tm}}_{n,jp^{k-tm}-1-n} & \text{otherwise} \end{cases}
$$

Corollary 2.3 *There are precisely $n(m+1)$ values of i for which $f_*(b_i) = 0$. They are*

$$s - 1 \text{ for } 1 \leq s \leq n$$

and

$$p^{km-m+k}s - 1, p^{km-m+k+1}s - 1, \ldots p^{km+k-1}s - 1 \text{ for } 1 + \frac{n}{p^k} \leq s \leq \frac{n}{p^{k-1}}$$

where $k > 0$. (Here the number s must be an integer, so for $p^{k-1} > n$, there are no values of s satisfying the inequalities.)

Proof. First we will show that the number of values of i is as indicated by induction on n. For $n = 1$ the only value of j with positive t is $j = 1$, which gives $t = 1$ and $m+1$ values of k, namely $0 \le k \le m$. The corresponding values of i are 0, $p-1$, $p^2-1, \dots p^m-1$. This starts the induction.

For the inductive step, suppose first that n is not divisible by p. Then for $j = n$ we have $t = 1$ and we get $m+1$ new values of i, namely $n-1$, $pn-1, \dots p^m n-1$. If n is divisible by p, we can write

$$n = jp^s \qquad \text{with } p \not| j.$$

The value of t associated with this value of j gets increased from s to $s+1$, and the $m+1$ new values of i we get are $np^{sm}-1$, $np^{sm+1}-1, \dots np^{sm+m}-1$.

The indicated values of i fall into various disjoint arithmetic progressions. The first of them ($0 \le i \le n-1$) contains precisely n values. For each positive k we have m arithmetic progressions each containing the same number of elements, which varies with k and is zero for large k.

We will argue by induction on n. The indicated values of i fall into various disjoint arithmetic progressions. The first of them ($0 \le i \le n-1$) contains precisely n values. For each positive k we have m arithmetic progressions each containing the same number of elements, which varies with k and is zero for large k.

We will argue by induction on n. Each time n is increased by one, the number of values of s in the first progression increases by one. The number of values of s in the other progressions remains the same (although the progressions themselves may shift) for all but a single value of k (which depends on the p–adic valuation of n), for which the number in each of the m progressions increases by one.

The following table illustrates this phenomenon in the case $p = 3$, $m = 2$ and $n \le 10$.

	New values	Values of s in progressions			
n	of i	First	$k=1$	$k=2$	$k=3$
1	0, 2, 8	0	1		
2	1, 5, 17	0, 1	1, 2		
3	26, 80, 242	0, 1, 2	2, 3	1	
4	3, 11, 35	$0, \cdots, 3$	2, 3, 4	1	
5	4, 14, 44	$0, \cdots, 4$	$2, \cdots, 5$	1	
6	53, 161, 485	$0, \cdots, 5$	$3, \cdots, 6$	1, 2	
7	6, 20, 62	$0, \cdots, 6$	$3, \cdots, 7$	1, 2	
8	7, 23, 71	$0, \cdots, 7$	$3, \cdots, 8$	1, 2	
9	728, 2186, 6560	$0, \cdots, 8$	$4, \cdots, 9$	2, 3	1
10	9, 29, 89	$0, \cdots, 9$	$4, \cdots, 10$	2, 3	1

Notice that there are always n values of s in the first progression. In the progressions for $k = 1$, one more value of s is added whenever n is not divisible by 3.

When n is divisible by 3, the number of values of s is fixed but the range of values shift by one. In this case the number of values of s increases by one for some higher value of k.

The generalization to other values of m, n and p is straightforward. ■

It follows that in the Serre E_∞–term, there is a factor of $E(x_{2i+1})$ where i ranges over the values described in 2.3. The other factor is a truncated polynomial algebra on the $y_{n,i-n}$ with $i = jp^{k+1} - 1$ for $j \le n$ and j not divisible by p. The height of such a generator depends on the number t, the smallest integer such that $jp^t > n$. The result is

Theorem 2.4 *In the Serre spectral sequence for $K(m)_*$ of the total space of the fibration*

$$\Omega^3 SU/SU(n+1) \longrightarrow \Omega^2 SU(n+1) \longrightarrow \Omega^2 SU,$$

the E_∞–term is

$$\bigotimes_{\substack{0<j\le n \\ p\nmid j}} \left(E(x_{2jp^k+1} : 0 \le k < t(m+1)) \otimes T_m(y_{n,jp^k-n-1} : k \ge t) \right)$$

where $T_h(y)$ denotes the truncated polynomial algebra on y of height p^h, and t is the smallest integer such that $p^j t > n$.

For odd p, $K(m)_*(\Omega^2 SU(n+1))$ has the same description.

This is the first part of Theorem B. For $p = 2$ there are some nontrivial multiplicative extensions in the Serre spectral sequence which we now describe. There are $\left[\frac{n+1}{2}\right]$ infinite families of even–dimensional truncated polynomial generators, one for each positive odd value of $j \le n$. In roughly half of them, the first $m+1$ elements are actually squares of odd–dimensional generators. Roughly a quarter of the $n(m+1)$ odd–dimensional generators have nontrivial squares.

In 2.3 we described the values of i corresponding to generators $x_{2i+1} \in K(m)_*(\Omega^2 SU(n+1))$ in terms of arithmetic progressions. However it is not convenient to describe the ones having nontrivial squares in these terms. We also indicated that each time n is increased by one, we get $m+1$ more generators. The ones with nontrivial squares the ones born on $\Omega^2 SU(\ell+1)$ with ℓ odd and more than $n/2$.

Theorem 2.5 *For $p = 2$, let $x_{2i+1} \in K(m)_*(\Omega^2 SU(n+1))$ be one of the odd–dimensional generators given by 2.3 above. Then $x_{2i+1}^2 = y_{n,2i+1-n}$ if $i = 2^k\ell - 1$ with $1 \le k \le m+1$, ℓ odd, and $n/2 < \ell \le n$; $x_{2i+1}^2 = 0$ for all other values of i.*

Proof. As in the proof of 1.13, our starting point is the structure of $K(m)_*(\Omega^2 S^{2\ell+1})$, which is given in 3.7 below. It is

$$T_{m+1}(u_{2\ell-1}, \cdots u_{2^{m+1}\ell-1}) \otimes T_m(y_{\ell,2^{m+2}\ell-1-\ell}, y_{\ell,2^{m+3}\ell-1-\ell}, \cdots),$$

where $y_{\ell,2^k\ell-1-\ell} = u_{2^k\ell-1}^2$ for $1 \le k \le m+1$.

As in the proof of 1.13, if ℓ is odd we have

$$x_{2^k\ell-1}^2 = y_{\ell,2^k\ell-1-\ell} \qquad \text{for } 1 \le k \le m+1.$$

and $y_{\ell,2^k\ell-1-\ell}$ maps to $y_{n,2^k\ell-1-n} \in K(m)_*(\Omega^2 SU(n+1))$ when $n < 2\ell$, and for larger n it maps to zero.

For even ℓ, $u_{2^i\ell-1}$ has a nontrivial transgression and we deduce that $x_{2^i\ell-1}^2 = 0$. The result follows. ∎

Using these extensions we get

Theorem 2.6 *With notation as above, the structure of $K(m)_*(\Omega^2 SU(n+1))$ for $p=2$ is*

$$\bigotimes_{\substack{0<j\le n/2 \\ j \text{ odd}}} (T_{tm}(y_{n,2^{k+1}j-n-1} : k \ge 0) \otimes E(x_{2^{k+1}j-1} : 0 \le k < t(m+1)))$$
$$\bigotimes$$
$$\bigotimes_{\substack{n/2<j\le n \\ j \text{ odd}}} (T_{m+1}(x_{2^{k+1}j-1} : 0 \le k \le m) \otimes T_{tm}(y_{n,2^{k+1}j-n-1} : k \ge m+1))$$

where $x_{2^{k+1}j-1}^2 = y_{n,2^{k+1}j-1-n}$ for $n/2 < j \le n$ and, as usual, $T_m(y)$ denotes the truncated polynomial algebra on y of height 2^m and t is the smallest integer such that $2^t j > n$.

This is the second part of Theorem B

3 $BP_*(\Omega^2 S^{2n+1})$ and $K(m)_*(\Omega^2 S^{2n+1})$

Now we will study $BP_*(\Omega^2 S^{2n+1})$ and prove Lemma 1.4. Our main tool will be the Adams spectral sequence converging to

$$\pi_*(BP \wedge \Omega^2 S^{2n+1}) = BP_*(\Omega^2 S^{2n+1}).$$

The E_2–term is

$$\mathrm{Ext}_A(\mathbf{Z}/(p),\, H_*(BP \wedge \Omega^2 S^{2n+1})),$$

where A is the dual Steenrod algebra and the homology is with coefficients in $\mathbf{Z}/(p)$. By a well known change-of-rings isomorphism this is the same as

$$\mathrm{Ext}_E(\mathbf{Z}/(p),\, H_*(\Omega^2 S^{2n+1})),$$

where E is the exterior algebra,

$$E = E(\tau_i : i \ge 0).$$

Recall that $|\tau_i| = 2p^i - 1$ and that

$$\mathrm{Ext}_E(\mathbf{Z}/(p),\, \mathbf{Z}/(p)) = P(a_i : i \ge 0),$$

where $\mathrm{bideg}(a_i) = (1, |\tau_i|)$. We will denote this ring below by R.

Thus we need to know the structure of $H_*(\Omega^2 S^{2n+1})$ as a comodule–algebra over E. Recall that for $p > 2$

$$H_*(\Omega^2 S^{2n+1}) = E(x_{(i)} : i \ge 0) \otimes P(y_{(i)} : i > 0),$$

where $|x_{(i)}| = 2np^i - 1$ and $|y_{(i)}| = 2np^i - 2$. For $p = 2$ we have

$$H_*(\Omega^2 S^{2n+1}) = P(x_{(i)} : i \geq 0),$$

and we will denote $x_{(i)}^2$ by $y_{(i+1)}$.

For the rest of this section we will assume that $p > 2$, leaving it to the reader to modify our statements appropriately for $p = 2$.

Lemma 3.1 *The right action of the Milnor primitive Q_j (dual to τ_j) is given by*

$$\begin{aligned} (x_{(i)})Q_j &= \begin{cases} y_{(i-j)}^{p^j} & \text{for } i > j \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \\ (y_{(i)})Q_j &= 0. \end{aligned}$$

Proof. This is an easy exercise with the Nishida relations, given the fact that each $x_{(i)}$ and each $y_{(i)}$ is defined as a certain Dyer–Lashof operation on $x_{(0)}$. Recall [CLJ76] that

$$\begin{aligned} x_{(i)} &= (Q^1)^i(x_{(0)}) \text{ and} \\ y_{(i)} &= \beta(x_{(i)}), \end{aligned}$$

where $(Q^1)^i$ is the i^{th} iterate of the first Dyer–Lashof operation Q^1 and β is the Bockstein.

The Nishida relation tells us that

$$(\beta Q^1(x))\mathcal{P}^1 = Q^0 \beta Q^1(x),$$

where $\mathcal{P}^1$ is the first Steenrod reduced power operation and Q^0 is the p^{th} power map in homology.

Applying this to $x_{(i)}$ for $i > 0$ gives

$$(y_{(i+1)})\mathcal{P}^1 = y_{(i)}^p.$$

With this in mind we can determine the E–module structure of

$$H_*(\Omega^2 S^{2n+1}).$$

Recall that the Milnor primitive Q_{j+1} can be defined as the commutator $[Q_j, \mathcal{P}^{p^j}]$. Then we can prove 3.1 by induction on j as follows.

We start the induction with the fact that $Q_0 = \beta$ and $y_{(i)} = \beta(x_{(i)})$.

Then we have

$$\begin{aligned} (x_{(i)})Q_{j+1} &= (x_{(i)})[Q_j, \mathcal{P}^{p^j}] \\ &= (x_{(i)})(Q_j \mathcal{P}^{p^j} - P^{p^j} Q_j) \\ &= (x_{(i)})Q_j \mathcal{P}^{p^j}. \end{aligned}$$

We know the second term vanishes because there are no primitives in the dimension of $(x_{(i)})\mathcal{P}^{p^j}$. The inductive hypothesis and the Cartan formula give

$$\begin{aligned}(x_{(i)})Q_j\mathcal{P}^{p^j} &= (y_{(i-j)}^{p^j})\mathcal{P}^{p^j} \\ &= y_{(i-j-1)}^{p^{j+1}}. \qquad \blacksquare\end{aligned}$$

To compute the required Ext group it is convenient to define an increasing multiplicative filtration $\{F_i\}$ on our homology ring and to study the resulting spectral sequence. We do this by setting

$$F_0 = P(y_{(i)} : i > 0)$$

and by defining the filtration degree of each $x_{(i)}$ to be one.

In this way we filter away the coaction of E. In other words the associated bigraded object $E^0H_*(\Omega^2S^{2n+1})$ has the trivial E-comodule structure. Thus our E_1–term is

$$H(\Omega^2S^{2n+1})\otimes R.$$

The differential d_1 is determined by the E–comodule structure of

$$H_*(\Omega^2S^{2n+1}),$$

which was given above. The formula one gets is

$$d_1(x_{(i)}) = \sum_{0\le j<i} a_j y_{(i-j)}^{p^j}. \tag{3.2}$$

Theorem 3.3 *(a) Let $r_i = d_1(x_{(i)})$ as given (3.2). Then the ideal $J = (r_1, r_2, \cdots) \subset P(y_{(i)} : i > 0)$ is regular. In other words r_{j+1} is not a zero divisor in $P(y_{(i)} : i > 0)/(r_1, \cdots r_j)$.*

(b) The filtration spectral sequence above collapses from E_2 and gives

$$E_\infty = E(x_{(0)}) \otimes P(y_{(i)} : i > 0)/J.$$

(c) The Adams spectral sequence for $BP_*(\Omega^2S^{2n+1})$ also collapses from E_2. The E_∞–term has the same description as in (b). (There are still extension problems in the BP_*–module structure which will be discussed below.)

Theorem C follows immediately from this result.

Proof. (a) is a reformulation of Lemma 4.15(b) of [RW77].

To show that (a) implies (b) we proceed as follows. Let

$$P_j = P(y_{(i)} : i > 0)\otimes E(x_{(0)}, x_{(1)}, \cdots x_{(j)})\otimes R \quad \text{for } j > 0.$$

Each P_j is a subcomplex of the E_1–term and there are short exact sequences

$$0 \longrightarrow P_j \longrightarrow P_{j+1} \longrightarrow \Sigma^{|x_{(j+1)}|}P_j \longrightarrow 0.$$

We will show by induction on j that

$$H_*(P_j) = P_0/(r_1, \cdots r_j).$$

This statement is trivial for $j = 0$ so we can start the induction. The short exact sequence above gives a long exact sequence of homology groups in which the connecting homomorphism is multiplication by r_{j+1}. By (a), r_{j+1} is not a zero divisor in $z_0/(r_1, \cdots r_j)$. Therefore the connecting homomorphism is monomorphic and the map $H_*(P_j) \to H_*(P_{j+1})$ is onto. This completes the inductive step.

For (c) we must make partial use of the Snaith splitting of the suspension spectrum of $\Omega^2 S^{2n+1}$. There is a corresponding splitting of the Adams spectral sequence that we are studying. We will see that in each Snaith summand the elements are concentrated either in even dimensions or in odd dimensions, so there is no room for any differentials.

More specifically, the Snaith splitting leads to a decomposition of the mod p homology in which $x_{(i)}$ and $y_{(i)}$ have degree p^i. Thus $x_{(0)}$ has degree one and all other generators have degree divisible by p. The stable summands whose degrees are not congruent to zero or one mod p have trivial mod p homology. From (b) we see that in each summand with degree divisible by p the Adams E_2–term is concentrated in even dimensions, while in the summands with degree congruent to one mod p it is concentrated in odd dimensions. ■

For the extensions in the BP_*-module structure we offer

Conjecture 3.4 *With notation as above,*

$$BP_*(\Omega^2 S^{2n+1}) = E(x_{(0)}) \otimes P(y_{(i)} : i > 0)/L,$$

where L is generated by the homogeneous components of the formal group law sum expression

$$\sum\nolimits^F v_j y_{(i-j)}^{p^j},$$

where v_j is the usual polynomial generator of BP_* with $v_0 = p$.

Now we will use Theorem 3.3 to prove Lemma 1.4. Recall the fibration

$$\Omega^3 SU/SU(n) \xrightarrow{f_n} \Omega^3 SU/SU(n+1) \longrightarrow \Omega^2 S^{2n+1}.$$

Let $a_{(i)} \in BP_*(\Omega^3 SU/SU(n))$ and $c_{(i)} \in BP_*(\Omega^3 SU/SU(n+1))$ be polynomial generators in dimension $2(np^i - 1)$ for $i \geq 0$ as before. They can be chosen such that the image of $c_{(i)}$ in $BP_*(\Omega^2 S^{2n+1})$ is $y_{(i)}$. The generator $a_{(0)}$ is the transgression of $x_{(0)}$ and it follows that

$$BP_*(\Omega^2 S^{2n+1}) = E(x_{(0)}) \otimes P(y_{(i)} : i > 0)/L,$$

where L now denotes the ideal generated by the images of the $f(a_{(i)})$ for $i > 0$.

Lemma 1.4 now follows by comparing this description with part (b) of Theorem 3.3.

Now we will consider the Morava K–theory of $\Omega^2 S^{2n+1}$. The Adams spectral sequence approach used above cam be modified to compute the connective Morava K–theory $k(m)_*(\Omega^2 S^{2n+1})$. Here $k(m)$ denotes the connective cover of $K(m)$, i.e.,

$$k(m)_* = \mathbf{Z}/(p)[v_m].$$

In this setting, (3.2) becomes

$$d_1(x_{(i)}) = a_m y_{(i-m)}^{p^m}. \tag{3.5}$$

The Adams E_2–term is a module over $P(a_m)$, and it is convenient to consider the a_m–torsion free quotient, namely

$$E_2/(a_m\text{–torsion}) = E(x_{(0)}, \cdots x_{(m)}) \otimes T_m(y_{(1)},\, y_{(2)}, \cdots). \tag{3.6}$$

Now we want to show that the Adams spectral sequence collapses modulo a_m–torsion from this point, i.e., that

$$a_m^{-1} E_\infty = a_m^{-1} E_2.$$

To do this we must show that for sufficiently large (depending on i) t, $a_m^t x_{(i)}$ for $0 \le i \le m$ and $a_m^t y(i)$ for $i \ge 1$ are permanent cycles.

We will need to make more extensive use of the Snaith splitting of $\Omega^2 S^{2n+1}$. Stably we can write (with everything localized at p)

$$\Omega^2 S_+^{2n+1} \simeq (S^0 \vee S^{2n-1}) \wedge \bigvee_{i \ge 0} \Sigma^{i(2pn-2)} D_{pi}$$

where X_+ denotes the suspension spectrum of the space X with a disjoint base point adjoined, and each D_{pi} is a certain (-1)–connected finite spectrum, which is independent of n. $H_*(D_{pi})$ is spanned by the monomials in $H_*(\Omega^2 S^{2n+1})$ of Snaith degree pi. The generators $x_{(j)}$ and $y_{(j)}$ each have Snaith degree p^j.

The top and bottom classes in $H_*(D_{p^j})$ for $j > 0$ are $x_{(j)}$ and $y_{(1)}^{p^{j-1}}$. The difference between the dimensions of these classes is $2p^{j-1} - 1$. It follows at once that the Adams spectral sequence for $k(m)_*(D_{p^j})$ collapses for $j \le m$, so $x_{(j)}$ and $y_{(j)}$ for $j \le m$ are permanent cycles.

For $j > m$, the top and bottom classes in E_2 for D_{p^j} not killed by a power of a_m are $y_{(j)}$ and $y_{(j-m+1)}^{p^{m-1}}$. Their difference in dimensions is $2p^{m-1} - 2$. It follows that if $y_{(j)}$ supports a differential, the target must be a_m–torsion. There are only finitely many such torsion elements, since D_{p^j} is a finite complex. It follows that for some $t > 0$, $a_m^t y_{(j)}$ is a permanent cycle. This is good enough for our purposes, and we get the following result, which was first proved by Yamaguchi [Yam88].

Theorem 3.7 *For $p > 2$,*

$$K(m)_*(\Omega^2 S^{2n+1}) = E(x_{(0)}, \cdots x_{(m)}) \otimes T_m(y_{(1)},\, y_{(2)}, \cdots),$$

and for p=2,

$$K(m)_*(\Omega^2 S^{2n+1}) = T_{m+1}(x_{(0)}, \cdots x_{(m)}) \otimes T_m(y_{(m+2)},\, y_{(m+3)}, \cdots),$$

where $y_{(i)} = x_{(i-1)}^2$ for $1 \le i \le m+1$.

Now we will examine the coalgebra structure of the Morava K–theory of $\Omega^2 S^{2n+1}$. We will describe the coproduct in $K(m)_*(\Omega^2 S^{2n+1})$ by giving the behavior of the Verschiebung map V, which by definition is dual to the p^{th} power map in $K(m)^*(\Omega^2 S^{2n+1})$.

Note that this map is *not* $K(m)_*$–linear. Suppose u and w in $K(m)^*(X)$ are dual to a and b in $K(m)_*(X)$ and that $u^p = v_m w$. Then we would have $V(v_m^{-1} b) = a$. This means that the coproduct expansion for b would contain the expression

$$v_m \sum_{0<i<p} \frac{1}{p} \binom{p}{i} a^i \otimes a^{p-i}.$$

With this in mind our result is the following.

Theorem 3.8 *In $K(m)_*(\Omega^2 S^{2n+1})$ the generators $x_{(i)}$ and $y_{(i)}$ for $i \leq m$ are primitive.*

For $i > m$,

$$V(v_m^{-1} y_{(i)}) = y_{(i-m)}^{p^{m-1}}.$$

Consequently we have

$$\begin{aligned}
K(1)^*(\Omega^2 S^{2n+1}) &= E(u_{(0)}, u_{(1)}) \otimes P(w_{(1)}), \\
K(2)^*(\Omega^2 S^{2n+1}) &= E(u_{(0)}, u_{(1)}, u_{(2)}) \otimes T_2(w_{(1)}, w_{(2)}, \cdots) \text{ and,} \\
K(m)^*(\Omega^2 S^{2n+1}) &= E(u_{(0)}, \cdots u_{(m)}) \otimes T_2(w_{(1)}, w_{(2)}, \cdots) \\
&\quad \otimes \quad T_1(w_{(i,j)} : i \geq 1 \text{ and } 1 < j < m) \text{ for } m > 2
\end{aligned}$$

where $u_{(0)}$ is dual to $x_{(0)}$, $w_{(i,j)}$ is dual to $y_{(i)}^{p^j}$, $w_{(i)} = w_{(i,0)}$ and $w_{(i)}^p$ is dual to $y_{(i+m)}$.

For $p = 2$ this says that $x_{(m)}$ is primitive while $y_{(m+1)} = x_{(m)}^2$ is not. To resolve this apparent contradiction, remember that mod 2 homology theories such as $K(m)_*$ tend to have noncommutative multiplications (see [AT65] and [AT66] for more details) even though we are dealing with a homotopy commutative H–space. If x is a primitive element in $K(m)_*(X)$ then the coproduct expansion for x^2 is

$$(x \otimes 1 + 1 \otimes x)^2 = x^2 \otimes 1 + 1 \otimes x^2 + (x \otimes 1)(1 \otimes x) + (1 \otimes x)(x \otimes 1).$$

Thus we have to deal with the commutator

$$[x \otimes 1,\, 1 \otimes x] \in K(m)_*(X \times X).$$

The commutativity of $K(m)_*(X)$ does not imply that of $K(m)_*(X \times X)$. In general there is a formula for the commutator in $K(m)^*(Y)$ (see [AT65] and [AT66] for $m = 1$ and [Wur86, 2.4] for $m > 1$) which says

$$[u,\, w] = v_m Q_{m-1}(u) Q_{m-1}(w),$$

where Q_{m-1} is a $K(m)$-cohomology operation in analogous to the Milnor primitive with the same name. Presumably a similar formula holds in the $K(m)$–homology of a homotopy commutative H–space. Thus we have

$$[x_{(m)} \otimes 1,\, 1 \otimes x_{(m)}] = v_m Q_{m-1}(x_{(m-1)}) \otimes Q_{m-1}(x_{(m-1)}) = v_m y_{(1)} \otimes y_{(1)}.$$

One way to understand this noncommutativity is the following. Let

$$X \times X \xrightarrow{t} X \times X$$

be the switching map that sends the point (x, y) to (y, x). The classical proof of the commutativity of the cup product (and of the Pontrjagin product in the case of a homotopy commutative H–space) rests on the fact that in $H^*(X \times X)$, $t^*(a \otimes b) = \pm(b \otimes a)$.

This is deduced from the analogous statement on the cochain level, which is accessible to direct geometric calculation. However, in the case of a generalized cohomology theory, *there are no cochains*, so this argument does not apply.

Since we do not actually require this calculation to prove Theorem 3.8 we need not make it more rigorous.

Now we begin the proof of Theorem 3.8. We will make use of the homology theory $E(m)_*/I_{m-1}$ which we abbreviate by $L(m)_*$. We have

$$L(m)_*(\text{pt.}) = \begin{cases} \mathbf{Z}_{(p)}[v_1] & \text{for } m = 1 \text{ and} \\ \mathbf{Z}/(p)[v_{m-1}, v_m, v_m^{-1}] & \text{for } m > 1. \end{cases}$$

There is a natural transformation $L(m)_* \to K(m)_*$ which sends v_{m-1} to zero.

We will also make use of the fibration

$$\Omega^3 SU/SU(n) \xrightarrow{f_n} \Omega^3 SU/SU(n+1) \longrightarrow \Omega^2 S^{2n+1}.$$

We will study the coalgebra structure in $L(m)_*(f_n)$. In general $L(m)_*(X)$ does not have a coproduct since the theory does not have a Künneth isomorphism. However when $X = \Omega^3 SU/SU(n)$, $L(m)_*(X)$ is a free $L(m)_*$–module by Lemma 1.2, so

$$L(m)_*(X \times X) = L(m)_*(X) \otimes L(m)_*(X)$$

and we have a coproduct.

For the sake of simplicity we begin with the case $n = 1$, i.e. with $\Omega^2 S^3$. Then we have $\Omega^3 SU/SU(n) = \mathbf{Z} \times BU$ and the coproduct structure is well known. $L(m)_*(BU)$ is a bipolynomial Hopf algebra, i.e.

$$L(m)_*(BU) = P(b_i : i > 0)$$

with $|b_i| = 2i$ and $V(b_{pi}) = b_i$.

Let

$$y_{2,i-2} \in L(m)_*(\Omega^3 SU/SU(2))$$

be an appropriate polynomial generator of in dimension $2i$, and let $a_{(j)}$ and $c_{(j)}$ as usual denote generators of $L(m)_*(BU)$ and $L(m)_*(\Omega^3 SU/SU(2))$ in dimension $2p^j - 1$. Then from Lemma 1.2 we have

$$f_{1*}(b_i) = \begin{cases} 0 & \text{if } i = 1 \\ y_{2,i-2} & \text{if } i \neq p^j - 1 \\ v_{m-1} c_{(j+1-m)}^{p^{m-1}} + v_m c_{(j-m)}^{p^m} & \text{if} i = p^j - 1. \end{cases}$$

Here it is to be understood that $c_{(j)} = 0$ for $j \leq 0$.

Now we have to argue by induction on m. For $m=1$ we have

$$\begin{aligned} f_{1*}(a_{(1)}) &= pc_{(1)} \text{ and} \\ f_{1*}(a_{(2)}) &= pc_{(2)} + v_1 c_{(1)}^p. \end{aligned}$$

The generators $a_{(1)}$ and $a_{(2)}$ are both primitive, so their images under f_{1*} are also. The primitivity of $pc_{(2)} + v_1 c_{(1)}^p$ implies that

$$V(v_1^{-1} c_{(2)}) = -c_{(1)}.$$

We can get rid of the minus sign by changing the definition of $c_{(j)}$. In a similar fashion we get

$$V(v_1^{-1} c_{(j)}) = c_{(j-1)} \text{ for } j > 2.$$

It follows immediately that

$$V(v_1^{-1} c_{(j)}) = c_{(j-1)} \text{ for } j > 1$$

in $K(1)_*(\Omega^2 S^{2n+1})$.

This proves Theorem 3.8 in the case $m = n = 1$ and starts our induction on m for the case $n = 1$. Now suppose inductively for $n = 1$ that

$$V(v_{m-1}^{-1} c_{(j)}) = c_{(j-m+1)}^{p^{m-2}} \text{ for } j \geq m$$

in $K(m-1)_*(\Omega^2 S^3)$.

Then the primitivity of

$$f_{1*}(a_{(j)}) = v_{m-1} c_{(j+1-m)}^{p^{m-1}} + v_m c_{(j-j)}^{p^m}$$

implies that

$$V(v_m^{-1} c_{(j)}) = -c_{(j-m)}^{p^{m-1}} \text{ for } j > m$$

in $K(m)_*(\Omega^2 S^3)$ as desired. This proves Theorem 3.8 for all values of m when $n = 1$.

This argument generalizes immediately to other values of n provided that the $a_{(j)}$ are primitive, but this is not always the case. They are primitive in the cases when they are the images of the generators in the same dimensions in $L(m)_*(BU)$. This will happens whenever the number $np^j - 1$ does not have the form $kp^j - 1$ for some $k < n$. It is elementary that this condition is equivalent to n not being divisible by p. Thus we have proved Theorem 3.8 in these cases.

If n is divisible by p then we can assume inductively that

$$V(a_{(j)}) = v_m a_{(j-m)}^{p^{m-1}}$$

in $K(m)_*(\Omega^3 SU/SU(n))$. In this case we can prove the Theorem 3.8 without using $L(m)$–theory. Lemma 1.4 still gives

$$f_{n*}(a_{(j)}) = v_m c_{(j-m)}^{p^m}.$$

The Verschiebung is natural so we get

$$V(c_{(j)}^{p^m}) = v_m^{p^m} c_{(j-m)}^{p^{2m-1}}.$$

Since $K(m)_*(\Omega^3 SU/SU(n+1))$ is a polynomial algebra by Lemma 1.2, this implies

$$V(c_{(j)}) = v_m c_{(j-m)}^{p^{m-1}}.$$

This completes the proof of Theorem 3.8.

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