ON THE 3-PRIMARY ARF-KERVAIRE INVARIANT PROBLEM

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WORK IN PROGRESS

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1. Introduction

For our purposes the Arf-Kervaire invariant problem for a prime p is to determine the fate of the elements

$$\theta_j \in \left\{ \begin{array}{ll} h_j^2 & \text{for } p=2 \\ b_{j-1} & \text{for } p>2 \end{array} \right\} \in \operatorname{Ext}_A^{2,2p^j(p-1)}(\mathbf{Z}/(p), \mathbf{Z}/(p)) \tag{1}$$

where A denotes the mod p Steenrod algebra. This Ext group is the E_2-term for the classical Adams spectral sequence converging to the p-component of the stable homotopy groups of spheres. In these bidegrees the groups are known (Adams [Ada60] for $p=2$ and Liulevicius [Liu62] for odd primes) to be isomorphic to $\mathbf{Z}/(p)$ in each case, generated by these elements.

Closely related to them are the elements

$$\beta_{p^{j-1}/p^{j-1}} \in \operatorname{Ext}_{BP_*(BP)}^{2,2p^j(p-1)}(BP_*, BP_*) \qquad \text{for } j>0.$$

This Ext group is the E_2-term of the Adams-Novikov spectral sequence converging to the p-local stable homotopy groups of spheres. The Thom reduction map sends this group to the one in (1). This generator maps to a unit multiple of the one in (1) in each case except $p=2$ and $j=1$, for which we have $\beta_{1/1}=0$ as explained in [Rav86, Theorem 5.1.22].

Browder's Theorem [Bro69] states that at $p=2$, h_j^2 is a permanent cycle in the Adams spectral sequence if and only if there is a framed manifold with nontrivial Kervaire invariant manifold in dimension $2^{j+1}-2$. Such manifolds are known to exist for $1 \leq j \leq 5$, the case $j=5$ (the most difficult) being given in [BJM84].

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In [HHR] we showed that for $p = 2$, θ_j does not exist for $j \geq 7$. The case $j = 6$ remains open.

There is no known intrepretation of the problem at odd primes in terms of manifolds. In [Rav78] (and in [Rav86, §6.4]) the third author showed for that for $p \geq 5$ the element θ_j for $j > 1$ is not a permanent cycle, while θ_1 is a permanent cycle representing $\beta_1 \in \pi_{2p^2-2p-1}S^0$. It is shown that modulo some indeterminacy there are differentials

$$d_{2p-1}(\theta_j) = h_0\theta_{j-1}^p \qquad \text{where } h_0 \in \operatorname{Ext}_A^{1,2p-1}(\mathbf{Z}/(p), \mathbf{Z}/(p)) \tag{2}$$

represents $\alpha_1 \in \pi_{2p-3}S^0$. The methods used there break down at $p = 3$, which is the subject of this paper.

In order to describe the problems at $p = 3$ we need to recall the methods of [Rav78] and [HHR]. We know now (but only suspected when [Rav78] was written) that the extended Morava stabilizer group $\mathbf{G}_n$ acts on the Morava spectrum E_n in such a way that the homotopy fixed point set $E_n^{h\mathbf{G}_n}$ is $L_{K(n)}S^0$, the Bousfield localization of the sphere spectrum with respect to the nth Morava K-theory. This is a corollary of the Hopkins-Miller theorem, for which we refer the reader to [Rez98]. For any closed subgroup $H \subset \mathbf{G}_n$ there is a homotopy fixed point spectral sequence

$$H^*(H; \pi_*E_n) \implies \pi_*E_n^{hH}$$

which coincides with the Adams-Novikov spectral sequence for E_n^{hH}. One has the expected restriction maps for subgroups.

The group $\mathbf{G}_n$ is known to have a subgroup of order p (unique up to conjugacy) when $p - 1$ divides n. This leads to a composite homomorphism

$$\operatorname{Ext}_{BP_*(BP)}(BP_*, BP_*) \longrightarrow H^*(C_p; \pi_*E_{p-1}) \longrightarrow H^*(C_p; \mathbf{F}_{p^{p-1}}[u, u^{-1}]) \tag{3}$$

where the second homomorphism is reduction modulo the maximal ideal in π_*E_{p-1} and $|u| = 2$. The action of C_p here is trivial, so the target is a bigraded form of the usual mod p cohomology of C_p. *Assume now that p is odd.* Then this cohomology is

$$E(\alpha) \otimes P(\beta) \otimes \mathbf{F}_{p^{p-1}}[u, u^{-1}]$$

where $\alpha \in H^1$ and $\beta \in H^2$ each have topological degree 0. It is shown that under the composite of (3) homomorphism we have

$$\begin{array}{rcl} \alpha_1 & \mapsto & u^{p-1}\alpha \\ \beta_{p^{j-1}/p^{j-1}} & \mapsto & u^{p^j(p-1)}\beta \end{array} \tag{4}$$

up to unit scalar. Hence all monomials in the $\beta_{p^{j-1}/p^{j-1}}$ and their products with α_1 have nontrivial images. We also show that there are relations

$$\beta_{p^{j-1}/p^{j-1}}\beta_{p/p}^{p^{j-1}} = \beta_{p^j/p^j}\beta_1^{p^{j-1}}. \tag{5}$$

In order to proceed further we need the following result of Toda ([Tod67] and [Tod68]): In the Adams-Novikov spectral sequence for an odd prime p there is a nontrivial differential

$$d_{2p-1}(\beta_{p/p}) = \alpha_1\beta_1^p. \tag{6}$$

Using (5-6) one can deduce that

$$d_{2p-1}(\beta_{p^{j-1}/p^{j-1}}) = \alpha_1\beta_{p^{j-2}/p^{j-2}}^p \qquad \text{for all } j \geq 2,$$

and (4) implies that this is nontrivial.

Now some comments are in order about why this approach fails for $p < 5$.

- For $p = 2$, (5) and a suitable modification of (4) both hold, but the right hand side of (6) is trivial, so this method does not show that any $\beta_{2^{j-1}/2^{j-1}}$ fails to be a permanent cycle.
- The group $\operatorname{Ext}^{2,2p^j(p-1)}_{BP_*(BP)}(BP_*, BP_*)$ is known to have $[(j-1)/2]$ other generators besides $\beta_{p^{j-1}/p^{j-1}}$. The sum of $\beta_{p^{j-1}/p^{j-1}}$ with any linear combination of them maps to θ_j under the Thom reduction map. For $p \geq 5$, they are all in the kernel of (3). This means that any element on the Adams-Novikov 2-line mapping to θ_j supports a nontrivial differential, so θ_j does not exist as a homotopy element.
- For $p = 3$ these other generators, such as β_7 in the bidegree of $\beta_{9/9}$, can have nontrivial images under (3). This has to do with the fact that they are v_2-periodic and hence v_{p-1}-periodic. It turns out that $\beta_{9/9} \pm \beta_7$ and hence θ_3 are permanent cycles even though θ_2 is not.

In order to describe the way out of these difficulties we need to say more about finite subgroups of $\mathbf{G}_n$. It is by definition an extension of the Morava stabilizer group $\mathbf{S}_n$ by $\operatorname{Gal}(\mathbf{F}_{p^n} : \mathbf{F}_p)$. The Galois group (which is cyclic of order n) is there for technical reasons but plays no role on our calculations. $\mathbf{S}_n$ is the group of units in the maximal order of a certain division algebra over the p-adic numbers $\mathbf{Q}_p$. Its finite subgroups have been classified by Hewett [Hew95].

$\mathbf{S}_n$ has an element of order p iff $p-1$ divides n, a condition that is trivial when $p = 2$. More generally $\mathbf{S}_n$ has an element of order p^{k+1} iff $p^k(p-1)$ divides n. For such n we could replace (3) by

$$\operatorname{Ext}_{BP_*(BP)}(BP_*, BP_*) \longrightarrow H^*(C_{p^{k+1}}; \pi_* E_n) \longrightarrow H^*(C_{p^{k+1}}; ?), \tag{7}$$

where the coefficient ring in the target will be named later. The naive choice of $\mathbf{F}_{p^n}[u, u^{-1}]$ for this ring turns out not to detect $\beta_{p^{j-1}/p^{j-1}}$ for $n > p-1$. Experience has shown two things:

(i) In order to flush out the spurious elements (which are v_2-periodic) having the same bidegree as $\beta_{p^{j-1}/p^{j-1}}$, we need to have $n > 2$.
(ii) In order to detect the $\beta_{p^{j-1}/p^{j-1}}$ itself, we need to have n be equal to $p^{k+1}(p-1)$ for some $k \geq 0$, not just be divisible by it. Then it will map to an element of order p in a cohomology group isomorphic to $\mathbf{Z}/p^{k+1}$. We cannot detect higher powers of it for $k > 0$.

For $p = 2$ these considerations suggest using the group C_8 and $n = 4$, which is the approach used in [HHR].

For $p = 3$ we need to use the group C_9 with $n = 6$.

For the prime 2, our strategy in [HHR] was to construct a ring spectrum Ω with a unit map $S^0 \to \Omega$ satisfying three properties:

(i) Detection Theorem. If θ_j exists, its image in $\pi_*\Omega$ is nontrivial.
(ii) Periodicity Theorem. $\pi_k\Omega$ depends only on the congruence class of k modulo 256.
(iii) Gap Theorem. $\pi_{-2}\Omega = 0$.

The nonexistence of θ_j for $j \geq 7$ follows from the fact that its dimension is congruent to -2 modulo 256.

Ever since the discovery of the Hopkins-Miller theorem, it has been possible to prove that $E_4^{hC_8}$ satisfies the first two of these properties *without the use of equivariant stable homotopy theory.*

For $p = 3$, the same goes for $E_6^{hC_9}$ with the periodicity dimension being 972 (2 more than the dimension of θ_5) instead of 256. If all goes well, we will get a theorem saying θ_j does not exist for $j \geq 5$, leaving the status of θ_4 open. We already know that θ_1 (in the 10-stem) and θ_3 (in the 106-stem) exist and θ_2 (in the 34-stem) does not.

For $p \geq 5$, the same holds for $E_{p-1}^{hC_p}$ with periodicity $2p^2(p-1)$, which is 2 more than the dimension of θ_2. In this case the spectrum also detects the product of α_1 with any monomial in the θ_js. As explained above, this enables us to use Toda's differential to show that none of the θ_j for $j > 1$ exists.

We cannot use Toda's differential for $p < 5$ because

(a) for $p = 2$ its target is trivial, and
(b) since we cannot detect products of the θ_js, we cannot make an inductive argument.

The proof of the Gap Theorem requires the use of equivariant stable homotopy theory and the slice filtration. Unfortunately the Morava spectrum E_n is not an equivariant spectrum for finite subgroups $H \subset \mathbf{S}_n$ as we would like, but only for certain groups homotopy equivalent to such H. The actions of $\mathbf{S}_n$ and its subgroups are defined only up to homotopy, and the content of the Hopkins-Miller theorem is that the group G_0 of E_∞ self-equivalences of E_n homotopic to the identity is contractible. This means that we have an action of an extension of G_0 by any such H. This is good enough for forming homotopy fixed point spectra E_n^{hH} with the expected properties, but not for the more delicate equivariant constructions needed for the Gap Theorem.

This difficulty led us to replace the homotopy action of $C_{2^{n+1}}$ on E_{2^n} by a pointwise action on a relative of the smash power $MU^{(2^n)}$, induced via the norm construction from the action of C_2 on MU by complex conjugation. This action of $C_{2^{n+1}}$ has the additional advantage of a transparent action on the homotopy of the spectrum. The action of $\mathbf{S}_n$ on $\pi_* E_n$ is problematic.

In order to do a similar thing at an odd prime we need an analog MU_{C_p} of the C_2-spectrum $MU_{\mathbf{R}}$. It should be a C_p-spectrum underlain by roughly (but not precisely, as will be explained below) $MU^{(p-1)}$ with two properties:

(i) It should have a tractable slice filtration that enables us to prove a gap theorem for certain periodic spectra derived from it. As a C_p-module, $Q\pi_k^u MU_{C_p}$, the indecomposable quotient of the kth homotopy group of the underlying spectrum, will be
$$Q\pi_{2k}^u MU_{C_p} = \begin{cases} 0 & \text{for } k \text{ odd} \\ J & \text{for } k = 2(p^n - 1) \\ \mathbf{Z}[C_p] & \text{otherwise} \end{cases}$$
where J denotes augmentation ideal J in the group ring $\mathbf{Z}[C_p]$.
(ii) The geometric fixed point spectrum $\Phi^{C_p} MU_{C_p}$ should be a wedge of suspensions of H/p, the mod p Eilenberg-Mac Lane spectrum. For $p = 2$ we have $\Phi^{C_2} MU_{\mathbf{R}} = MO$, the unoriented cobordism spectrum, which fits this description. This identification is a pivotal step in determining differentials in the slice spectral sequence needed to prove the Periodicity Theorem.

Alternatively, we could look for a C_p-spectrum BP_{C_p} underlain by $BP^{(p-1)}$ with similar properties including $\Phi^{C_p}BP_{C_p} = H/p$. It would be nice if MU_{C_p} were also an E_∞-ring spectrum like $MU_{\mathbf{R}}$, but this is out of reach at the moment. Once we have such an MU_{C_p} or BP_{C_p} for $p = 3$, we can use the norm to get a C_9-spectrum underlain by (approximately) $MU^{(6)}$ or $BP^{(6)}$. *The spectar we actaually construct have a extara smash factors of MU and BP respectively, but thiis is harmless for our purposes.*

We will make extensive use of the old and new equivariant methods recalled and introduced in [HHR]. The reader would be well advised to have a copy of it within easy reach while reading this paper.

2. Odd primary analogs of $MU_{\mathbf{R}}$ and $BP_{\mathbf{R}}$

We now describe a program for constructing MU_{C_p} and BP_{C_p}. We start with the p-fold smash power $MU^{(p)}$ or $BP^{(p)}$. In both cases C_p acts by permuting the factors, and the spectrum is obtained by norming up from the action of the trivial group on MU or BP. For technical reasons we need to smash them with MU or BP equipped with the trivial group action.

To derive MU_{C_p} (respectively BP_{C_p}) from $MU^{(p)}\wedge MU$ ($BP^{(p)}\wedge BP$) we will use the *method of polynomial algebras* introduced in [HHR, §2.4]. Roughly speaking, it gives us a way of killing off a polynomial sub-$\mathbf{Z}[C_p]$-algebra R of $\pi_*^u X$ (where X is $MU^{(p)}$or $BP^{(p)}$) equivariantly. One forms an associative C_p-ring spectrum A underlain by a wedge of spheres, one for each monomial in R. There is a map $A \to X$ representing the inclusion $R \to \pi_*^u X$. This makes X an A-module. There is also a map $A \to S^0$ obtained by sending all positive dimensional summands of A to a point. This makes S^0 into an A-module. These two module structures enable us to form the C_p-spectrum

$$Y = X \underset{A}{\wedge} S^0 \quad \text{with} \quad \pi_*^u Y = \pi_*^u X \otimes_R \mathbf{Z}.$$

Its geometric fixed point spectrum is

$$\Phi^{C_p} Y = \Phi^{C_p} X \underset{\Phi^{C_p} A}{\wedge} S^0,$$

where $\Phi^{C_p} X$ is MU or BP. Y and $\Phi^{C_p} Y$ do *not* inherit ring structures from X.

Let $G = C_p$ or $\{e\}$. Given a suitable wedge of spheres W with G-action, we let $S^0[W]$ denote the corresponding equivariant polynomial algebra. (In [HHR], each sphere is assumed to be invariant, but we are not assuming that here.) Let

$$\begin{aligned} \overline{A}_0 &= S^0\left[\bigvee_{\substack{k>0 \\ k\neq p^n-1}} S^{2k}\right] \\ \text{and} \qquad A_0 &= N_1^p \overline{A}_0 \;=\; S^0\left[C_{p+} \wedge \bigvee_{\substack{k>0 \\ k\neq p^n-1}} S^{2k}\right]. \end{aligned}$$

Then we have

$$MU \underset{\overline{A}_0}{\wedge} S^0 = BP,$$

which implies that

$$MU^{(p)} \underset{A_0}{\wedge} S^0 = N_1^p \left(MU \underset{\overline{A}_0}{\wedge} S^0 \right) = N_1^p BP = BP^{(p)} \qquad \text{as } C_p\text{-spectra}$$

and

$$\left(MU^{(p)} \wedge MU \right) \underset{A_0 \wedge \overline{A}_0}{\wedge} S^0 = BP^{(p)} \wedge BP \qquad \text{as } C_p\text{-spectra.}$$

Theorem 2.1. Changing the geometric fixed points of $MU^{(p)} \wedge MU$ and $BP^{(p)} \wedge BP$. *For each $n > 0$ there is an equivariant map*

$$f_n : S^{2p^{n-1}\rho - 2} \to MU^{(p)} \wedge MU$$

such that the composite map

$$S^{2p^{n-1}\rho-2} \xrightarrow{f_n} MU^{(p)} \wedge MU \longrightarrow BP^{(p)} \wedge BP \xrightarrow{m} BP$$

(where m is multiplication) is v_n modulo decomposables and

$$S^{2p^{n-1}-2} \xrightarrow{\Phi^{C_p} f_n} MU \wedge MU \longrightarrow BP \wedge BP \xrightarrow{m} BP$$

is v_{n-1} (where $v_0 = p$) modulo decomposables.

Let

$$A_1 = S^0 \left[\bigvee_{n>0} S^{2p^{n-1}\rho - 2} \right]$$

and use the maps f_n to define A_1-module structures on $MU^{(p)} \wedge MU$ and $BP^{(p)} \wedge BP$. Then we define

$$MU_{C_p} = \left(MU^{(p)} \wedge MU \right) \underset{A_1}{\wedge} S^0 \qquad \textit{and} \qquad BP_{C_p} = \left(BP^{(p)} \wedge BP \right) \underset{A_1}{\wedge} S^0.$$

The second of these is underlain by $BP^{(p-1)} \wedge BP$ with trivial action on the second factor, and we have

$$\begin{aligned} \Phi^{C_p} MU_{C_p} &= (MU \wedge MU) \underset{\Phi^{C_p} A_1}{\wedge} S^0 &&= H/p \wedge \overline{A}_0 \wedge MU \\ \textit{and} \qquad \Phi^{C_p} BP_{C_p} &= (BP \wedge BP) \underset{\Phi^{C_p} A_1}{\wedge} S^0 &&= H/p \wedge BP. \end{aligned}$$

In [HHR] we used $MU_{\mathbf{R}}$ rather than $BP_{\mathbf{R}}$ because the former is an E_∞-ring spectrum. We do not have that luxury here. When explicit computations are needed, we will use BP_{C_p} rather than MU_{C_p}.

Proof. We will construct the map f_n geometrically by defining a C_p-action on Milnor hypersurface

$$H = H^{p^{n-1}, \dots, p^{n-1}} \subset \left(\mathbf{C}P^{p^{n-1}} \right)^p$$

whose cobordism class will be shown to represent v_n modulo decomposables. Its fixed point set H^{C_p} will be the degree p Fermat hypersurface V in $\mathbf{C}P^{p^{n-1}}$, whose cobordism class will be shown to represent v_{n-1} modulo decomposables. We will do these cobordism calculations separately in Lemma 2.2 below.

Let $\nu : \mathbf{C}P^{p^{n-1}} \to BU$ be the map inducing the stable normal bundle and let

$$\left(\mathbf{C}P^{p^{n-1}} \right)^p \xrightarrow{\nu^p} BU^p$$

be its p-fold Cartesian product. Let C_p act by permuting the factors of source and target. Thomification leads to an equivariant map

$$S^{2p^{n-1}\rho} \xrightarrow{\nu^p} MU^{(p)}.$$

A Milnor hypersurface

$$H = H^{p^{n-1},\ldots,p^{n-1}} \subset \left(\mathbf{C}P^{p^{n-1}}\right)^p$$

can be chosen to be invariant under the C_p-action. To see this, let

$$\left[x_0^{(j)}, x_1^{(j)}, \ldots, x_{p^{n-1}}^{(j)}\right]$$

for $1 \le j \le p$ denote a point in the jth factor $\mathbf{C}P^{p^{n-1}}$. C_p acts on the product by permuting the homogeneous coordinates $x_k^{(j)}$ for each k. We define our hypersurface by the equation

$$\sum_{0\le k\le p^{n-1}} \left(\prod_{1\le j\le p} x_k^{(j)}\right) = 0.$$

It is is invariant under the group action since each term in the sum is.

The normal bundle of H is induced by a map

$$H \to BU^p \times BU(1)$$

which Thomifies to a map

$$S^{2p^{n-1}\rho-2} \to MU^{(p)} \wedge MU(1)$$

whose composition with the multiplication map into MU represents the cobordism class of H.

The fixed point set of the product $\left(\mathbf{C}P^{p^{n-1}}\right)^p$ is the diagonal copy of $\mathbf{C}P^{p^{n-1}}$. On it the equation becomes

$$\sum_{0\le k\le p^{n-1}} x_k^p = 0,$$

which defines the Fermat hypersurface V. Note that when $n = 1$, it consists of the p points

$$\left\{\left[1, e^{(2j-1)\pi i/p}\right] : 1 \le j \le p\right\}.$$

Note that $\pi_* MU^{(p)} \wedge MU$ is the cobordism group of complex manifolds M equipped with a decomposition of their normal bundles into $p+1$ Whitney summands. Equivariantly M can be equipped with a C_p-action that lifts to a bundle map permuting the first p of them.

For the element of $\pi_\star^{C_p} MU^{(p)} \wedge MU$ at hand, the manifold is H with the group action inherited from the Cartesian product of projective spaces and the $p+1$ bundles are the pullbacks of the normal bundles of the p projective spaces and the line bubdle that defines the hypersurface.

The composite map mf_n represents the cobordism class of the Milnor hypersurface, which is v_n modulo decomposables.

Passing to geometric fixed points gives

$$S^{2p^{n-1}-2} \xrightarrow{\Phi f_n} MU \wedge MU \xrightarrow{m} MU$$

The fixed point set of the C_p-action on the Milnor hypersurface H is the degree p hypersurface in $\mathbf{C}P^{p^{n-1}}$. The s-number (see (8) below) of the latter is a unit multiple of p, so the map Φf_n represents v_{n-1} modulo decomposables. □

Lemma 2.2. Cobordism classes of Milnor and Fermat hypersurfaces. *The Milnor and Fermat hypersufaces H and V of the proof above represent v_n and v_{n-1} modulo decomposables respectively.*

Before proving this we recall some characteristic classes of of complex vector bundles. Let ξ be a complex k-plane bundle over a manifold M. Write its total Chern class formally as follows:

$$c_t(\xi) = 1 + c_1(\xi)t + \cdots + c_k(\xi)t^k = (1 + x_1 t) \cdots (1 + x_k t)$$

so that $c_i(\xi) = \sigma_i(x_1 \cdots , x_k)$ is the ith elementary symmetric function in the formal indeterminates x_j. Consider the polynomial

$$P_n(x_1, \ldots , x_k) = x_1^n + \cdots + x_k^n$$

and express it via the elementary symmetric functions:

$$P_n(x_1, \ldots , x_k) = s_n(\sigma_1, \ldots , \sigma_k).$$

Substituting the Chern classes for the elementary symmetric functions we obtain a certain characteristic class of ξ:

$$s_n(\xi) = s_n(c_1(\xi), \ldots , c_k(\xi)) \in H^{2n}(M).$$

The following properties of characteristic class follow immediately from the definition:

Proposition 2.3. Properties of the s-class.

(i) $s_n(\xi) = 0$ *when* $2n$ *exceeds the dimension of* M.
(ii) $s_n(\xi \oplus \eta) = s_n(\xi) + s_n(\eta)$.
(iii) *If* ξ *is a line bundle, then* $s_n(\xi) = c_1(\xi)^n$.

Now we define the *s-number* of a $2k$-dimensional complex manifold M to be

$$s_k[M] := s_k(\tau)\langle M\rangle \in \mathbf{Z}, \tag{8}$$

where τ and $\langle M\rangle$ are the complex tangent bundle and fundamental homology class of M. It could equivalently be defined in terms of the normal bundle, giving the negative of the integer above. This characteristic number is useful because it vanishes on Cartesian products and thus detects indecomposable cobordism classes. See Stong's book [Sto68, Chapter 7] for details. It is known that a $2k$-dimensional manifold M represents a polynomial generator of $\pi_{2k}MU_{(p)}$ iff

$$s_k[M] = \left\{ \begin{array}{ll} u & \text{for } k \neq p^n - 1 \\ up & \text{for } k = p^n - 1 \end{array} \right\} \qquad \text{for some unit } u \in \mathbf{Z}_{(p)}^{\times}. \tag{9}$$

Proof of Lemma 2.2. Recall that the fundamental homology class of a hypersurface $S \subset T$ of dimension $2k$ is Poincaré dual of the first Chern class of the normal line bundle ν that defines it. Moreover for $y \in H^{2k}T$, we have

$$y\langle S\rangle = yc_1(\nu)\langle T\rangle. \tag{10}$$

In the case of the degree p Fermat hypersurface in $\mathbf{C}P^{p^{n-1}}$, let $x \in H^2$ denote the first Chern class of the tautological line bundle η and $b_i \in H_{2i}$ the linear dual of x^i. The normal line bundle in this case is

$$\nu = \eta^{\otimes p},$$

and the fundamental class is $pb_{p^{n-1}-1}$, the Poincaré dual of $px = c_1(\eta^{\otimes p})$. Let τ be the tangent bundle of the ambient space $\mathbf{C}P^{p^{n-1}}$. Then the tangent bundle of V is the restriction of

$$\tau' = \tau \oplus -\nu.$$

The total Chern classes of these two bundles are

$$\begin{aligned} c(\tau) &= (1+x)^{1+p^{n-1}} \\ c(\tau') &= (1+x)^{1+p^{n-1}}(1+px)^{-1} \end{aligned}$$

so

$$\begin{aligned} s_{p^{n-1}-1}(\tau) &= (1+p^{n-1})x^{p^{n-1}-1} \\ s_{p^{n-1}-1}(\tau') &= (1+p^{n-1}-p^{p^{n-1}-1})x^{p^{n-1}-1} \\ s_{p^{n-1}-1}[V] &= p(1+p^{n-1}-p^{p^{n-1}-1}) \end{aligned}$$

Since this is a p-local unit multiple of p, V represents v_{n-1} modulo decomposables as claimed.

We now make a similar calculation for the Milnor hypersurface H. Here the ambient space is is the p-fold Cartesian product $X = \left(\mathbf{C}P^{p^{n-1}}\right)^p$ with cohomology

$$H^*X = \mathbf{Z}[x_1, \ldots, x_p]/\left(x_j^{1+p^{n-1}}\right).$$

Let η_j denote the pullback of the tautological line bundle on the jth factor. The normal bundle is

$$\nu = \eta_1 \otimes \cdots \otimes \eta_p,$$

so the fundamental class of the hypersurface H is the Poincaré dual of $x_1 + \cdots + x_p$.

The total Chern classes of these two bundles are

$$\begin{aligned} c(\tau) &= \prod_{j=1}^{p}(1+x_j)^{1+p^{n-1}} \\ c(\tau') &= \left(1+\sum_{j=1}^{p} x_j\right)^{-1} \prod_{j=1}^{p}(1+x_j)^{1+p^{n-1}} \end{aligned}$$

so

$$\begin{aligned} s_{p^n-1}(\tau) &= (1+p^{n-1})\sum_{j=1}^{p} x_j^{p^n-1} \;=\; 0 \\ s_{p^n-1}(\tau') &= -\left(\sum_{j=1}^{p} x_j\right)^{p^n-1} + (1+p^{n-1})\sum_{j=1}^{p} x_j^{p^n-1} \\ &= -c_1(\nu)^{p^n-1} \\ s_{p^n-1}[H] &= -c_1(\nu)^{p^n-1}\langle H\rangle \;=\; -c_1(\nu)^{p^n}\langle X\rangle \qquad \text{by (10)} \\ &= -\binom{p^n}{p^{n-1},\dots,p^{n-1}} \;=\; -\frac{p^n!}{(p^{n-1}!)^p}. \end{aligned}$$

This is also a unit multiple of p, so H represents v_n modulo decomposables as claimed. □

3. The slice filtration

We will define the slice filtration of the category of G-equivariant spectra as in [HHR, §4]. The relevant groups G for us are C_3 and C_9. For an integer n and a subgroup H of G, let

$$\widehat{S}(n,H) = G_+ \underset{H}{\wedge} S^{n\rho_H}.$$

where ρ_H is the regular representation of H.

Definition 3.1. Slice cells. Slice cells *are members of the set*

$$\mathcal{A} = \left\{\widehat{S}(n,H),\ \Sigma^{-1}\widehat{S}(n,H)\colon\ H\subset G,\ n\in\mathbf{Z}\right\}.$$

The dimension of a slice cell is that of its underlying spheres, i.e., $n|H|$ *or* $n|H|-1$.

Definition 3.2. Slice null and slice positive. *A G-spectrum Y is* slice n-null, *written*

$$Y < n \quad \text{or} \quad Y \le n-1$$

if for every slice cell $\widehat{S}$ with $\dim\widehat{S}\ge n$ *the G-space*

$$\mathcal{S}_G(\widehat{S},Y)$$

is equivariantly contractible. A G-spectrum X is slice n-positive, *written*

$$X > n \quad \text{or} \quad X \ge n+1$$

if

$$\mathcal{S}_G(X,Y)$$

is equivariantly contractible for every Y with $Y\le n$.

The full subcategory of $\mathcal{S}^G$ consisting of X with $X>n$ will be denoted $\mathcal{S}^G_{>n}$ or $\mathcal{S}^G_{\ge n+1}$. Similarly, the full subcategory of $\mathcal{S}^G$ consisting of X with $X<n$ will be denoted $\mathcal{S}^G_{<n}$ or $\mathcal{S}^G_{\le n-1}$.

Remark 3.3. Subcategories associated with the slice filtration. *The category $\mathcal{S}^G_{>n}$ is the smallest full subcategory of $\mathcal{S}^G$ containing the slice cells $\widehat{S}$ with* $\dim\widehat{S}>n$ *and possessing the following properties:*

i) *If X is weakly equivalent to an object of $\mathcal{S}^G_{>n}$, then X is in $\mathcal{S}^G_{>n}$.*
ii) *Arbitrary wedges of objects of $\mathcal{S}^G_{>n}$ are in $\mathcal{S}^G_{>n}$.*
iii) *If $X \to Y \to Z$ is a cofibration sequence and X and Y are in $\mathcal{S}^G_{>n}$ then so is Z.*
iv) *If $X \to Y \to Z$ is a cofibration sequence and X and Z are in $\mathcal{S}^G_{>n}$ then so is Y.*

More briefly, these properties are that $\mathcal{S}^G_{>n}$ is closed under weak equivalences, homotopy colimits (properties ii) and iii)), and extensions.

Let P^nX be the Bousfield localization, or Dror Farjoun nullification ([Far96, Hir03]) of X with respect to the class $\mathcal{S}^G_{>n}$, and $P_{n+1}X$ the homotopy fiber of $X \to P^nX$. Thus, by definition, there is a functorial fibration sequence

$$P_{n+1}X \to X \to P^nX.$$

When there is more than one group involved, we will denote these fucntors by P^n_G and P^G_{n+1}. In the nonequivariant case, P^nX is the nth Postnikov section of X, i.e., the spectrum obtained by killing all homotopy groups above dimension n. The fiber $P_{n+1}X$ is the n-connected cover of X.

Definition 3.4. The slice tower. *The* slice tower *of X is the tower $\{P^nX\}_{n\in\mathbf{Z}}$. The spectrum P^nX is the n^{th}* slice section *of X.*

When considering more than one group, we will write $P^nX = P^n_GX$ and $P_nX = P^G_nX$.

Let P^n_nX be the fiber of the map

$$P^nX \to P^{n-1}X.$$

Definition 3.5. n-slices. *The n-*slice *of a spectrum X is P^n_nX. A spectrum X is an n-*slice *if $X = P^n_nX$.*

In [HHR] we considered certain G-spectra (with G a finite cycylic 2-group) related to MU and found that their oddly indexed slices were contractible, and their evenly indexed slices each had the form $W \wedge H\mathbf{Z}_{(2)}$, with $H\mathbf{Z}_{(2)}$ being the 2-local integer Eilenberg-Mac Lane spectrum with trivial action and W a certain wedge of slice cells as defined above. In the case at hand we get a similar result involving $H\mathbf{Z}_{(3)}$ in each W could also have summands equivalent to the codimension 1 skeleton of some $\Sigma^{-1}\widehat{S}(2n, C_3)$. When $G = C_p$, this skeleton is defined in terms of an equivariant cellular structure in which there is a single cell in dimension $2n-1$ and p cells (permuted cyclically by the group) in each higher dimension up to $2pn-1$. It is underlain by a wedge of two copies of S^{6n-2} when $G = C_3$ and six of them when $G = C_9$.

For $G = C_p$ for an odd prime p, let λ^k denote the 2-dimensional representation corresponding to rotation through an angle of $2\pi k/p$. Let $q = p-1$ and $r = q/2$. For $G = C_{p^2}$ for an odd prime p, let λ^k for $k \neq 0$ denote the 2-dimensional representation corresponding to rotation through an angle of $2\pi k/p^2$, and let $\lambda' = \lambda^p$.

Proposition 3.6. p-local representation spheres. *With notation as above, in $RO(C_p)$ we have*

$$\rho = 1 + \sum_{k=1}^{r} \lambda^k.$$

For k prime to p, S^{λ^k} is p-locally equivalent to S^λ, and S^ρ is p-locally equivalent to $S^{1+r\lambda}$.

In $RO(C_{p^2})$,

$$\rho = 1 + \sum_{k=1}^{(p^2-1)/2} \lambda^r = 1 + \sum_{k=1}^{r} (\lambda')^k + \sum_{\substack{0<k<p^2/2 \\ p \not| \ k}} \lambda^k.$$

and S^ρ is p-locally equivalent to $S^{1+r\lambda'+pr\lambda}$.

Definition 3.7. Half representation spheres at odd primes. *For $G = C_p$, $S^{(2m-1)\lambda/2}$ is the finite G-spectrum underlain by $\vee_{p-1} S^{(2m-1)}$ with $H^u_{2m-1}(S^{(2m-1)\lambda/2})$ isomorphic to the augmentation ideal as a $\mathbf{Z}[G]$-module, with fixed point set S^0. Equivalently, it is the codimension 1 skeleton of $S^{m\lambda}$ under its standard cellular structure, namely the one with p cells in each positive dimension up to $2m$.*

A similar definition can be made for larger cyclic groups.

Our justification for this notation is the fact that

$$S^{(2m-1)\lambda/2} \wedge S^{(2n-1)\lambda/2} = S^{(m+n-1)\lambda} \vee \left(G_+ \wedge \bigvee_{p-2} S^{2(m+n-1)} \right).$$

We will often ignore the free summand on the right. For $p = 2$ it is trivial, and since $\lambda = 2\sigma$ (where σ is the sign representation), $\lambda/2 = \sigma$. Alternatively a half representation sphere can be thought of as the odd primary analog of a nonoriented representation sphere, namely one for which the matrix corresponding to a generator of the cyclic group has determinant -1.

Lemma 3.8. A $(2pn-2)$-slice. *For $G = C_p$, let $\overline{S}^{2n\rho-1}$ denote the $(2pn-2)$-skeleton of $S^{2n\rho-1}$, where ρ denotes the regular representation. Then $\overline{S}^{2n\rho-1} \wedge H\mathbf{Z}$ is a $(2pn-2)$-slice.*

The p-localization of $\overline{S}^{2n\rho-1}$ is $S^{(2qn-1)\lambda/2+2n-1}$.

Hill [Hil, Theorem 3.1] has shown that the k-skeleton of a n-dimensional slice cell for $n > 0$ is $\geq k$ for any $0 < k < n$. The methods below can be used to show that its smash product with $H\mathbf{Z}$ is a k-slice.

Proof. Let $X = \overline{S}^{2n\rho-1} \wedge H\mathbf{Z}$. We have a cofiber sequence

$$G_+ \wedge S^{2pn-2} \longrightarrow \overline{S}^{2n\rho-1} \longrightarrow S^{2n\rho-1} \longrightarrow G_+ \wedge S^{2pn-1}.$$

It follows that $\overline{S}^{2n\rho-1} \geq 2pn-2$ and hence $X \geq 2pn-2$. We also know that $X \leq 2pn-1$ since the same holds for $G_+ \wedge S^{2pn-2} \wedge H\mathbf{Z}$ and $S^{2n\rho-1} \wedge H\mathbf{Z}$. Hence showing that X is a $(2pn-2)$-slice reduces to showing that the groups $\pi^G_{2n\rho-1}X$ and $\pi^G_{2pn-1}X$ both vanish.

For the latter we have

$$\begin{aligned} \pi^G_{2pn-1}X &= \pi_{2pn-1}X^G \\ &= \pi_{2pn-1}S^{2n-1} \wedge H\mathbf{Z} \\ &= H_{2pn-1}S^{2n-1} \ = \ 0. \end{aligned}$$

For the former we have an exact sequence

$$\begin{array}{ccccc} \pi^G_{2n\rho-1}(G_+ \wedge S^{2pn-2} \wedge H\mathbf{Z}) & = & \pi^u_1(G_+ \wedge H\mathbf{Z}) & = & 0 \\ \downarrow & & & & \\ \pi^G_{2n\rho-1}(X) & & & & \\ \downarrow & & & & \\ \pi^G_{2n\rho-1}(S^{2n\rho-1} \wedge H\mathbf{Z}) & = & \pi^G_0(H\mathbf{Z}) & = & \mathbf{Z} \\ \downarrow & & & & \\ \pi^G_{2n\rho-1}(G_+ \wedge S^{2pn-1} \wedge H\mathbf{Z}) & = & \pi^u_0(G_+ \wedge H\mathbf{Z}) & = & \mathbf{Z}[C_p] \end{array}$$

The bottom map is monomorphic, so $\pi^G_{2n\rho-1}(X) = 0$ and X is a $(2pn-2)$-slice. $\square$

With the above example in mind, we make the following definition.

Definition 3.9. Generalized slice cells and refinements. *A generalized n-dimensional slice cell is a (p-local) n-dimensional G-CW complex W such that ($W \wedge H\mathbf{Z}_{(p)}$) $W \wedge H\mathbf{Z}$ is an n-slice.*

For a (p-local) G-spectrum X with $\pi^i_n X$ a free ($\mathbf{Z}_{(p)}$-module) abelian group, a generalized refinement of $\pi^u_n X$ is a map $f : W \to X$ where W as above is underlain by a wedge of (p-local) n-spheres such that $\pi^u_n(f)$ is an isomorphism of G-modules.

4. The detection theorem

The calculations for the 3-primary detection theorem are similar to those at $p = 2$ subject to some obvious changes. In the bigrading of θ_j we have

$$\begin{array}{rcl} \{\beta_{9/9},\, \beta_7\} & \text{for} & j = 3 \\ \{\beta_{27/27},\, \beta_{21/3}\} & \text{for} & j = 4 \\ \{\beta_{81/81},\, \beta_{63/9},\, \beta_{62}\} & \text{for} & j = 5 \end{array}$$

and so on, generalizing to

$$\left\{\beta_{c(j,k)/3^{j-1-2k}} : 0 \le k \le (j-1)/2)\right\}$$

where $c(j,k) = (3^j + 3^{j-1-2k})/4$. As before we will construction a detection homomorphism which kills each element in this set other than $\beta_{3^{j-1}/3^{j-1}}$. It leads to a valuation on BP_* with

$$||v_n|| = \max(0, (6-n)/6),$$

which for $k > 0$ gives

$$\begin{aligned} ||\beta_{c(j,k)/3^{j-1-2k}}|| &= \left|\left|\frac{v_2^{c(j,k)}}{3v_1^{3^{j-1-2k}}}\right|\right| \\ &= \frac{2}{3}\frac{(3^j + 3^{j-1-2k})}{4} - 1 - \frac{5 \cdot 3^{j-1-2k}}{6} \\ &= \frac{2 \cdot 3^j - 8 \cdot 3^{j-1-2k}}{12} - 1 \\ &= \frac{(2 \cdot 3^{2k+1} - 8) \cdot 3^{j-1-2k}}{12} - 1 \\ &\ge \frac{54-8}{12} - 1 \ > \ 2, \end{aligned}$$

which shows that the element in question maps to one that is divisible by 9 and therefore trivial.

We replace the ring $A = \mathbf{Z}_2[\zeta_8]$ by $\mathbf{Z}_3[\zeta_9]$, where the maximal ideal is generated by $\pi = \zeta_9 - 1$ with π^6 being a unit multiple of 3. There is a formal A-module F over $R_* = A[w^{\pm 1}]$ with logarithm

$$\log_F(x) = x + \sum_{k>0} \frac{w^{3^k-1}x^{3^k}}{\pi^k}$$

The analog of HHR Lemma 11.8 is the statement the v_i maps to a unit multiple of π^{6-i} for $1 \le i \le 6$. Hazewinkel's formula

$$3\ell_n = \sum_{0\le i<n} \ell_i v_{n-i}^{3^i}$$

can be rewritten as

$$3\ell_n = \sum_{1\le i\le n} \ell_{n-i} v_i^{3^{n-i}}$$

maps under $\phi : BP_* \to R_*/(w-1)$ to

$$\frac{3}{\pi^n} = \sum_{1\le i\le n} \frac{\phi(v_i)^{3^{n-i}}}{\pi^{n-i}} = \phi(v_n) + \sum_{1\le i\le n-1} \frac{\phi(v_i)^{3^{n-i}}}{\pi^{n-i}}$$

For $n \le 6$, assume inductively that $\phi(v_n) = u_i\pi^{6-n}$ for a unit $u_n \in A$. Then we have

$$\begin{aligned} \frac{3}{\pi^n} &= \phi(v_n) + \sum_{1\le i\le n-1} \frac{(u_i\pi^{6-i})^{3^{n-i}}}{\pi^{n-i}} \\ &= \phi(v_n) + \sum_{1\le i\le n-1} u_i^{3^{n-i}} \pi^{3^{n-i}(6-i)+i-n} \end{aligned}$$

The exponent of π is each term of the sum exceeds $6-n$, and the result follows.

Now we can mimic the calculation of HHR §11.5 as follows. Let $G = C_9$ and let $H \subset G$ be a nontrivial subgroup of order h. There are elements $r_i^H \in \pi_{2i}MU^{(6)}$ satisfying

$$x + \sum_{i>0} m_i x^{i+1} = \left(x + \sum_{k>0} \gamma^{9/h}(m_{3^k-1})x^{3^k} \right) \circ \left(x + \sum_{i>0} r_i^H x^{i+1} \right)$$

Applying the homomorphism $\lambda^{(6)} : \pi_*MU^{(6)} \to R_*$, we get

$$x + \sum_{k>0} \frac{w^{3^k-1}}{\pi^k} x^{3^k} = \left(x + \sum_{j>0} \frac{\zeta^{9/h} w^{3^j-1}}{\pi^j} x^{3^j} \right) \circ \left(x + \sum_{i>0} \lambda^{(6)}(r_{H,i}) x^{i+1} \right). \tag{11}$$

Let $s_{H,i} = \lambda^{(6)}(r_{H,i})$ and

$$f_H(x) = x + \sum_{i>0} s_{H,i} x^{i+1}.$$

Then (11) reads

$$\begin{aligned} x+\sum_{k>0}\frac{w^{3^k-1}}{\pi^k}x^{3^k} &= \left(x+\sum_{j>0}\frac{\zeta^{9/h}w^{3^j-1}}{\pi^j}x^{3^j}\right)\circ f_H(x) \\ &= f_H(x)+\sum_{j>0}\frac{\zeta^{9/h}w^{3^j-1}}{\pi^j}f_H(x)^{3^j} \qquad (12) \end{aligned}$$

We see immediately that $f_H(x)$ is odd, so $S_{H,i}=0$ for odd i. Modulo x^4, (12) reads

$$x+\frac{w^2x^3}{\pi}=x+s_{H,2}x^3+\zeta^{9/h}\frac{w^2}{\pi}x^3$$

so

$$s_{H,2}=\frac{(1-\zeta^{9/h})w^2}{\pi}=\begin{cases} -w^2 & \text{for } h=9 \\ -(3+3\pi+\pi^2)w^2 & \text{for } h=3 \end{cases}$$

For $h=3$ we need to find the smallest k such that $s_{H,2k}$ is a unit. We will compute modulo x^{28} and it suffices to replace $f_H(x)^{3^j}$ by its mod 3 approximation,

$$(f_H(x))^{3^j}=\left(\sum_{k\geq 0}s_{H,2k}x^{2k+1}\right)^{3^j}\equiv\sum_{k\geq 0}s_{H,2k}^{3^j}x^{(2k+1)3^j}.$$

where $s_{H,0}=1$. Thus we have

$$\begin{aligned} x+\sum_{k>0}\frac{w^{3^k-1}}{\pi^k}x^{3^k} &= f_{C_3}(x)+\sum_{j>0}\frac{\zeta^3w^{3^j-1}}{\pi^j}f_{C_3}(x)^{3^j} \\ &\equiv x+\sum_{k>0}s_{C_3,2k}x^{2k+1}+\sum_{j>0}\frac{\zeta^3w^{3^j-1}}{\pi^j}\sum_{k\geq 0}s_{C_3,2k}^{3^j}x^{(2k+1)3^j} \\ \sum_{k>0}\frac{w^{3^k-1}}{\pi^k}x^{3^k} &\equiv \sum_{k>0}s_{C_3,2k}x^{2k+1}+\sum_{j>0}\frac{\zeta^3w^{3^j-1}}{\pi^j}\sum_{k\geq 0}s_{C_3,2k}^{3^j}x^{(2k+1)3^j} \\ \sum_{k>0}\frac{w^{3^k-1}}{\pi^k}x^{3^k} &\equiv \sum_{k>0}s_{C_3,2k}x^{2k+1}+\sum_{j>0}\frac{\zeta^3w^{3^j-1}x^{3^j}}{\pi^j} \\ &\quad +\sum_{j>0}\frac{\zeta^3w^{3^j-1}}{\pi^j}\sum_{k>0}s_{C_3,2k}^{3^j}x^{(2k+1)3^j} \\ (1-\zeta^3)\sum_{j>0}\frac{w^{3^j-1}}{\pi^j}x^{3^j} &\equiv \sum_{k>0}s_{C_3,2k}x^{2k+1}+\sum_{j>0}\frac{\zeta^3w^{3^j-1}}{\pi^j}\sum_{k>0}s_{C_3,2k}^{3^j}x^{(2k+1)3^j} \end{aligned}$$

From this we see that $S_{C_3,2k}$ is congruent to zero unless $2k+1$ is divisible by 3, so we replace k by $3\ell+1$, and we have

$$(1-\zeta^3)\sum_{j>0}\frac{w^{3^j-1}}{\pi^j}x^{3^j} \equiv \sum_{\ell\geq 0}s_{C_3,6\ell+2}x^{6\ell+3}+\sum_{j>0}\frac{\zeta^3w^{3^j-1}}{\pi^j}\sum_{\ell\geq 0}s_{C_3,6\ell+2}^{3^j}x^{(6\ell+3)3^j}$$

Equating coefficients of x^3 gives

$$s_{C_3,2}=\frac{(1-\zeta^3)w^2}{\pi}=-(\,pi^2+3\pi+3)w^2,$$

which we abbreviate by $\pi^2 u_1 w^2$ for a unit $u_1 \in A$. Subtracting this from both sides gives

$$\begin{aligned}
(1-\zeta^3)\sum_{j\geq 2}\frac{w^{3^j-1}}{\pi^j}x^{3^j} &\equiv \sum_{\ell>0} s_{C_3,6\ell+2}x^{6\ell+3} + \sum_{j>0}\frac{\zeta^3 w^{3^j-1}}{\pi^j}s_{C_3,2}^{3^j}x^{3^{j+1}} \\
&\quad + \sum_{j>0}\frac{\zeta^3 w^{3^j-1}}{\pi^j}\sum_{\ell>0} s_{C_3,6\ell+2}^{3^j}x^{(6\ell+3)3^j} \\
&\equiv \sum_{\ell>0} s_{C_3,6\ell+2}x^{6\ell+3} + \sum_{j\geq 2}\frac{\zeta^3 w^{3^{j-1}-1}}{\pi^{j-1}}(\pi^2 u_1 w^2)^{3^{j-1}}x^{3^j} \\
&\quad + \sum_{j>0}\frac{\zeta^3 w^{3^j-1}}{\pi^j}\sum_{\ell>0} s_{C_3,6\ell+2}^{3^j}x^{(6\ell+3)3^j} \\
&\equiv \sum_{\ell>0} s_{C_3,6\ell+2}x^{6\ell+3} + \sum_{j\geq 2}\frac{\zeta^3 \pi^{2\cdot 3^{j-1}+1} w^{3^j-1}}{\pi^j}u_1^{3^{j-1}}x^{3^j} \\
&\quad + \sum_{j>0}\frac{\zeta^3 w^{3^j-1}}{\pi^j}\sum_{\ell>0} s_{C_3,6\ell+2}^{3^j}x^{(6\ell+3)3^j} \\
\sum_{j\geq 2}\frac{w^{3^j-1}(1-\zeta^3-\zeta^3\pi^{2\cdot 3^{j-1}+1})}{\pi^j}x^{3^j} & \\
&= \sum_{j\geq 2}\frac{w^{3^j-1}a_j}{\pi^j}x^{3^j} \qquad \text{where } a_j = 1-\zeta^3-\zeta^3\pi^{2\cdot 3^{j-1}+1} \\
&\equiv \sum_{\ell>0} s_{C_3,6\ell+2}x^{6\ell+3} \\
&\quad + \sum_{j>0}\frac{\zeta^3 w^{3^j-1}}{\pi^j}\sum_{\ell>0} s_{C_3,6\ell+2}^{3^j}x^{(6\ell+3)3^j}
\end{aligned}$$

Equating coefficients of x^9 gives

$$s_{C_3,8} = \pi w^8 u_2$$

for a unit $u_2 = a_2/\pi^3 \in A$. Subtracting this from both sides gives

$$\begin{aligned}
&\sum_{j\geq 3}\frac{w^{3^j-1}a_j}{\pi^j}x^{3^j} \\
&\qquad \equiv \sum_{\ell>1} s_{C_3,6\ell+2}x^{6\ell+3} + \sum_{j>0}\frac{\zeta^3 w^{3^j-1}}{\pi^j}\sum_{\ell>0} s_{C_3,6\ell+2}^{3^j}x^{(6\ell+3)3^j}
\end{aligned}$$

From this we see that $s_{C_3,14}$ and $s_{C_3,20}$ are congruent to zero, and

$$\begin{aligned} s_{C_3,26} &\equiv w^{26}\left(\frac{a_3}{\pi^3} - \frac{\zeta^3}{\pi}\pi^3 u_2^3\right) \\ &= w^{26}\left(\frac{a_3}{\pi^3} - \frac{\zeta^3 a_2^3}{\pi^7}\right) \\ &= w^{26}\left(\frac{a_3}{\pi^3} - \zeta^3\pi^2 u_2^3\right) \\ &= w^{26}u_3 \end{aligned}$$

for a unit $u_3 \in A$.

5. The odd primary periodicity theorem

This is intended to be an outline without detailed proofs. We assume there is a C_p-spectrum $BP_{\mathbf{R}}$ underlain by $BP^{(q)}$, where as usual $q = p-1$. We also define $r = q/2$.

λ will denote a degree 2 representation of a nontrivial cyclic p-group G that sends a generator to a rotation of order p. We will let

$$a = a_\lambda \in \pi^G_{-\lambda}S^0 \qquad \text{and} \qquad u = u_\lambda \in \pi^G_{2-\lambda}H\mathbf{Z}.$$

These elements will figure in the statement of the Slice Differentials Theorem. For $G = C_p$, the regular representation ρ decomposes as $\rho = 1 + r\lambda$.

More generally $\lambda(p^i)$ will denote a degree 2 representation of a cyclic p-group G containing C_{p^i} that sends a generator to a rotation of order p^i. We will denote by ρ_{p^i} the composite of the regular representation of C_{p^i} with any surjection $G \to C_{p^i}$. It satisfies

$$\rho_{p^i} = \rho_{p^{i-1}} + rp^{i-1}\lambda(p^i).$$

We will make use of the C_p-map $f : N_1^p(BP) \to BP_{\mathbf{R}}$ which is onto in underlying homotopy. It induces maps

$$N_p^{p^i}(f) : N_1^{p^i}(BP) \to N_p^{p^i}(BP_{\mathbf{R}})$$

for all $i \geq 1$. Recall that the indecomposable quotient of $\pi^u_{2p^n-2}BP_{\mathbf{R}}$ is refined by a map

$$\overline{S}^{2p^{n-1}\rho-1} \xrightarrow{\overline{v}_n} BP_{\mathbf{R}},$$

where ρ denotes the regular representation of C_p and $\overline{S}^V$ denotes the codimension 1 skeleton of S^V. Since $\overline{S}^V$ is not a sphere, $\overline{v}_n$ is not an element in $\pi^{C_p}_\star BP_{\mathbf{R}}$. The codimension 1 skeleton of $\overline{S}^V$ *is* a sphere, so we have an element

$$\overline{y}_n \in \pi^{C_p}_{2p^{n-1}+(qp^{n-1}-1)\lambda}BP_{\mathbf{R}}$$

defined to be the composite

$$\begin{tikzcd} S^{2p^{n-1}+(qp^{n-1}-1)\lambda} \arrow[r] \arrow[d, equal] & \overline{S}^{2p^{n-1}-1+qp^{n-1}\lambda} \arrow[d, equal] & \\ S^{(2p^{n-1}-1)\rho+(r-1)\lambda} \arrow[r] \arrow[rr, bend right, "\overline{y}_n"'] & \overline{S}^{2p^{n-1}\rho-1} \arrow[r, "\overline{v}_n"] & BP_{\mathbf{R}}. \end{tikzcd}$$

This represents a permanent cycle in $E_2^{1,1+(2p^{n-1}-1)\rho}$ of the slice SS, and

$$f_n = a^{qp^{n-1}-1}\overline{y}_n \in E_2^{2qp^{n-1}-1,2p^n}$$

is on the vanishing line.

We also have maps

$$\begin{tikzcd} S^{2(p^n-1)\rho} \arrow[r, "N_1^p(v_n)"] \arrow[rr, bend right, "\overline{x}_n"'] & N_1^p(BP) \arrow[r, "f"] & BP_{\mathbf{R}} \end{tikzcd}$$

representing a permanent cycle in $E_2^{0,2(p^n-1)\rho}$ with

$$e_n = a^{q(p^n-1)}\overline{x}_n \in E_2^{2q(p^n-1),2p(p^n-1)}$$

also on the vanishing line.

The Slice Differentials Theorem for $G = C_p$ says

$$\begin{array}{rrcl} & d_{2p^n-1}(u^{p^{n-1}}) & = & a^{p^{n-1}}f_n \\ \text{and} & d_{2q(p^n-1)+1}(u^{qp^{n-1}}f_n) & = & a^{qp^{n-1}}e_n. \end{array}$$

For a finite cyclic p-group G of order g, let $G' = G/C_p$ with order $g' = g/p$. We need to replace $\overline{x}_n$ and $\overline{v}_n$ by their norms. The former is an element in $\pi^G_{2(p^n-1)\rho_G} N_p^g(BP_{\mathbf{R}})$ that we will denote by $\overline{x}_n^G$. The corresponding element on the vanishing line is

$$e_n^G = a_{2(p^n-1)\overline{\rho}_G}\overline{x}_n^G \in E_2^{2(p^n-1)(g-1),2(p^n-1)g}.$$

It lies on the line through the origin of slope $g-1$.

For the latter we need to determine $X = N_p^g\left(\overline{S}^{2p^{n-1}\rho-1}\right)$. It is underlain by the g'-fold smash power of the wedge of q copies of S^{2p^n-2}. The number of wedge summands here will be congruent to -1 modulo g. This means it is the wedge of some free G-spheres, which we will ignore, and a wedge of $g-1$ copies of $S^{2g'(p^n-1)}$. Since

$$\left(\overline{S}^{2p^{n-1}\rho-1}\right)^{C_p} = S^{2p^{n-1}-1},$$

it follows that

$$X^{C_p} = N_p^g\left(S^{2p^{n-1}-1}\right) = S^{(2p^{n-1}-1)\rho_{g'}},$$

which is a G'-spectrum. It reveals all of the fixed point data about X showing that (modulo free summands) it has the form

$$\overline{S}^{(2p^{n-1}-1)\rho_{G'}+m\lambda(g)}$$

for some m. Equating dimensions of the above and $S^{2g'(p^n-1)}$, we get

$$\begin{array}{rcl} 2(p^n-1)g'+1 & = & (2p^{n-1}-1)g'+2m \\ m & = & (qp^{n-1}-1)g'+(g'+1)/2. \end{array}$$

With this in mind we define $\overline{y}_n^G \in \pi^G_{(2p^{n-1}-1)\rho_{G'}+(m-1)\lambda(g)} N_p^g(BP_{\mathbf{R}})$ to be the composite

$$\begin{tikzcd}
S^{(2p^{n-1}-1)\rho_{G'}+(m-1)\lambda(g)} \arrow[r] & \overline{S}^{(2p^{n-1}-1)\rho_{G'}+m\lambda(g)} \arrow[d] & \\
& N_p^g\left(\overline{S}^{2p^{n-1}\rho-1}\right) \arrow[r, "N_p^g(\overline{v}_n)"] & N_p^g(BP_{\mathbf{R}}).
\end{tikzcd}$$

The corresponding element on the vanishing line is

$$\begin{aligned} f_n^G &= a_{(2p^{n-1}-1)\overline{\rho}_{G'}+(m-1)\lambda(g)} \overline{y}_n^G \\ &\in E_2^{2m-1+(2p^{n-1}-1)(g'-1),2m-1+(2p^{n-1}-1)g'}. \end{aligned}$$

This lies on the line with slope $g-1$ and vertical intercept $(q-1)g'$.

Slice Differentials Theorem. *Let G be a finite cyclic p-group of order g. In the slice SS for $N_p^g(BP_{\mathbf{R}})$ there are differentials*

$$\begin{aligned} d_{2(p^n-1)g'+1}(u^{p^{n-1}}) &= a^{p^{n-1}} f_n^G \\ &= a^{p^{n-1}} a_{(2p^{n-1}-1)\overline{\rho}_{G'}+(m-1)\lambda(g)} \overline{y}_n^G \\ \textit{and}\qquad d_{2q(p^n-1)g'+1}(u^{qp^{n-1}} f_n^G) &= a^{qp^{n-1}} e_n^G \\ &= a^{qp^{n-1}} a_{2(p^n-1)\overline{\rho}_G} \overline{x}_n^G \end{aligned}$$

for each $n \geq 1$, where m, $\overline{y}_n^G$ and $\overline{x}_n^G$ are as above, $a = a_{\lambda(p)}$ and $u = u_{\lambda(p)}$.

In particular inverting $\overline{x}_n^G$ kills $a^{qp^{n-1}} a_{2(p^n-1)\overline{\rho}_G}$ and makes u^{p^n} a permanent cycle.

Now we specialize to the case $p = 3$ and $G = C_9$. In order to detect our θ_js, we need to invert $\overline{x}_1^{C_9}$ and $N_3^9(\overline{x}_3^{C_3})$. Inverting the former makes $u^3_{\lambda(3)}$ a permanent cycle.

Inverting the latter makes a permanent cycle out of

$$\begin{aligned} N_3^9(u^{27}_{\lambda(3)}) &= u^{27}_{3\lambda(9)-2\rho_3} \\ &= u^{27}_{3\lambda(9)-2\lambda(3)} \\ &= u^{27}_{\rho_9-3\lambda(3)} \qquad \text{because } \rho_9 = 1+\lambda(3)+3\lambda(9). \end{aligned}$$

Since $u_{3\lambda(3)} = u^3_{\lambda(3)}$ is a permanent cycle, so is $u^{27}_{\rho_9}$. Then the periodicity we want is given by

$$(u_{4\rho_9}\overline{x}_1^{C_9})^{27} \in \pi_{972} N_3^9(BP_{\mathbf{R}}).$$

6. Toward a reduction theorem for odd primes

We follow the proof in HHR. The statement we need to prove is that

$$MU^{(p)} \underset{A_0 \wedge A}{\wedge} S^0 = H\mathbf{Z}_{(p)}.$$

It is complicated by the fact that we do not have a precise handle on A. We only know it is an A_1-algebra satisfying $A \underset{A_1}{\wedge} S^0 = A_2$, where

$$\begin{aligned} A_0 &= S^0 \left[G_+ \wedge \bigvee_{\substack{k>0 \\ k\neq p^n-1}} S^{2k} \right] = N_1^p S^0 \left[\bigvee_{\substack{k>0 \\ k\neq p^n-1}} S^{2k} \right] \\ A_1 &= \bigwedge_{n>0} \bigvee_{k\geq 0} S^{k(2p^{n-1}\rho-2)} = \bigwedge_{n>0} S^0 \left[S^{2p^{n-1}\rho-2} \right] \\ &= S^0 \left[\bigvee_{n>0} S^{2p^{n-1}\rho-2} \right], \end{aligned}$$

which we map to $BP^{(p)}$ using the generators $\overline{v}_n$. Let

$$E = BP^{(p)} \underset{A_1}{\wedge} S^0$$

be the C_p-spectrum obtained from $BP^{(p)}$ by killing the $\overline{v}_n$. It appears to be underlain by $BP^{(p-1)}$ and to have H/p as its geometric fixed points. Its slice spectral sequence is depicted on 5/25/11.

Let

$$A_2 = S^0 \left[\bigvee_{n>0} S^{\tau_n} \right]$$

where

(13) $\tau_n = 2p^{n-1} - 1 + (2qp^{n-1} - 1)\lambda/2 = (p-2)\lambda/2 + (2p^{n-1} - 1)\rho \in RO_{1/2}(G).$

Note that for $p = 2$, τ_n as defined above is $(2^n - 1)\rho \in RO(G)$. There is a map

(14) $$\bar{t}_n : S^{\tau_n} \to E$$

refining $\pi^u_{2p^n-2}E$.

We can still define the analogs of the auxiliary spectra $R(k)$ by

(15) $$R(k) = MU^{(p)} \underset{A_0 \wedge A}{\wedge} (A'_0 \wedge A')$$

where

$$A'_0 = N_1^p \left(\bigvee_{\substack{\ell>k \\ k\neq p^n-1}} S^{2\ell} \right)$$

and A' is a similar modification of A.

We know that the C_p-map

$$R(\infty) \to H\mathbf{Z}_{(p)}$$

is underlain by an equiavalence, so it suffices to so show that the corresponding map h of geometric fixed points is also an equivalence. As in the 2-primary case, we have

$$\pi_* \Phi^G H\mathbf{Z}_{(p)} = \mathbf{Z}/(p)[b] \qquad \text{where } b = a^{-1}u$$

For the moment, we will ignore the technicalities about the monoidal geometric fixed point functor and cofibrant replacements. Hopefully the following are true.

$$\begin{aligned}
\Phi^G MU^{(p)} &= MU \\
\Phi^G A_0 &= S^0\left[\bigvee_{\substack{k>0\\ k\neq p^n-1}} S^{2k}\right] \\
\Phi^G A_1 &= S^0\left[\bigvee_{n>0} S^{2p^{n-1}-2}\right] \\
\Phi^G A_2 &= \bigwedge_{n>0}\left(\left(S^0\vee S^{2p^{n-1}-1}\right)\wedge S^0\left[S^{2p^n-2}\right]\right)
\end{aligned}$$

It should follow that

$$\begin{aligned}
\Phi^G(MU^{(p)} \underset{A_0}{\wedge} S^0) &= MU \underset{\Phi^G A_0}{\wedge} S^0 \\
&= BP \\
\Phi^G(MU^{(p)} \underset{A_0\wedge A_1}{\wedge} S^0) &= BP \underset{\Phi^G A_1}{\wedge} S^0 \\
&= H/p \\
\Phi^G R(\infty) &= \Phi^G(MU^{(p)} \underset{A_0\wedge A}{\wedge} S^0) \\
&= H/p \underset{\Phi^G A_2}{\wedge} S^0 \\
&\overset{?}{=} H/p\wedge \bigwedge_{n>0}\left(\bigvee_{0\le j<p} S^{2jp^{n-1}}\right) \\
&= \bigvee_{k\ge 0}\Sigma^{2k}H/p.
\end{aligned}$$

The penultimate equality above depends on having an A_∞-structure on A_2, which is produced in the note of June 1

Hence the source and target of

$$h : \Phi^G R(\infty) \to \Phi^G H\mathbf{Z}_{(p)} \tag{16}$$

are equivalent as in the 2-primary case (Prop. 7.5 of HHR).

However we cannot form analogs of the spectra $MU^{(G)}/G\cdot\overline{r}_{2^n-1}$ used in Lemma 7.7, because of the twisted nature of A. To get around this we start with the spectrum

$$E = MU^{(p)} \underset{A_0\wedge A_1}{\wedge} S^0$$

instead. It is underlain by $BP^{(p-1)}$ and $\Phi^G E = H/p$. Unfortunately it is not known to have a good multiplicative structure. The same is true of the spectrum

$$\tilde{E} = MU^{(p)} \underset{A_1}{\wedge} S^0,$$

for which $E = \tilde{E} \underset{A_0}{\wedge} S^0$ and whose geometric fixed point set is

$$\Phi^G\tilde{E} = H/p\wedge A_0.$$

The reduction theorem is the statement that the map

$$\tilde{E} \underset{A_0 \wedge A_2}{\wedge} S^0 = E \underset{A_2}{\wedge} S^0 \to H\mathbf{Z}_{(p)}$$

is an equivariant equivalence.

Let

$$A_{2,n} = S^0\left[S^{\tau_n}\right] \qquad \text{where } \tau_n \text{ is as in (13);}$$

we know that modulo free summands this is (nonmultiplicatively) equivalent to

$$\left(S^0 \vee S^{\tau_n}\right) \wedge S^0\left[S^{2(p^n-1)\rho}\right].$$

The generator of $\pi^G_{2(p^n-1)\rho} A_{2,n}$ maps to $\overline{x}_n = N(v_n)$.

Then let

$$M_n = \left(MU^{(p)} \underset{A_0 \wedge A_1}{\wedge} S^0\right) \underset{A_{2,n}}{\wedge} S^0, \tag{17}$$

so we have

$$\Phi^G M_n = \bigvee_{0 \le j < p} \Sigma^{2jp^{n-1}} H/p.$$

Lemma 6.1. Analog of Lemma 7.7 of HHR. *If for every $n > 0$ and $0 < j < p$, the class $b^{jp^{n-1}}$ is in the image of*

$$\pi_{2jp^{n-1}} \Phi^G M_n \to \pi_{2jp^{n-1}} \Phi^G H\mathbf{Z}_{(p)},$$

then the map $\pi_*(h)$ of (16) is surjective and hence an isomorphism.

Proof. We argue as in HHR using that fact that

$$R(\infty) = M_1 \underset{MU^{(p)}}{\wedge} M_2 \underset{MU^{(p)}}{\wedge} \cdots$$

to show that all powers of b are in the image of (16). $\square$

Proposition 6.2. Analog of Prop. 7.10 of HHR. *For every $n > 0$ and $0 < j < p$, the class $b^{jp^{n-1}}$ is in the image of*

$$\pi_{2jp^{n-1}} \Phi^G M_n \to \pi_{2jp^{n-1}} \Phi^G H\mathbf{Z}_{(p)}.$$

We will use the M_n of (17) to mimic the diagram (7.11) in HHR, so we have

$$\begin{tikzcd}
& S^{\tau_n} \arrow[d, dotted, "y_n"] \arrow[ddl, dotted, "\tilde{y}_n"'] \arrow[dr, "\overline{t}_n"] \arrow[drr, "0"] \arrow[ddr, "0"'] & & \\
& EG_+ \wedge E \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \tilde{E}G \wedge E \arrow[d] \\
\Sigma^{-1}\tilde{E}G \wedge M_n \arrow[r] \arrow[d] & EG_+ \wedge M_n \arrow[r] \arrow[d] & M_n \arrow[r] \arrow[d] & \tilde{E}G \wedge M_n \arrow[d] \\
\Sigma^{-1}\tilde{E}G \wedge H\mathbf{Z}_{(p)} \arrow[r] & EG_+ \wedge H\mathbf{Z}_{(p)} \arrow[r] & H\mathbf{Z}_{(p)} \arrow[r] & \tilde{E}G \wedge H\mathbf{Z}_{(p)}
\end{tikzcd} \tag{18}$$

The map $\overline{t}_n$ of (14) has trivial composition with the map $E \to M_n$ by construction. Its composition with $E \to \tilde{E}G \wedge E$ is trivial because $\Phi^G E = H/p$. Hence the indicated liftings to $EG_+ \wedge E$ and $\Sigma^{-1}\tilde{E}G \wedge M_n$ exist. There is a similar diagram with E replaced by $R(p^n - 2)$ as defined in (15).

We will assume for now (HHR 7.12) that the composite of y_n with the map to $EG_+ \wedge H\mathbf{Z}_{(p)}$ is essential. This implies the same about the composite of $\tilde{y}_n$ with the map to $\Sigma^{-1}\tilde{E}G$. Then we have a diagram

$$\begin{tikzcd}
\pi^G_{\tau_n+1}\tilde{E}G \wedge M_n \arrow[r, equal] \arrow[d] & \pi_{2p^{n-1}}\tilde{E}G \wedge M_n \arrow[d] \\
\pi^G_{\tau_n+1}\tilde{E}G \wedge H\mathbf{Z}_{(p)} \arrow[r, equal] & \pi_{2p^{n-1}}\tilde{E}G \wedge H\mathbf{Z}_{(p)}
\end{tikzcd}$$

so $\tilde{y}_n$ maps to a nontrivial element of

$$\pi_{2p^{n-1}}\tilde{E}G \wedge H\mathbf{Z}_{(p)} = \pi_{2p^{n-1}}\Phi^G H\mathbf{Z}_{(p)} = \mathbf{Z}/(p),$$

namely a unit multiple of $b^{p^{n-1}}$.

To get the required jth power (for $0 < j < p$) of $b^{p^{n-1}}$, recall that

$$EG_+ \wedge EG_+ \simeq EG_+ \qquad \text{and} \qquad \tilde{E}G \wedge \tilde{E}G \simeq \tilde{E}G.$$

For brevity let

$$X = \tilde{E}G \qquad \text{and} \qquad Y = EG_+.$$

From (18) we have a diagram

$$\begin{tikzcd}
 & S^{\tau_n} \arrow[dl, "\tilde{z}_n"'] \arrow[dr, "z_n"] & \\
\Sigma^{-1}X \wedge H\mathbf{Z}_{(p)} \arrow[rr] & & Y \wedge H\mathbf{Z}_{(p)}
\end{tikzcd}$$

Taking j-fold smash products and using the multiplication on $H\mathbf{Z}_{(p)}$ gives

(19)
$$\begin{tikzcd}
 & S^{j\tau_n} \arrow[dl, "\tilde{z}_n^{(j)}"'] \arrow[dr, "z_n^{(j)}"] \arrow[ddl, bend right, "\tilde{z}_{n,j}"] \arrow[ddr, bend left, "z_{n,j}"'] & \\
\Sigma^{-j}X^{(j)} \wedge H\mathbf{Z}_{(p)}^{(j)} \arrow[rr] \arrow[d, equal] & & Y^{(j)} \wedge H\mathbf{Z}_{(p)}^{(j)} \arrow[d, equal] \\
\Sigma^{-j}X \wedge H\mathbf{Z}_{(p)}^{(j)} \arrow[rr] \arrow[d] & & Y \wedge H\mathbf{Z}_{(p)}^{(j)} \arrow[d] \\
\Sigma^{-j}X \wedge H\mathbf{Z}_{(p)} \arrow[rr] & & Y \wedge H\mathbf{Z}_{(p)}
\end{tikzcd}$$

Then if $z_{n,j}$ is essential, it follows that $\Phi^G\tilde{z}_{n,j}$ is $b^{jp^{n-1}}$. Thus it remains to prove

Proposition 6.3. Analog of HHR Prop. 7.12. *The map $z_{n,j}$ of (19) is essential for $0 < j < p$ and $n > 0$.*

Following HHR we use the spectrum $R(p^n-2)$. We will denote its jth smash power by R_j for short. We have maps

$$\begin{tikzcd}
 & S^{\tau_n} \arrow[dl, dotted, "y_n"'] \arrow[d, "\bar{t}_n"] \\
EG_+ \wedge R(p^n-2) \arrow[r] & R(p^n-2)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & S^{j\tau_n} \arrow[dl, dotted, "y_n^{(j)}"'] \arrow[d, "\bar{t}_n^{(j)}"] \arrow[ddl, dotted, "z_{n,j}"] \\
EG_+ \wedge R_j \arrow[r] \arrow[d] & R_j \\
EG_+ \wedge H\mathbf{Z}_{(p)}. &
\end{tikzcd}$$

We need to show that any choice of y_n leads a nontrivial $z_{n,j}$. It suffices to do this for $j=p-1$. Note that since $G=C_p$, the G-equivariant homotopy type of $EG_+ \wedge X$ is determined by the ordinary homotopy type of X for any G-equivariant spectrum X. We will study it by examining the smash product of EG_+ with the ordinary (nonequivariant) Postnikov tower for X.

Lemma 6.4. Analog of HHR Lemma 7.14. *For $0<m<2p^n-2$ and $0<j<p$,*

$$\pi^G_{j\tau_n} EG_+ \wedge P^m_m R_j = 0$$

and there is an exact sequence

$$\begin{tikzcd}
\pi^G_{j\tau_n} EG_+ \wedge P_{2p^n-2} R_j \arrow[r] & \pi^G_{j\tau_n} EG_+ \wedge R_j \arrow[d] \\
 & \pi^G_{j\tau_n} EG_+ \wedge H\mathbf{Z}_{(p)} = \mathbf{Z}/(p).
\end{tikzcd}$$

Proof. We can identify the slices of R_j (with respect to the trivial group) as in HHR and derive the first assertion since the slices in question are contractible. The 0th slice is $H\mathbf{Z}_{(p)}$ and the first nontrivial positively indexed slice is $P^{2p^n-2}_{2p^n-2} R_j$. Hence

$$P_1 R_j = P_{2p^n-2} R_j.$$

The second assertion follows from the exact sequence of the smash product of EG_+ with the fibration

$$P_1 R_j \longrightarrow R_j \longrightarrow P^0_0 R_j.$$

$\square$

Thus we need to show that $y_n^{(j)}$ is not in the image of

$$\pi^G_{j\tau_n} EG_+ \wedge P_{2p^n-2} R_j.$$

What follows is still suspect for $j>1$.

Proposition 6.5. Analog of HHR Prop. 7.15. *The image of the vertical composite map of*

$$\begin{tikzcd}
 & \pi^G_{j\tau_n} EG_+ \wedge P_{2p^n-2} R_j \arrow[d] \\
y_n^{(j)} \in \pi^G_{j\tau_n} EG_+ \wedge R_j \arrow[r] & \pi^G_{j\tau_n} R_j \arrow[d, "i_0^*"] \\
 & \bigoplus_{p-1} \pi^u_{2j(p^n-1)} R_j
\end{tikzcd}$$

is contained in the image of $1-\gamma$ *(for a generator* γ *of* G*) while that of* $y_n^{(j)}$ *is not.*

The image of y_n generates a rank one summand of the indicated $\mathbf{Z}[G]$-module, which is

$$\mathrm{Hom}_{\mathbf{Z}}(I, \mathbf{Z} \oplus I) = I \oplus \mathbf{Z} \oplus \bigoplus_{p-2} \mathbf{Z}[G].$$

It jth smash power is a nontrivial fixed element of the jth tensor power of this module. Hence it is not in the image of $1-\gamma$.

To prove the first assertion of Proposition 6.5 we need two steps as in HHR. First it is shown (Corollary 6.8) that the image of

$$\pi^G_{j\tau_n} EG_+ \wedge P_{2p^n-2} R_j \to \pi^G_{j\tau_n} R_j$$

is contained in the image of the transfer map

$$\pi_{j\tau_n} R_j \to \pi^G_{j\tau_n} R_j$$

from the trivial subgroup of G. We then show (Lemma 6.9) that the image of the transfer map in $\pi^u_{j(2p^n-2)} R_j$ is in the image of $(1-\gamma)$. We now turn to these steps.

Lemma 6.6. Analog of HHR Lemma 7.16. *Let* $M \geq 0$ *be a* G*-spectrum. The image of*

$$\pi^G_0 EG_+ \wedge M \to \pi^G_0 M$$

is the image of the transfer map

$$\pi_0 M \to \pi^G_0 M$$

from the trivial subgroup of G*.*

Proof: Since M is (-1)-connected the cell decomposition of EG_+ implies that $\pi^G_0 G_+ \wedge M \to \pi^G_0 EG_+ \wedge M$ is surjective. The composite

$$\pi^G_0 G_+ \wedge M \to \pi^G_0 EG_+ \wedge M \to \pi^G_0 M$$

is the transfer. □

Corollary 6.7. Analog of HHR Cor. 7.17. *The image of*

$$\pi^G_{j\tau_n} EG_+ \wedge P_{2p^n-2} R_j \to \pi^G_{j\tau_n} P_{2p^n-2} R_j$$

is contained in the image of the transfer map.

Proof: This follows from Lemma 6.6 above, after the identification

$$\pi^G_{j\tau_n} \left(P_{2p^n-2} R(p^n-2)\right)^{(j)} \approx \pi^G_0 S^{-j\tau_n} \wedge \left(P_{2p^n-2} R(p^n-2)\right)^{(j)}$$

and the observation that

$$S^{-j\tau_n} \wedge \left(P_{2p^n-2} R(p^n-2)\right)^{(j)} \approx \left(P_0 \left(S^{-\tau_n} \wedge R(p^n-2)\right)\right)^{(j)}$$

is ≥ 0. □

Corollary 6.8. Analog of HHR Cor. 7.18. *The image of*

$$\pi^G_{\tau_n} EG_+ \wedge P_{2p^n-2} R(p^n-2) \to \pi^G_{\tau_n} R(p^n-2)$$

is contained in the image of the transfer map.

Proof: Immediate from Corollary 6.7 and the naturality of the transfer. □

The remaining step is the special case $X = P_{2p^n-2}R(p^n-2)$, $V = \tau_n$ of the next result.

Lemma 6.9. Analog of HHR Lemma 7.19. *Let X be a G-spectrum, V a virtual representation of G of virtual dimension d, and $H \subset G$ the subgroup of index 2. Write $\epsilon \in \{\pm 1\}$ for the degree of*

$$\gamma : i_0^* S^V \to i_0^* S^V.$$

The image of

$$\pi_V X \xrightarrow{\mathrm{Tr}} \pi_V^G X \to \pi_d^u X$$

is contained in the image of

$$(1+\epsilon\gamma) : \pi_d^u X \to \pi_d^u X.$$

Proof: Consider the diagram

$$\begin{tikzcd} \pi_V^G(G_+ \wedge X) \arrow[r] \arrow[d] & \pi_V^G X \arrow[d] \\ \pi_d^u(G_+ \wedge X) \arrow[r] & \pi_d^u X, \end{tikzcd}$$

in which the map of the top row is induced by the projection $G_+ \to S^0$. By the Wirthmüller isomorphism, the term in the upper left is isomorphic to $\pi_V X$ and the map of the top row can be identified with the transfer map. The non-equivariant identification

$$G_+ \approx S^0 \vee S^0$$

gives an isomorphism of groups of non-equivariant stable maps

$$[G_+ \wedge S^V, X] \approx [S^V, X] \oplus [S^V, X],$$

and so an isomorphism of the group in the lower left hand corner with

$$\pi_d^u X \oplus \pi_d^u X$$

under which the generator $\gamma \in G$ acts as

$$(a, b) \mapsto (\epsilon\gamma b, \epsilon\gamma a).$$

The map along the bottom is $(a, b) \mapsto a + b$. Now the image of the left vertical map is contained in the set of elements invariant under γ which, in turn, is contained in the set of elements of the form

$$(a, \epsilon\gamma a).$$

□

Proof of Proposition **??**: As described after its statement, Proposition **??** is a consequence of Corollary 6.8 and Lemma 6.9. □

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