The Periodicity Theorem in the solution to the Arf-Kervaire invariant problem

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1 Review of our strategy

Review of our strategy

Our goal is to prove

Main Theorem. *The Arf-Kervaire elements* $\theta_j \in \pi_{2^{j+1}-2}(S^0)$ *do not exist for* $j \geq 7$.

Our strategy is to find a map $S^0 \to \Omega$ to a nonconnective spectrum Ω with the following properties.

(i) It has an Adams-Novikov spectral sequence in which the image of each θ_j is nontrivial. This is the Detection Theorem discussed by Hopkins yesterday.
(ii) $\pi_{-2}(\Omega) = 0$. This is the Gap Theorem discussed by Hill earlier today.
(iii) It is 256-periodic, meaning $\Sigma^{256}\Omega \cong \Omega$. This is the Periodicity Theorem.

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Our strategy (continued)

(ii) and (iii) imply that $\pi_{254}(\Omega) = 0$.

If θ_7 exists, (i) implies it has a nontrivial image in this group, so it cannot exist.

The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$. 1.3

2 The spectrum Ω

The spectrum Ω

As explained previously, there is an action of the cyclic group C_8 on the 4-fold smash product $MU^{(4)}$. It is derived using a norm induction from the action of C_2 on MU by complex conjugation.

We will construct a C_8-spectrum $\tilde{\Omega}$ by inverting a certain element $D \in \pi_\star(MU^{(4)})$, the $RO(C_8)$-graded homotopy of $MU^{(4)}$. We have a theorem (not to be treated in this talk) equating its homotopy fixed point $\tilde{\Omega}^{hC_8}$ with its actual fixed point set $\tilde{\Omega}^{C_8}$, which we denote by Ω. We will see that $\tilde{\Omega}^{C_8}$ has the gap property while $\tilde{\Omega}^{hC_8}$ has the periodicity and detection properties. 1.4

The spectrum Ω (continued)

The homotopy of $(MU^{(4)})^{hC_8}$ can be computed using the homotopy fixed point spectral sequence, for which

$$E_2 = H^*(C_8; \pi_*(MU^{(4)})).$$

In this case it coincides with the Adams-Novikov spectral sequence for $\pi_*((MU^{(4)})^{hC_8})$. Algebraic methods available since the 1990s can be used to show that it detects the θ_js. D has to be chosen so that this is still true after we invert it.

The homotopy of $(MU^{(4)})^{C_8}$ and $\Omega = D^{-1}(MU^{(4)})^{C_8}$ can be also computed using the slice spectral sequence described by Hill. It has the convenient property that π_{-2} vanishes in the E_2-term. In fact π_k vanishes for $-4 < k < 0$.

This is our main motivation for developing the slice spectral sequence. We do not know how to show this vanishing using the other spectral sequence.

In order to identify D we need to study the slice spectral sequence in more detail. 1.5

3 The slice spectral sequence

The slice spectral sequence

Recall that for $G = C_8$ we have a slice tower

$$\begin{array}{ccccccccc} \cdots \to & P_G^{n+1}MU^{(4)} & \to & P_G^{n}MU^{(4)} & \to & P_G^{n-1}MU^{(4)} & \to \cdots \\ & \uparrow & & \uparrow & & \uparrow & \\ & {}^GP_{n+1}^{n+1}MU^{(4)} & & {}^GP_n^nMU^{(4)} & & {}^GP_{n-1}^{n-1}MU^{(4)} & \end{array}$$

in which

- the inverse limit is $MU^{(4)}$,
- the direct limit is contractible and
- ${}^GP_n^nMU^{(4)}$ is the fiber of the map $P_G^nMU^{(4)} \to P_G^{n-1}MU^{(4)}$.

${}^GP_n^nMU^{(4)}$ is the nth slice and the decreasing sequence of subgroups of $\pi_*(MU^{(4)})$ is the slice filtration. We also get slice filtrations of the $RO(G)$-graded homotopy $\pi_\star(MU^{(4)})$ and the homotopy groups of fixed point sets $\pi_*((MU^{(4)})^H)$ for each subgroup H. 1.6

The slice spectral sequence (continued)

This means the slice filtration leads to a slice spectral sequence converging to $\pi_*(MU^{(4)})$ and its variants.

One variant has the form

$$E_2^{s,t} = \pi_{t-s}^G({}^GP_t^tMU^{(4)}) \implies \pi_{t-s}^G(MU^{(4)}).$$

Recall that $\pi_*^G(MU^{(4)})$ is by definition $\pi_*((MU^{(4)})^G)$, the homotopy of the fixed point set. 1.7

The slice spectral sequence (continued)

Slice Theorem . *In the slice tower for $MU^{(4)}$, every odd slice is contractible and $P_{2n}^{2n} = \hat{W}_n \wedge H\mathbf{Z}$, where $H\mathbf{Z}$ is the integer Eilenberg-Mac Lane spectrum and $\hat{W}_n$ is a certain wedge of the following three types of finite G-spectra:*

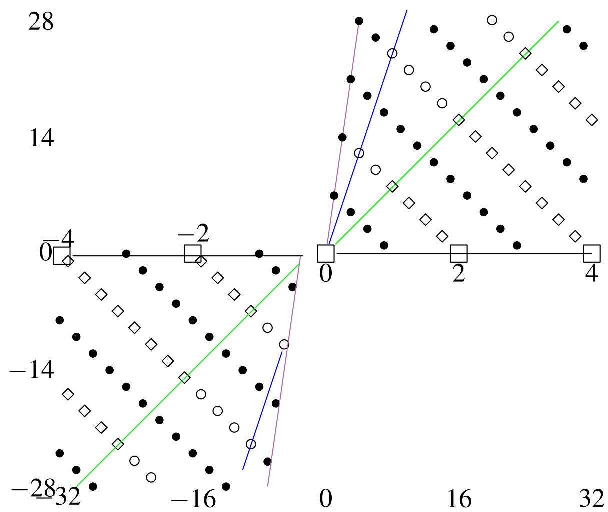
- *$S^{(n/4)\rho_8}$ (when n is divisible by 4), where ρ_g denotes the regular real representation of C_g,*
- *$C_8 \wedge_{C_4} S^{(n/2)\rho_4}$ (when n is divisible by 2) and*
- *$C_8 \wedge_{C_2} S^{n\rho_2}$.*

The same holds after we invert D, in which case negative values of n can occur. 1.8

3.1 Slices of the form $S^{m\rho_8} \wedge H\mathbf{Z}$

Slices of the form $S^{m\rho_8} \wedge H\mathbf{Z}$

Here is a picture of some slices $S^{m\rho_8} \wedge H\mathbf{Z}$.



1.9

Slices of the form $S^{m\rho_8} \wedge H\mathbf{Z}$ (continued)

- Note that all elements are in the first and third quadrants between certain black lines with slopes 0 and orchid lines with slope 7, and are concentrated on diagonals where t is divisible by 8.
- Bullets, circles and diamonds indicate cyclic groups of order 2, 4 and 8, and boxes indicate copies of the integers.
- A similar picture for $S^{m\rho_4} \wedge H\mathbf{Z}$ would be confined to the regions between the black lines and blue lines with slope 3 and concentrated on diagonals where t is divisible by 4.
- A similar picture for $S^{m\rho_2} \wedge H\mathbf{Z}$ would be confined to the regions between the black lines and green lines with slope 1 and concentrated on diagonals where t is divisible by 2.

1.10

3.2 Implications for the slice spectral sequence

Implications for the slice spectral sequence

These calculations imply the following.

- The slice spectral sequence for $MU^{(4)}$ is concentrated in the first quadrant and confined by the same vanishing lines.
- Later we will invert elements in $\pi_{m\rho_8}(MU^{(4)})$. The fact that

$$S^{-\rho_8} \wedge (C_8 \wedge_H S^{m\rho_h}) = C_8 \wedge_H S^{(m-8/h)\rho_h}$$

  means that the resulting slice spectral sequence is confined to the regions of the first and third quadrants shown in the picture.

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4 Geometric fixed points

Geometric fixed points

In order to proceed further, we need another concept from equivariant stable homotopy theory.

Unstably a G-space X has a fixed point set,

$$X^G = \{x \in X : \gamma(x) = x \,\forall\, \gamma \in G\}.$$

This is the same as $F(S^0, X_+)^G$, the space of based equivariant maps $S^0 \to X_+$, which is the same as the space of unbased equivariant maps $* \to X$.

The homotopy fixed point set X^{hG} is the space of based equivariant maps $EG_+ \to X_+$, where EG is a contractible free G-space. The equivariant homotopy type of X^{hG} is independent of the choice of EG.

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Geometric fixed points (continued)

Both of these definitions have stable analogs, but the fixed point functor is awkward for two reasons:

- it fails to commute with smash products and
- it fails to commute with infinite suspensions.

The geometric fixed set $\Phi^G X$ is a convenient substitute that avoids these difficulties. In order to define it we need the isotropy separation sequence, which in the case of a finite cyclic 2-group G is the cofiber sequence

$$EC_{2+} \to S^0 \to \tilde{E}C_2.$$

Here EC_2 is a G-space via the projection $G \to C_2$ and S^0 has the trivial action, so $\tilde{E}C_2$ is also a G-space.

1.13

Geometric fixed points (continued)

$$EC_{2+} \to S^0 \to \tilde{E}C_2.$$

Under this action EC_2^G is empty while for any proper subgroup H of G, $EC_2^H = EC_2$, which is contractible. For an arbitrary finite group G it is possible to construct a G-space with the similar properties.

Definition. *For a finite cyclic 2-group G and G-spectrum X, the geometric fixed point spectrum is*

$$\Phi^G X = (X \wedge \tilde{E}C_2)^G.$$

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Geometric fixed points (continued)

$$\Phi^G X = (X \wedge \tilde{E}C_2)^G.$$

This functor has the following properties:

- For G-spectra X and Y, $\Phi^G(X \wedge Y) = \Phi^G X \wedge \Phi^G Y$.
- For a G-space X, $\Phi^G \Sigma^\infty X = \Sigma^\infty(X^G)$.
- A map $f : X \to Y$ is a G-equivalence iff $\Phi^H f$ is an ordinary equivalence for each subgroup $H \subset G$.

From the suspension property we can deduce that

$$\Phi^{C_8} MU^{(4)} = MO,$$

the unoriented cobordism spectrum.

Geometric Fixed Point Theorem. *Let σ denote the sign representation. Then for any G-spectrum X, $\pi_\star(\tilde{E}C_2 \wedge X) = a_\sigma^{-1}\pi_\star(X)$, where $a_\sigma : S^0 \to S^\sigma$ is the inclusion of the fixed point set.*

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Geometric fixed points (continued)

Recall that $\pi_*(MO) = \mathbf{Z}/2[y_i : i > 0, i \neq 2^k - 1]$ where $|y_i| = i$. It is not hard to show that

$$\pi_*(MU^{(4)}) = \mathbf{Z}[r_i, \gamma(r_i), \gamma^2(r_i), \gamma^3(r_i) : i > 0]$$

where $|r_i| = 2i$, γ is a generator of G and $\gamma^4(r_i) = (-1)^i r_i$. In $\pi_{i\rho_8}(MU^{(4)})$ we have the element

$$Nr_i = r_i \gamma(r_i) \gamma^2(r_i) \gamma^3(r_i).$$

Applying the functor Φ^G to the map $Nr_i : S^{i\rho_8} \to MU^{(4)}$ gives a map $S^i \to MO$.

Lemma. *The generators r_i and y_i can be chosen so that*

$$\Phi^G Nr_i = \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise.} \end{cases}$$

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5 Some slice differentials

Some slice differentials

We know that the slice spectral sequence for $MU^{(4)}$ has a vanishing line of slope 7. We will describe the subring of elements lying on it.

Let $f_i \in \pi_i(MU^{(4)})$ be the composite

$$S^i \xrightarrow{a_{i\rho_8}} S^{i\rho_8} \xrightarrow{Nr_i} MU^{(4)},$$

where $a_{i\rho_8}$ is the inclusion of the fixed point set. The following facts about f_i are easy to prove.

- It appears in the slice spectral sequence in $E_2^{7i,8i}$, which is on the vanishing line.
- The subring of elements on the vanishing line is the polynomial algebra on the f_i.

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Some slice differentials (continued)

- Under the map
$$\pi_*(MU^{(4)}) \to \pi_*(\Phi^G MU^{(4)}) = \pi_*(MO)$$
we have
$$f_i \mapsto \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise} \end{cases}$$
- Any differential landing on the vanishing line must have a target in the ideal $(f_1, f_3, f_7, \dots)$. A similar statement can be made after smashing with $S^{2^k\sigma}$.

1.18

Some slice differentials (continued)

Recall that for an oriented representation V there is a map $u_V : S^{|V|} \to \Sigma^V H\mathbf{Z}$, which lies in $\pi_{V-|V|}(H\mathbf{Z})$. It satisfies $u_{2V} = u_V^2$, so $u_{2^k\sigma} = u_{2\sigma}^{2^{k-1}}$.

Slice Differentials Theorem. *In the slice spectral sequence for $\Sigma^{2^k\sigma} MU^{(4)}$ for $k > 0$, we have $d_r(u_{2^k\sigma}) = 0$ for $r < 1 + 8(2^k - 1)$, and*

$$d_{1+8(2^k-1)}(u_{2^k\sigma}) = a_\sigma^{2^k} f_{2^k-1}.$$

A similar statement holds for the G-spectrum $MU^{(g/2)}$ for a cyclic 2-group G of order g.

Sketch of proof: Inverting a_σ in the slice spectral sequence will make it converge to $\pi_*(MO)$. This means each power of $u_{2\sigma}$ has to support a nontrivial differential. The only way this can happen is as indicated in the theorem.

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6 Some $RO(G)$-graded calculations

Some $RO(G)$-graded calculations

For a cyclic 2-group G let

$$\begin{aligned}\overline{\Delta}_k^{(g)} = N_2^g r_{2^k-1} &= r_{2^k-1}\gamma(r_{2^k-1})\dots\gamma^{g/2-1}(r_{2^k-1}) \\ &\in \pi_{(2^k-1)\rho_g}(MU^{(g/2)})\end{aligned}$$

We want to invert this element and study the resulting slice spectral sequence. As explained previously, for $G = C_8$ it is confined to the first and third quadrants with vanishing lines of slopes 0 and 7.

The differential d_r on $u_{2\sigma}^{2^k}$ described in the theorem is the last one possible since its target, $a_\sigma^{2^{k+1}} f_{2^{k+1}-1}$, lies on the vanishing line. If we can show that this target is killed by an earlier differential after inverting $\overline{\Delta}_k^{(g)}$, then $u_{2\sigma}^{2^k}$ will be a permanent cycle. 1.20

Some $RO(G)$-graded calculations (continued)

We have

$$\begin{aligned}f_{2^{k+1}-1}\overline{\Delta}_k^{(g)} &= (a_{\rho_g}^{2^{k+1}-1} N r_{2^{k+1}-1})(N r_{2^k-1}) \\ &= a_{\rho_g}^{2^k} N r_{2^{k+1}-1}(a_{\rho_g}^{2^k-1} N r_{2^k-1}) \\ &= a_{\rho_g}^{2^k}\overline{\Delta}_{k+1}^{(g)} f_{2^k-1} \\ &= a_V^{2^k}\overline{\Delta}_{k+1}^{(g)} a_\sigma^{2^k} f_{2^k-1} \qquad \text{where } V = \rho_g - \sigma \\ &= a_V^{2^k} p\overline{\Delta}_{k+1}^{(g)} d_{1+8(2^k-1)}(u_{2^k\sigma}).\end{aligned}$$

Corollary. *In the $RO(G)$-graded slice spectral sequence for $\left(\overline{\Delta}_k^{(g)}\right)^{-1} MU^{(g/2)}$, the class $u_{2^{k+1}\sigma} = u_{2\sigma}^{2^k}$ is a permanent cycle.*

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7 An even trickier $RO(G)$-graded calculation

An even trickier $RO(G)$-graded calculation

The corollary shows that inverting a certain element makes a power of $u_{2\sigma}$ a permanent cycle. We need to invert something to make a power of $u_{2\rho_8}$ a permanent cycle.

We will get this by using the norm property of u. It says that if V is an oriented representation of a subgroup $H \subset G$ with $V^H = 0$ and V' is the induced representation of V, then the norm functor N_h^g from H-spectra to G-spectra satisfies $N_h^g(u_V)u_{V''} = u_{V'}$, where V'' is the induced representation of the trivial representation of degree $|V|$.

From this we can deduce that $u_{2\rho_8} = u_{8\sigma_8} N_4^8(u_{4\sigma_4}) N_2^8(u_{2\sigma_2})$, where σ_g denotes the sign representation on C_g. 1.22

An even trickier $RO(G)$-graded calculation (continued)

We have $u_{2\rho_8} = u_{8\sigma_8} N_4^8(u_{4\sigma_4}) N_2^8(u_{2\sigma_2})$.

By the Corollary we can make a power of each factor a permanent cycle by inverting some $\overline{\Delta}_{k_m}^{(2^m)}$ for $1 \le m \le 3$. If we make k_m too small we will lose the detection property, that is we will get a spectrum that does not detect the θ_j. It turns out that k_m must be chosen so that $8|2^m k_m$.

- Inverting $\overline{\Delta}_4^{(2)}$ makes $u_{32\sigma_2}$ a permanent cycle.

- Inverting $\overline{\Delta}_2^{(4)}$ makes $u_{8\sigma_4}$ a permanent cycle.
- Inverting $\overline{\Delta}_1^{(8)}$ makes $u_{4\sigma_8}$ a permanent cycle.
- Inverting the product D of the norms of all three makes $u_{32\rho_8} = u_{2\rho_8}^{16}$ a permanent cycle.

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An even trickier $RO(G)$-graded calculation (continued)

Let
$$D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)}) \in \pi_{19\rho_8}(MU^{(4)}).$$
The we define $\tilde{\Omega} = D^{-1}MU^{(4)}$ and $\Omega = \tilde{\Omega}^{C_8}$.

Since the inverted element is represented by a map from $S^{m\rho_8}$, the slice spectral sequence for $\pi_*(\Omega) = \pi_*^{C_8}(\tilde{\Omega})$ has the usual properties:

- It is concentrated in the first and third quadrants and confined by vanishing lines of slopes 0 and 7.
- It has the gap property, i.e., no homotopy between dimensions -4 and 0.

1.24

8 The proof of the Periodicity Theorem

The proof of the Periodicity Theorem

Preperiodicity Theorem. *Let $\Delta_1^{(8)} = u_{2\rho_8}\left(\overline{\Delta}_1^{(8)}\right)^2 \in E_2^{16,0}(D^{-1}MU^{(4)}) = E_2^{16,0}(\tilde{\Omega})$. Then $\left(\Delta_1^{(8)}\right)^{16}$ is a permanent cycle.*

To prove this, note that $\left(\Delta_1^{(8)}\right)^{16} = u_{32\rho_8}\left(\overline{\Delta}_1^{(8)}\right)^{32}$. Both $u_{32\rho_8}$ and $\overline{\Delta}_1^{(8)}$ are permanent cycles, so $\left(\Delta_1^{(8)}\right)^{16}$ is also one.

Hence we have an equivariant map $\Pi : \Sigma^{256}\tilde{\Omega} \to \tilde{\Omega}$ where

- $u_{32\rho_8} : S^{256-32\rho_8} \to \tilde{\Omega}$ induces to the unit map from S^0 on the underlying ring spectrum and
- $\Delta_1^{(8)}$ is invertible because it is a factor of D.

1.25

The proof of the Periodicity Theorem (continued)

The above imply that the underlying map $i_0\Pi$ of ordinary spectra is a homotopy equivalence. It is known that any such map induces an equivalence of homotopy fixed point sets, so

$$\Sigma^{256}\tilde{\Omega}^{hC_8} \xrightarrow[\simeq]{\Pi^{hC_8}} \tilde{\Omega}^{hC_8}$$

Unfortunately the slice spectral sequence tells us nothing about this homotopy fixed point set. We know it detects all of the θ_j, but there is no direct way of showing that it has the gap property.

Fortunately we have a theorem stating that in this case the homotopy fixed set is equivalent to the actual fixed point set Ω. The slice spectral sequence tells us that the latter has the gap property. Thus we have proved

Periodicity Theorem. *Let $\Omega = (D^{-1}MU^{(4)})^{C_8}$. Then $\Sigma^{256}\Omega$ is equivalent to Ω.*

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9 Recap

Recap of the proof

- $\tilde{\Omega}$ is obtained from the C_8-spectrum $MU^{(4)}$ by inverting a certain element

$$D = \overline{\Delta}_1^{(8)} N_4^8 \left(\overline{\Delta}_2^{(4)}\right) N_2^8 \left(\overline{\Delta}_4^{(2)}\right) \in \pi_{19\rho_8}(MU^{(4)}).$$

- Since we are inverting an element in $\pi_{m\rho_8}$, the resulting slice spectral sequence has the gap property.
- Inverting D makes

$$\left(u_{2\rho_8}\left(\overline{\Delta}_1^{(8)}\right)^2\right)^{16} \in E_2^{256,0}(\tilde{\Omega})$$

a permanent cycle. We used geometric fixed points and $RO(G)$-graded homotopy to prove this.

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Recap of the proof (continued)

- The resulting equivariant map

$$\Pi : \Sigma^{256}\tilde{\Omega} \to \tilde{\Omega}$$

is an equivalence of the underlying spectra.
- This means that we have an equivalence of homotopy fixed point spectra

$$\Pi^{hC_8} : \Sigma^{256}\tilde{\Omega}^{hC_8} \to \tilde{\Omega}^{hC_8}.$$

- $\pi_*(\tilde{\Omega}^{hC_8})$ is accessible via the Adams-Novikov spectral sequence, and we know that it detects each θ_j, in addition to being 256-periodic.
- Our Homotopy Fixed Point Theorem (not covered in this talk) equates $\tilde{\Omega}^{hC_8}$ with $\Omega = \tilde{\Omega}^{C_8}$, which is known to have the gap property.

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