

# THE 7-CONNECTED COBORDISM RING AT $p = 3$

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ABSTRACT. In this paper, we study the cobordism spectrum  $MO\langle 8 \rangle$  at the prime 3. This spectrum is important because it is conjectured to play the role for elliptic cohomology that Spin cobordism plays for real K-theory. We show that the torsion is all killed by 3, and that the Adams-Novikov spectral sequence collapses after only 2 differentials. Many of our methods apply more generally.

## INTRODUCTION

In algebraic topology, the complex cobordism spectrum  $MU$  is a sort of universal example of a well-behaved cohomology theory. Virtually every commonly studied theory admits an orientation from  $MU$ . The most significant exception is real K-theory,  $KO$ . Now elliptic cohomology is supposed to be a higher analog of  $KO$ . So one would not expect it to admit an orientation from  $MU$  either. In Witten's interpretation [Wit] of the level 1 elliptic genus as the index of the equivariant Dirac operator on the free loop space  $LM$  of a manifold  $M$ , one needs  $LM$  to be Spin. The easiest way to guarantee this is to take  $M$  to be a manifold such that the classifying map of its tangent bundle,  $M \rightarrow BO$ , lifts to  $BO\langle 8 \rangle$ , the 7-connected cover of  $BO$ . This indicates that whatever the final version of elliptic cohomology is, it should admit an orientation of  $MO\langle 8 \rangle$ , which is the Thom spectrum built from  $BO\langle 8 \rangle$ .

Now  $MO\langle 8 \rangle$  is not very well understood. It has torsion in its homotopy at the primes 2 and 3, and if we localize it by inverting 2 and 3, it splits into a wedge of suspensions of the Brown-Peterson spectrum  $BP$ . This means that the homotopy groups modulo torsion are easily computed, but the multiplicative structure is unknown. It is certainly not polynomial. There have been low-dimensional calculations of  $MO\langle 8 \rangle$  at the prime 2, the most recent of which is due to Gorbunov and Mahowald [GM]. However, at the prime 3 virtually nothing is known. We attempt to remedy that in this paper.

There is actually a candidate for level 1 elliptic cohomology at each of the primes 2 and 3. In each case, the spectrum involved is called  $EO_2$ , and they are special cases

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of a more general construction due to Hopkins and Miller [HM]. Their homotopy groups are completely known, and much of the algebra structure is known as well. But at the moment, there is no solid evidence relating them to elliptic cohomology. It would strengthen the case considerably if there was an orientation  $MO\langle 8 \rangle \rightarrow EO_2$ . This is another reason to try to understand  $MO\langle 8 \rangle$ . In fact, we show in this paper that  $MO\langle 8 \rangle$  behaves very similarly to  $EO_2$ . Hopkins and Mahowald (personal communication) have constructed a connective, uncompleted version of  $EO_2$ , called  $eo(2)$ , at  $p = 3$ . The conjecture is that  $MO\langle 8 \rangle$  at  $p = 3$  should be an amalgam of  $BP$  and  $eo(2)$  just as  $MSU$  at  $p = 2$  is an amalgam of  $BP$  and  $ko$  [Pen].

There is a standard strategy to try to compute the homotopy of a Thom spectrum such as  $MO\langle 8 \rangle$ . First one computes the cohomology of the base, in our case  $BO\langle 8 \rangle$ , as a module over the Steenrod algebra  $\mathcal{A}$ . Here we would take mod 3 cohomology, so use the mod 3 Steenrod algebra. Then one computes the operations on the Thom class, or equivalently, the homomorphism  $\mathcal{A} \rightarrow H^*MO\langle 8 \rangle$ . These two things together give the structure of  $H^*MO\langle 8 \rangle$  as a module over  $\mathcal{A}$ , and one then applies the Adams spectral sequence to get at the homotopy.

The main problem with this strategy for  $MO\langle 8 \rangle$  at  $p = 3$  is in calculating the  $\mathcal{A}$ -module structure on  $H^*BO\langle 8 \rangle$ . One certainly knows the structure of  $H^*BSO$ , and there is a fibration

$$K(\mathbf{Z}, 3) \rightarrow BO\langle 8 \rangle \rightarrow BSO$$

but the Serre spectral sequence only gives limited information about the  $\mathcal{A}$ -action. There are many  $\mathcal{A}$ -extensions that are hard to compute. So we need a different method. The method we use is based on Hopf rings. It turns out that  $BO\langle 8 \rangle$  localized at 3 is homotopy equivalent to  $\mathbf{BP}\langle \mathbf{1} \rangle_8$ , the 8th space in the  $\Omega$ -spectrum for  $BP\langle \mathbf{1} \rangle$ . One can then use the results in [Wil, RW] to calculate the  $\mathcal{A}$ -action.

This approach also sheds light on the homomorphism  $\mathcal{A} \rightarrow H^*MO\langle 8 \rangle$  defined by the Thom class. First, notice that this homomorphism is the map induced on cohomology by

$$MO\langle 8 \rangle \rightarrow MSO \rightarrow H\mathbf{F}_3.$$

At  $p = 3$ , this map factors through  $BP$ , so it will certainly kill the 2-sided ideal generated by the Bockstein. Denote by  $\mathcal{P}$  the subHopf algebra of  $\mathcal{A}$  generated by the reduced powers. Note that  $P^1$  in dimension 4 must go to 0 in  $H^*MO\langle 8 \rangle$  for dimensional reasons, so we have a map

$$\mathcal{P}/\mathcal{P}(P^1) \rightarrow H^*MO\langle 8 \rangle.$$

We show in the first section that this map is in fact injective. It turns out to be not very much harder to calculate the kernel of the corresponding map for all  $MO\langle k \rangle$  at odd primes, and all  $MU\langle k \rangle$ , so we do so. This calculation is originally due to Rosen, who used a different method in his unpublished thesis [Ros].

Now it follows from the general theory of  $\mathcal{A}$ -module coalgebras that Rosen's result puts some fairly tight restrictions on the  $\mathcal{A}$ -module structure of  $H^*MO\langle 8 \rangle$ . We also

know, in the particular case of  $MO\langle 8 \rangle$  at  $p = 3$ , that the cohomology is evenly graded. Let  $X$  denote the 8-skeleton of  $BP$ , which is a 3-cell complex which is  $P^1$ -free. Then these considerations lead to a proof that  $MO\langle 8 \rangle \wedge X$  is a wedge of suspensions of  $BP$ . Essentially we just have to put back in the  $P^1$  that  $H^*MO\langle 8 \rangle$  doesn't see.

This result has a number of nice corollaries about the global structure of the homotopy of  $MO\langle 8 \rangle$ . For example, we show that the 3-torsion is all killed by 3 itself. There are also elements in  $\pi_*MO\langle 8 \rangle$  analogous to  $v_1^k$ , for each  $k > 1$ , as constructed in [Hov]. We show that the 3-torsion coincides with the  $v_1$ -torsion, and that the  $v_1$ -torsion is all killed by any of the  $v_1$ -elements above. This is all contained in the second section.

Since  $MO\langle 8 \rangle \wedge X$  is a wedge of suspensions of  $BP$ , any  $X$ -resolution of the sphere becomes an Adams-Novikov resolution of  $MO\langle 8 \rangle$  upon smashing with  $MO\langle 8 \rangle$ . But there is a very simple  $X$ -resolution of the sphere, used by the second author in [Rav, Chapter 7]. The resulting Adams-Novikov resolution of  $MO\langle 8 \rangle$  puts severe restrictions on its Adams-Novikov spectral sequence. We find in the third section that the spectral sequence must collapse at  $E_{10}$  after at most  $d_5$  and  $d_9$ . This is precisely what happens in  $EO_2$ , and the pattern of differentials appears to be the same.

At this point, we must do some calculation to learn more. In the fourth section we calculate far enough to see that the Adams-Novikov spectral sequence does not collapse at  $E_6$ , so one really does need two differentials. Here we use the Adams spectral sequence to compute with. This is certainly not the best method for computing in  $MO\langle 8 \rangle$ . We have a better spectral sequence for doing so which we will describe in a future paper. Here we calculate through dimension 32. This calculation leads to several conjectures about the behavior further out. The main thing stopping us from proving these conjectures at this time is a more thorough understanding of the homology of  $MO\langle 8 \rangle$ .

Our methods do not apply to  $MO\langle k \rangle$  for  $k$  larger than 8, for two closely related reasons. Firstly, the homology of  $\mathbf{BP}\langle \mathbf{1} \rangle_n$  is, so far as we know, not known as a comodule over the dual Steenrod algebra when  $n$  is larger than  $2p + 2$ . Secondly, one expects torsion in the homology of  $MO\langle k \rangle$  for  $k > 8$  and in the homology of  $MU\langle k \rangle$  for  $k > 6$ . One might ask if there is a Thom spectrum mapping to  $MO\langle k \rangle$  for which these problems disappear. In fact, there is. One can build  $p$ -local Thom spectra over the spaces of the  $BP\langle r \rangle$  spectrum, and if  $r$  is large enough relative to the connectivity  $k$ , these problems do not arise. We then get bounded torsion results for these Thom spectra. This is explained in the last section.

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Let us fix notation we will use throughout the paper. We will mostly be working in the  $p$ -local category, whether we are dealing with spaces or spectra. When we need a notation for the  $p$ -localization of  $X$ , we will use  $X_{(p)}$ . If  $E$  is a ring spectrum,  $\mu : E \wedge E \rightarrow E$  will denote its multiplication, and  $\eta : S^0 \rightarrow E$  will denote its unit. We reserve the letter  $H$  for the mod  $p$  homology spectrum, so that  $H_*H$  is the dual Steenrod algebra.  $T : X \wedge Y \rightarrow Y \wedge X$  will denote the twist map.

### 1. ROSEN'S THEOREM

Our goal in this section is to determine which Steenrod reduced powers act trivially on the Thom class in  $H^0MO\langle k \rangle$ . First let us recall some well-known facts and notation.

The mod  $p$  Steenrod algebra will be denoted by  $\mathcal{A}$ , and its dual by  $\mathcal{A}_*$ . For  $p = 2$ ,  $\mathcal{A}$  is generated by the Steenrod squares  $Sq^{2^i}$ , and  $\mathcal{A}_*$  is a polynomial algebra

$$\mathcal{A}_* = P(\xi_1, \xi_2, \dots) \text{ with } |\xi_n| = 2^n - 1.$$

The diagonal is defined by

$$\Delta(\xi_n) = \sum_{i=0}^n \xi_{n-i}^{2^i} \otimes \xi_i.$$

There is a canonical anti-automorphism  $c$ , and we let  $\zeta_n$  denote  $c\xi_n$ , so that we have

$$\Delta(\zeta_n) = \sum_{i=0}^n \zeta_i \otimes \zeta_{n-i}^{2^i}.$$

As usual,  $A(n)$  denotes the subHopf algebra of  $\mathcal{A}$  generated by  $Sq^{2^i}$  for  $i \leq n$ , and it is convenient to let  $A(-1) = 0$ .

We denote by  $\mathcal{P}$  the quotient of  $\mathcal{A}$  by the 2-sided ideal generated by  $Sq^1$ .  $\mathcal{P}$  is also a Hopf algebra, and its dual  $\mathcal{P}_* = P(\xi_1^2, \xi_2^2, \dots)$ . We denote by  $P(n)$  the subHopf algebra of  $\mathcal{P}$  generated by  $Sq^{2^i}$  for  $1 \leq i \leq n+1$  and let  $P(-1) = 0$ .

For  $p$  odd, let us denote by  $\mathcal{P}$  the sub-Hopf algebra of the mod  $p$  Steenrod algebra generated by the reduced powers. Recall that  $\mathcal{P}^*$  is generated by the  $P^{p^n}$ , which have degree  $2p^n(p-1)$ . The dual  $\mathcal{P}_*$  is a polynomial algebra  $P(\xi_1, \xi_2, \dots)$  with  $|\xi_n| = 2(p^n - 1)$  and

$$\Delta(\xi_n) = \sum_{i=0}^n \xi_{n-i}^{p^i} \otimes \xi_i.$$

There is a canonical anti-automorphism  $c$  as at  $p = 2$  and we denote  $c\xi_n$  by  $\zeta_n$ . Then we have

$$\Delta\zeta_n = \sum_{i=0}^n \zeta_i \otimes \zeta_{n-i}^{p^i}.$$

Denote by  $P(n)$  the subHopf algebra of  $\mathcal{P}$  generated by  $P^{p^i}$  for  $i \leq n$ , and let  $P(-1) = 0$ . Notice that, for any prime  $p$ ,  $H^*BP = \mathcal{P}$ .

Let us recall the result of Bahri-Mahowald [BM]. Given an integer  $r$ , let  $\phi(r)$  denote the dimension of the  $r$ th nonzero homotopy group of  $BSO$ . An explicit formula for  $\phi(r)$  is as follows: write  $r = 4a + b$ , where  $0 \leq b \leq 3$ , and let  $\phi(r) = 8a + 2^b$ . Let  $f$  denote the map of ring spectra

$$f : MO\langle\phi(r)\rangle \rightarrow MO \rightarrow H\mathbf{F}_2.$$

Then Bahri and Mahowald show that the kernel of  $H^*(f)$  is the left ideal generated by the augmentation ideal of  $A(r-1)$ .

The goal of this section is to investigate the analogous question when  $p$  is odd, and also for  $MU\langle k \rangle$ . If  $k \geq 2$  and  $p$  is odd, let  $f$  denote the map of ( $p$ -local) ring spectra

$$f : MO\langle k \rangle \rightarrow MSO \rightarrow BP.$$

Similarly, at any prime, let  $f$  denote the map of ring spectra

$$f : MU\langle k \rangle \rightarrow MU \rightarrow BP.$$

**Theorem 1.1 (Rosen [Ros]).** (1) *Suppose  $p$  is odd and  $tq + 4 \leq k \leq (t+1)q$ . Then the kernel of  $H^*f : \mathcal{P} \rightarrow H^*MO\langle k \rangle$  is the left ideal generated by the augmentation ideal of  $P(t-1)$ .*

(2) *Suppose  $p$  is an arbitrary prime, and  $tq + 2 \leq k \leq (t+1)q$ . Then the kernel of  $H^*f : \mathcal{P} \rightarrow H^*MU\langle k \rangle$  is the left ideal generated by the augmentation ideal of  $P(t-1)$ .*

To prove this theorem, we will also need the equivalent dual statement: under the hypotheses above the image of  $H_*f : H_*MO\langle k \rangle \rightarrow \mathcal{P}_*$  is

$$P(\zeta_1^{p^t}, \zeta_2^{p^{t-1}}, \dots, \zeta_t^p, \zeta_{t+1}, \dots).$$

Rosen's proof of this theorem is similar to the proof of Bahri-Mahowald [BM]. Because Rosen's proof is unpublished, and because we need the formalism of our proof later, we present our proof here.

We will first prove the easy half, that the image is contained within the polynomial algebra above.

**Lemma 1.2.** (1) *Suppose  $p$  is odd, and  $k \geq tq + 2$ . Then  $P^{p^{t-1}}U = 0$ , where  $U$  is the Thom class in either  $H^0MO\langle k \rangle$  or in  $H^0MU\langle k \rangle$ .*

(2) *Suppose  $p = 2$ , and  $k \geq 2t + 2$ . Then  $Sq^{2^t}U = 0$ , where  $U$  is the Thom class in  $H^0MU\langle k \rangle$ .*

*Proof.* First assume  $p$  is odd. It suffices to prove that  $P^{p^{t-1}}U = 0$  in  $H^*MO\langle tq + 4 \rangle$ , since there is a map  $MU\langle tq + 2 \rangle \rightarrow MO\langle tq + 4 \rangle$  compatible with the Thom class. For this we use Giambalvo's calculation of  $H^*BO\langle k \rangle$  [Giam]. Given an integer  $n$ , let  $\alpha(n)$  denote the sum of the digits in the  $p$ -adic expansion of  $n$ . He shows that the image

of  $H^*BO$  in  $H^*BO\langle tq + 4 \rangle$  is a polynomial algebra on classes  $\Theta_i$  in degree  $4i$ , where  $\alpha(2i - 1) \geq (p - 1)t + 1$ . In particular, the image is 0 in positive degrees less than  $4p^t$ . By the Thom isomorphism theorem, the image of  $H^*MSO$  in  $H^*MO\langle tq + 4 \rangle$  is also 0 in positive degree less than  $4p^t$ . Since  $P^{p^t-1}U$  is in degree  $2p^t(p - 1) < 4p^t$ , it must be 0. A similar argument works when  $p = 2$  using the results of Stong [St].  $\square$

To prove the other half of Rosen's theorem, we use a completely different method. We will outline it here. We first point out that the space  $BO\langle k \rangle$  is the  $k$ th space in the  $\Omega$ -spectrum for connective real  $K$ -theory,  $ko$ . When localized at  $p$ ,  $ko$  splits as a wedge of  $(p - 1)/2$  shifted copies of  $BP\langle 1 \rangle$ . There is a corresponding decomposition of the  $p$ -localization of  $BO\langle k \rangle$  into a product of spaces in the  $\Omega$ -spectrum of  $BP\langle 1 \rangle$ . The homology of these spaces form a Hopf ring which is not completely understood, but there is a map from the homology Hopf ring of  $BP$ , which is completely understood. We then use the Hopf ring distributive law to deduce the rest of Rosen's theorem.

We first recall the  $n$ th space functor from spectra to spaces. Given a spectrum  $E$ , the functor  $E^n(-)$  on spaces is representable by the Brown representability theorem, so it has a representing space  $\mathbf{E}_n$ , the  $n$ th space of  $E$ . The  $n$ th space functor is a right adjoint to the functor that takes the space  $X$  to the  $n$ th suspension of its suspension spectrum. The main properties of this functor are summarized in the following proposition.

- Proposition 1.3.** (1) *If  $E$  is a CW spectrum, then  $\mathbf{E}_n$  has the homotopy type of a CW complex.*  
 (2) *The  $n$ th space functor takes cofiber sequences to fiber sequences, and locally finite wedges to products.*  
 (3)  $\Sigma\mathbf{E}_n \simeq \mathbf{E}_{n+1}$ .  
 (4) *If  $E$  is connective, then  $\mathbf{E}_n$  is  $(n - 1)$ -connected.*  
 (5) *The  $n$ th space functor commutes with  $p$ -localization.*

The proof of this proposition will be left to the reader, except for the first part, which can be found in [LMM, p. 52]. The second part is a consequence of the general facts that right adjoints preserve limits, and that many colimits in spectra are also limits. We will also comment on the last part. There is no reason that  $\mathbf{E}_n$  should be connected, but if we write  $\mathbf{E}_n = \pi_{-n}E \times E'$ , then  $E'$  is nilpotent since its fundamental group is abelian. We can therefore define the  $p$ -localization of  $\mathbf{E}_n$  to be  $\pi_{-n}(E_{(p)}) \times E'_{(p)}$ . But we will only need this for connected spaces.

Recall that  $\mathbf{Z} \times BU = \mathbf{ku}_0$  and  $\mathbf{Z} \times BO = \mathbf{ko}_0$  are the 0th spaces of the connective  $K$ -theory spectra. We need to identify some of the other spaces in these  $\Omega$ -spectra.

**Lemma 1.4.** *We have the following homotopy equivalences of  $H$ -spaces.*

- (1)  $BU\langle 2i \rangle \simeq \mathbf{ku}_{2i}$ .
- (2)  $BO\langle 8i \rangle \simeq \mathbf{ko}_{8i}$ .
- (3)  $BO\langle 4i \rangle_{(p)} \simeq (\mathbf{ko}_{(p)})_{4i}$ . (Recall  $p$  is odd here.)

*Proof.* Using the cofibre sequence

$$\Sigma^2 ku \xrightarrow{\times v} ku \rightarrow H\mathbf{Z}$$

we get a fibration of infinite loop spaces

$$\mathbf{ku}_{n+2} \rightarrow \mathbf{ku}_n \rightarrow K(\mathbf{Z}, n).$$

By induction, we have an H-space equivalence  $BU\langle 2n \rangle \simeq \mathbf{ku}_{2n}$ . Obstruction theory shows the composite

$$BU\langle 2n + 2 \rangle \rightarrow BU\langle 2n \rangle \rightarrow \mathbf{ku}_{2n}$$

lifts to an H-map

$$BU\langle 2n + 2 \rangle \rightarrow \mathbf{ku}_{2n+2}$$

which is an isomorphism on homotopy groups. The first part follows. The second and third parts are similar. Multiplication by  $v \in ku_2$  is replaced by multiplication by  $v \in ko_8$  and  $w \in ko_4$  respectively. The base of the fibration is no longer an Eilenberg-MacLane space, but the obstruction theory still works.  $\square$

Now we must recall the well-known  $p$ -local splittings of  $ku$  and  $ko$ . Recall the  $p$ -local spectrum  $BP\langle 1 \rangle$ , whose homotopy is  $BP\langle 1 \rangle_* = \mathbf{Z}_{(p)}[v_1]$ , where  $|v_1| = 2(p - 1)$ . There is an obvious ring homomorphism  $\pi_* BP\langle 1 \rangle \rightarrow \pi_* ku_{(p)}$  that takes  $v_1$  to  $v^{p-1}$ , and we then have  $\pi_* ku_{(p)} \simeq \pi_* BP\langle 1 \rangle[v]/(v^{p-1} - v_1)$ . We have a corresponding multiplicative splitting of spectra

$$ku_{(p)} \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} BP\langle 1 \rangle.$$

The multiplication on the right-hand side is defined as follows. If  $i + j < p - 1$ , we have

$$\Sigma^{2i} BP\langle 1 \rangle \wedge \Sigma^{2j} BP\langle 1 \rangle \xrightarrow{\mu} \Sigma^{2(i+j)} BP\langle 1 \rangle$$

and if  $i + j \geq p - 1$  we have

$$\Sigma^{2i} BP\langle 1 \rangle \wedge \Sigma^{2j} BP\langle 1 \rangle \xrightarrow{\mu} \Sigma^{2(i+j)} BP\langle 1 \rangle \xrightarrow{\times v_1} \Sigma^{2(i+j-p+1)} BP\langle 1 \rangle.$$

In particular, multiplication by  $v \in \pi_2 ku$  corresponds in the splitting to the identity map on the summands  $\Sigma^{2i} BP\langle 1 \rangle$  where  $0 < i < p - 1$  and takes the summand  $\Sigma^{2(p-1)} BP\langle 1 \rangle$  to  $BP\langle 1 \rangle$  by multiplication by  $v_1$ . Similar remarks hold for  $ko$  at odd primes  $p$ , except there are  $(p - 1)/2$  summands.

By the preceding lemma, we then get a product decomposition of  $BU\langle k \rangle_{(p)}$  and, if  $p$  is odd, of  $BO\langle k \rangle_{(p)}$ . We summarize in the following corollary, where we use the notation  $[v_1]$  for the map  $\mathbf{BP}\langle 1 \rangle_{i+q} \rightarrow \mathbf{BP}\langle 1 \rangle_i$  corresponding to multiplication by  $v_1$ .

**Corollary 1.5.** (1) *If  $k$  is even, there is a  $p$ -local decomposition of  $H$ -spaces*

$$BU\langle k \rangle_{(p)} \simeq \prod_{i=0}^{p-2} \mathbf{BP}\langle \mathbf{1} \rangle_{k+2i}.$$

*The map  $BU\langle k+2 \rangle \rightarrow BU\langle k \rangle$  corresponds to the identity map on the factors  $\mathbf{BP}\langle \mathbf{1} \rangle_{k+2i}$  when  $0 < i < p-1$  and to  $[v_1] : \mathbf{BP}\langle \mathbf{1} \rangle_{k+q} \rightarrow \mathbf{BP}\langle \mathbf{1} \rangle_k$  on the remaining factor.*

(2) *If  $k$  is divisible by 4 and  $p$  is odd, there is a  $p$ -local decomposition of  $H$ -spaces*

$$BO\langle k \rangle \simeq \prod_{i=0}^{(p-3)/2} \mathbf{BP}\langle \mathbf{1} \rangle_{k+4i}.$$

*The map  $BO\langle k+4 \rangle \rightarrow BO\langle k \rangle$  corresponds to the identity map on the factors  $\mathbf{BP}\langle \mathbf{1} \rangle_{k+4i}$  when  $0 < i < (p-1)/2$  and to  $[v_1] : \mathbf{BP}\langle \mathbf{1} \rangle_{k+q} \rightarrow \mathbf{BP}\langle \mathbf{1} \rangle_k$  on the remaining factor.*

Recall that we are trying to determine the image of the map

$$H_*MO\langle k \rangle \rightarrow H_*MSO \rightarrow H_*BP = \mathcal{P}_*.$$

By the Thom isomorphism theorem, it suffices to determine the image of the algebra map

$$H_*BO\langle k \rangle \rightarrow H_*BSO \rightarrow H_*MSO \rightarrow H_*BP.$$

However, by the above corollary, we have

$$H_*BSO = H_*BO\langle 4 \rangle = H_*\mathbf{BP}\langle \mathbf{1} \rangle_4 \otimes \cdots \otimes H_*\mathbf{BP}\langle \mathbf{1} \rangle_q.$$

There is a natural map  $H_*\mathbf{BP}_i \rightarrow H_*\mathbf{BP}\langle \mathbf{1} \rangle_i$ , which, by [Wil], is a surjection for  $i \leq 2q$ .

The Hopf algebra  $H_*\mathbf{BP}_k$  is computed by Ravenel-Wilson in [RW]. To describe it, we need to introduce some notation. First, given a class  $z \in \pi_i BP = BP^{-i}(*),$  there is a corresponding map  $* \rightarrow \mathbf{BP}_{-i}$ . If we choose a generator for  $\mathbf{F}_p = H_0(*),$  we get a class  $[z] \in H_0\mathbf{BP}_{-i}$ . From the canonical orientation in  $BP^2CP^\infty$  we get a map  $CP^\infty \rightarrow \mathbf{BP}_2$ . Let  $\beta_j \in H_{2j}CP^\infty$  be dual to  $x^j \in H^*CP^\infty = Z[x]$ . Define  $b_j \in H_{2j}\mathbf{BP}_2$  to be its image. These elements together generate  $H_*\mathbf{BP}_k$  in the following sense. There is a circle product

$$\mathbf{BP}_i \times \mathbf{BP}_j \xrightarrow{\circ} \mathbf{BP}_{i+j}$$

coming from the ring spectrum structure of  $BP$ . On the other hand, there is the usual product

$$\mathbf{BP}_k \times \mathbf{BP}_k \xrightarrow{*} \mathbf{BP}_k$$

coming from the infinite loop space structure on  $\mathbf{BP}_k$  that makes  $H_*\mathbf{BP}_k$  into a ring. Together, these structures make  $H_*\mathbf{BP}_*$  or  $H_*\mathbf{E}_*$  for any ring spectrum  $E$ , into a Hopf ring [RW].

One of the results of Ravenel-Wilson is that  $H_*\mathbf{BP}_k$  is a polynomial algebra on generators of the form  $[v^I] \circ b^J$ . Here  $I$  and  $J$  are finite sequences of nonnegative integers,  $[v^I] = [v_1^{i_1} v_2^{i_2} \cdots]$ , and  $b^J = b_{p^0}^{j_0} \circ b_{p^1}^{j_1} \circ \cdots$ . There are conditions on  $I$  and  $J$  which we will not state yet, as they become much simpler in  $H_*\mathbf{BP}\langle \mathbf{1} \rangle_k$ .

Under the map  $H_*\mathbf{BP}_k \rightarrow H_*\mathbf{BP}\langle \mathbf{1} \rangle_k$ , the elements  $[v_i]$  go to 0 for  $i > 1$ . Therefore, for  $k \leq 2q$ ,  $H_*\mathbf{BP}\langle \mathbf{1} \rangle_k$  is generated multiplicatively by elements  $[v_1^i] \circ b^J$ , where the conditions on  $i$  and  $J$  are that if  $i > 0$  then all elements  $j_n$  of  $J$  must be less than  $p$ , and that

$$2 \sum_n j_n - qi = k.$$

Now let us consider again the map

$$H_*BSO \rightarrow H_*MSO \rightarrow H_*BP.$$

The generator  $[v_1^i] \circ b^J \in H_*\mathbf{BP}\langle \mathbf{1} \rangle_k$  occurs in dimension  $\sum 2j_n p^n \cong \sum j_n \pmod{q}$ . Using the identity  $2 \sum j_n - qi = k$ , we find that the only generators occurring in dimensions divisible by  $q$  in  $H_*BSO$  come from the factor  $H_*\mathbf{BP}\langle \mathbf{1} \rangle_q$ . Therefore, all the other factors must go to 0 for dimensional reasons.

Putting the last few paragraphs together, we have proved the following lemma.

**Lemma 1.6.** (1) *If  $p$  is odd, and  $tq + 4 \leq k \leq (t + 1)q$ , the image of  $H_*MO\langle k \rangle$  in  $\mathcal{P}_*$  is the same as the image of the composite*

$$H_*\mathbf{BP}\langle \mathbf{1} \rangle_{tq} \xrightarrow{[v_1^{t-1}]} H_*\mathbf{BP}\langle \mathbf{1} \rangle_q \rightarrow H_*BSO \rightarrow H_*MSO \rightarrow H_*BP.$$

(2) *If  $p$  is arbitrary and  $tq + 2 \leq k \leq (t + 1)q$ , the image of  $H_*MU\langle k \rangle$  in  $\mathcal{P}_*$  is the same as the image of the composite*

$$H_*\mathbf{BP}\langle \mathbf{1} \rangle_{tq} \xrightarrow{[v_1^{t-1}]} H_*\mathbf{BP}\langle \mathbf{1} \rangle_q \rightarrow H_*BU \rightarrow H_*MU \rightarrow H_*BP.$$

We know from Lemma 1.2 that the image in question is contained in

$$P(\zeta_1^{p^{t-1}}, \zeta_2^{p^{t-2}}, \dots, \zeta_{t-1}^p, \zeta_t, \dots),$$

and we must now show that the image is that large.

Now the map  $H_*BSO \rightarrow H_*MSO \rightarrow H_*BP$  is onto, and since all the other factors go to 0, the map  $H_*\mathbf{BP}\langle \mathbf{1} \rangle_q \rightarrow H_*BP$  must be onto. There is only one generator in  $H_*\mathbf{BP}\langle \mathbf{1} \rangle_q$  in dimension  $2(p^n - 1)$ , namely  $x_n = [v_1^{n-1}] \circ b_1^{op-1} \circ \cdots \circ b_{p^{n-1}}^{op-1}$ . Thus, the image of  $x_n$  must be congruent modulo decomposables to  $\zeta_n \in \mathcal{P}_*$ .

It is evident that  $x_n$  is in the image of  $[v_1^{t-1}]$  when  $n \geq t$ . But we can use properties of Hopf rings to find more elements in the image. Suppose we have a Hopf ring  $B(*)$  defined over a ring  $R$ . Then each  $B(n)$  is a Hopf algebra over  $R$ , equipped with a counit  $\epsilon : B(n) \rightarrow R$ , and a unit for the  $*$ -product, which we denote  $[0_n]$ . Note that,

if our Hopf ring is of the form  $E_*\mathbf{G}_*$  for two ring spectra  $E$  and  $G$ , and if  $z \in \pi_*G$ , then  $\epsilon([z]) = 1$ . For  $x \in B(m)$ , we have

$$[0_n] \circ x = \epsilon(x)[0_{n+m}].$$

Denote by  $I(*)$  the kernel of  $\epsilon$ . Then  $I(*)$  is a Hopf ideal, which we call the augmentation ideal and typically denote by simply  $I$ . We will need to consider the Hopf ideal  $I^{*k}$  as well, which is the ideal consisting of sums of terms of the form  $x_1 * x_2 * \cdots * x_k$ , where each  $x_k$  is in  $I$ . To see that this is really a Hopf ideal, one needs the Hopf ring distributive law, which we will recall here. Given  $a \in B(n)$ , write

$$\Delta a = \sum a' \otimes a''.$$

Then

$$a \circ (b * c) = \sum \pm(a' \circ b) * (a'' \circ c).$$

Since we will only be considering elements in even degree in each  $B(n)$  and only for even  $n$ , we can ignore the signs. There is also a right distributive law of the same form. Finally, given a ring homomorphism  $R \rightarrow S$ , one can extend scalars to get a Hopf ring defined over  $S$ .

Now, form a power series  $b(s) = \sum b_i s^i$ . Here  $b_0 = [0_2]$ , and we think of  $b(s) \in H_*\mathbf{BP}\langle \mathbf{1} \rangle_2[[s]]$  as an element in a Hopf ring defined over  $Z[[s]]$  using the extension of scalars above. That is to say, the circle and star products on  $Z[[s]]$  are just the usual product. Now  $b(s)$  is a group-like element in the Hopf algebra  $H_*\mathbf{BP}\langle \mathbf{1} \rangle_2[[s]]$ , so that

$$\psi(b(s)) = b(s) \otimes b(s).$$

(This is an easy computation in  $H_*\mathbf{CP}^\infty$ .) In this circumstance, the Hopf ring distributive law reduces to the ordinary distributive law. So if  $x, y \in H_*\mathbf{BP}\langle \mathbf{1} \rangle_*$ , we have

$$(x * y) \circ b(s) = (x \circ b(s)) * (y \circ b(s)).$$

In particular,

$$y^{*p^k} \circ b(s) = (y \circ b(s))^{*p^k}.$$

If we take  $y$  to be constant and look at the coefficient of  $s^{p^i}$ , we find that

$$y^{*p^k} \circ b_{p^i} = (y \circ b_{p^{i-k}})^{*p^k}.$$

Here, if  $i < k$ ,  $y^{*p^k} \circ b_{p^i} = 0$ .

The other ingredient we need is the main relation of Ravenel-Wilson [RW]. Recall this says that if  $E$  and  $G$  are complex oriented ring spectra, then in  $E_*\mathbf{G}_*[[s]]$ , we have

$$b([p]_E(s)) = [p]_{[G]}(b(s)).$$

Here the series on the left is just  $b(ps) = [0_2]$  if  $E$  is mod  $p$  homology. But the  $p$ -series on the right is not the usual one—ordinary addition is replaced by the star product, and multiplication is replaced by the circle product.

**Lemma 1.7.** *We have*

$$[0_2] \equiv b(s)^{*p} * ([v_1] \circ b(s)^{\circ p}) \pmod{I^{*p^2}}.$$

Thus,

$$b_i^{*p} \equiv -[v_1] \circ b_i^{\circ p} \pmod{I^{*p+1}}.$$

*Proof.* Recall that  $[p](x) = px +_F v_1 x^p$  in the formal group law associated to  $BP\langle 1 \rangle$ . We therefore have

$$[p]_{BP\langle 1 \rangle}(b(s)) = b(s)^{*p} * ([v_1] \circ b(s)^{\circ p}) * \prod ([a_{kl}] \circ (b(s)^{*p})^{\circ k} \circ [v_1^l] \circ b(s)^{\circ pl}).$$

Note that each term in this  $*$ -product is congruent to  $[0_2]$  modulo  $I^{*p}$ , except  $[v_1] \circ b(s)^{\circ p}$ . Thus, using the main relation above, we have

$$[v_1] \circ b(s)^{\circ p} \equiv [0_2] \pmod{I^{*p}}.$$

Now, by the distributive law,  $(b(s)^{*p})^{\circ k} = (b(s)^{\circ k})^{*p^k}$ . Since each  $\circ$ -factor  $[a_{kl}]$ ,  $[v_1^l]$ , and  $b(s)$  is group-like, the distributive law gives

$$[a_{kl}] \circ (b(s)^{\circ k})^{*p^k} \circ [v_1^l] \circ b(s)^{\circ pl} = ([a_{kl}] \circ b(s)^{\circ k} \circ [v_1^l] \circ b(s)^{\circ pl})^{*p^k}.$$

But, since  $[v_1] \circ b(s)^{\circ p} \equiv [0_2] \pmod{I^{*p}}$ , the factor inside the parentheses is congruent to  $[0_2] \pmod{I^{*p}}$  as well. Since we are working in characteristic  $p$ , raising to either the  $p$ th  $*$ -power or the  $p$ th  $\circ$ -power is additive, and we find that

$$[a_{kl}] \circ (b(s)^{*p})^{\circ k} \circ [v_1^l] \circ b(s)^{\circ pl} \equiv [0_2] \pmod{I^{*p^{k+1}}}.$$

This completes the first part of the lemma.

The second part of the lemma is obtained by expanding the series in the first part. We have

$$\left( \sum b_i^{*p} s^{pi} \right) * \left( \sum [v_1] \circ b_i^{\circ p} s^{pi} \right) \equiv [0_2] \pmod{I^{*p^2}}.$$

If we work mod  $I^{*p+1}$  instead, all the cross terms in this  $*$ -product disappear, and we are left with

$$b_i^{*p} \equiv -[v_1] \circ b_i^{\circ p} \pmod{I^{*p+1}}. \quad \square$$

Note that, in dimension  $2p$ , there is no room for  $p + 1$ -fold  $*$ -decomposables in  $H_*\mathbf{BP}\langle 1 \rangle_2$ . Thus we have

$$b_1^{*p} = -[v_1] \circ b_1^{\circ p}.$$

Using this, we can now complete the proof of Rosen's theorem.

**Lemma 1.8.** *In the Hopf ring  $H_*\mathbf{BP}\langle 1 \rangle_*$  we have the following relations.*

(1)

$$b_1^{*p^n} = (-1)^n [v_1^n] \circ b_1^{\circ p} \circ b_p^{\circ p-1} \circ \dots \circ b_{p^{n-1}}^{\circ p-1}.$$

(2)

$$x_k^{*p^n} = [v_1^{n+k-1}] \circ b_1^{\circ p} \circ b_p^{\circ p-1} \circ \cdots \circ b_{p^{n-1}}^{\circ p-1} \circ b_{p^n}^{\circ p-2} \circ b_{p^{n+1}}^{\circ p-1} \circ \cdots \circ b_{p^{n+k-1}}^{\circ p-1}.$$

*Proof.* We proceed by induction. We have

$$b_1^{*p^n} = (b_1^{*p^{n-1}})^{*p} = ((-1)^{n-1} [v_1^{n-1}] \circ b_1^{\circ p} \circ b_p^{\circ p-1} \circ \cdots \circ b_{p^{n-2}}^{\circ p-1})^{*p}.$$

Using the distributive law one element at a time starting from the right, we get

$$b_1^{*p^n} = (-1)^{n-1} b_1^{*p} \circ b_p^{\circ p-1} \circ \cdots \circ b_{p^{n-1}}^{\circ p-1}.$$

Then applying the relation  $b_1^{*p} = -[v_1] \circ b_1^{\circ p}$  completes the proof of the first part.

For the second part, recall that

$$x_k = [v_1^{k-1}] \circ b_1^{\circ p-1} \circ \cdots \circ b_{p^{k-1}}^{\circ p-1}.$$

Because  $[v_1]$  is primitive, we find that

$$x_k^{*p^n} = [v_1^{k-1}] \circ (b_1^{\circ p-1} \circ \cdots \circ b_{p^{k-1}}^{\circ p-1})^{*p^n}.$$

Using the Hopf ring distributive law to remove as many factors from the right, one at a time as possible, we get

$$x_k^{*p^n} = [v_1^{k-1}] \circ (b_1^{*p^n}) \circ b_{p^n}^{\circ p-2} \circ b_{p^{n+1}}^{\circ p-1} \circ \cdots \circ b_{p^{n+k-1}}^{\circ p-1}.$$

Now the first part completes the proof.  $\square$

This lemma completes the proof of Rosen's theorem, since we know the image has the same rank as  $P(\zeta_1^{p^{t-1}}, \zeta_2^{p^{t-2}}, \dots, \zeta_{t-1}^p, \zeta_t, \dots)$ , within which it is contained.

## 2. CONSEQUENCES OF ROSEN'S THEOREM

In this section, we will see that Rosen's theorem implies that one can smash  $MO\langle 8 \rangle$  with a small finite spectrum  $X$  and get a wedge of suspensions of  $BP$ . This in turn gives bounds on the torsion and the  $v_1$ -torsion in the homotopy of  $MO\langle 8 \rangle$ . We begin with a more general theorem.

- Theorem 2.1.** (1) *Let  $R$  denote a  $p$ -local finite type connective commutative ring spectrum equipped with a map  $f : R \rightarrow H\mathbf{F}_p$  of ring spectra such that the kernel of  $H^*f : \mathcal{A} \rightarrow H^*R$  is  $\mathcal{A}(IA(n))$ . Let  $X$  denote a finite spectrum such that  $H^*X$  is a free  $A(n)$ -module. Then  $R \wedge X$  is a wedge of suspensions of  $H\mathbf{F}_p$ .*
- (2) *Let  $R$  denote a  $p$ -local finite type connective commutative ring spectrum equipped with a map  $f : R \rightarrow BP$  of ring spectra such that the kernel of  $H^*f : \mathcal{P} \rightarrow H^*R$  is  $\mathcal{P}(IP(n))$ . Suppose also that  $H^*R$  is evenly graded. Let  $X$  denote a finite spectrum such that  $H^*X$  is evenly graded and a free  $P(n)$ -module. Then  $R \wedge X$  is a wedge of suspensions of  $BP$ .*

Note that finite spectra  $X$  as in the above theorem do exist, by the results of Mitchell [Mit] for the  $A(n)$  case and J. Smith [Sm] for the  $P(n)$  case.

*Proof.* We will begin with the first part of the theorem, and we will show that  $H^*(R \wedge X) = H^*R \otimes H^*X$  is a free  $\mathcal{A}$ -module. To do this we use a characterization of free  $\mathcal{A}$ -modules, due to Adams-Margolis [AM] at  $p = 2$ , and Moore-Peterson [MP] at odd primes, but proved most cleanly by Miller-Wilkerson in [MW]. We begin at  $p = 2$ . Let  $P_t^s \in \mathcal{A}$  denote the dual of  $\xi_t^{2^s}$ . Then if  $s < t$ ,  $(P_t^s)^2 = 0$ , so we can take the  $P_t^s$ -homology group of an  $\mathcal{A}$ -module. At  $p$  odd, we have two kinds of differentials:  $P_t^s$  and  $Q_t$ , where  $P_t^s$  is as above and  $Q_t$  is the dual of  $\tau_t$ . We again have  $Q_t^2 = 0$ , but if  $s < t$ , we now have  $(P_t^s)^p = 0$ . We then define the  $P_t^s$  homology groups of an  $\mathcal{A}$ -module by taking the kernel of  $P_t^s$  modulo the image of  $(P_t^s)^{p-1}$ .

Miller and Wilkerson show that, if  $B$  is a subHopf algebra of  $\mathcal{A}$ , and  $M$  is a bounded below  $B$ -module, then  $M$  is free if and only if  $H(M, x) = 0$  for all differentials  $x \in B$ . In particular, for  $X$  as in the first part of the theorem, we have  $H(H^*X, x) = 0$  for all  $x \in A(n)$ . On the other hand, Margolis shows in [Mar, pp.356-358] that, under the hypotheses on  $R$  as in the first part of the theorem,  $H(H^*R, x) = 0$  for  $x \notin A(n)$ . Margolis restricts himself to the prime 2, but in fact his argument works for an arbitrary prime. The crucial step is Theorem 19.21 of [Mar]. Now in general there is no Kunneth theorem for  $x$ -homology, but there is a spectral sequence, so that  $H(H^*R \otimes H^*X, x) = 0$  for all  $x \in \mathcal{A}$ , so  $H^*(R \wedge X)$  is a free  $\mathcal{A}$ -module. Choosing generators gives a map to a wedge of suspensions of  $H\mathbf{F}_p$ , which is an isomorphism on mod  $p$  homology. Since  $R$  is finite type, we then get a homotopy equivalence as required.

We use a similar method to prove the second part of the theorem, where we replace  $\mathcal{A}$  by  $\mathcal{P}$ . The theorem of Miller and Wilkerson does not apply directly to  $\mathcal{P}$  at  $p = 2$ , since we defined  $\mathcal{P}$  as a quotient Hopf algebra rather than a sub Hopf algebra. But one can use the doubling isomorphism between the category of  $\mathcal{A}$ -modules and the category of evenly graded  $\mathcal{P}$ -modules. We get that a  $\mathcal{P}$ -module  $M$  is free over a sub Hopf algebra  $B$  of  $\mathcal{P}$  if and only if  $H(M, x) = 0$  for all differentials  $x \in B$ . Note that the differentials are the doubles of the old ones: that is, they are the  $P_t^{s+1}$  for  $s < t$ . Doubling the theorems of Margolis, we find that, if the kernel of  $H^*f : \mathcal{P} \rightarrow H^*R$  is  $\mathcal{P}(IP(n))$ , then  $H(H^*R, x) = 0$  for all differentials  $x \notin P(n)$ . Smashing with  $X$  then gives us a free  $\mathcal{P}$ -module, though not a free  $\mathcal{A}$ -module. At odd primes, of course, we do not need the doubling. In any case, choosing a generator gives a map to (a suspension of)  $H\mathbf{F}_p$ , which will lift to  $BP$  since  $R \wedge X$  is evenly graded. Then we get a mod  $p$  homology equivalence from  $R \wedge X$  to a wedge of suspensions of  $BP$  which is then a homotopy equivalence.  $\square$

We can apply the first part of this theorem directly to  $MO\langle k \rangle$  at  $p = 2$ , using the result of Bahri-Mahowald mentioned in the previous section [BM]. But to apply the second part of this theorem to  $MO\langle k \rangle$  at an odd prime or to  $MU\langle k \rangle$ , we have to

know the homology is evenly graded. Now  $H_*\mathbf{BP}_n$  is always evenly graded, and the map

$$H_*\mathbf{BP}_n \rightarrow H_*\mathbf{BP}\langle 1 \rangle_n$$

is onto for  $n \leq 2p + 2$  by [Wil]. Thus  $H_*MO\langle 8 \rangle$  is evenly graded for  $p$  odd, and  $H_*MU\langle 6 \rangle$  is evenly graded for arbitrary  $p$ .

In these small cases, we can find explicit models for the finite spectra  $X$  used in Theorem 2.1. Indeed, for  $p > 3$ , we can take  $X = S^0$  and we recover the fact that  $MO\langle 8 \rangle$  and  $MU\langle 6 \rangle$  split into a wedge of suspensions of  $BP$  when localized at such a prime. For  $p = 3$ , we have to find a finite spectrum which is free over  $P(0)$ , which is the subalgebra generated by  $P^1$ . Let  $Y$  denote the 8-skeleton of  $BP$ , a 3-cell complex where the 4-cell is attached to the 0-cell by  $\alpha_1$  and the 8-cell to the 4-cell by  $\alpha_1$ . In fact, the 8-cell is also attached to the 0-cell, but we will see that this attachment is irrelevant to us.  $Y$  is then obviously free over  $P(0)$ . At  $p = 2$ , we need to find a finite spectrum that is free over  $P(1)$ , which is the double of  $A(1)$ . Here one can double the construction of  $A(1)$  in [DM] to get a complex  $Z$  with 8 cells in dimensions 0 through 12. We have then proved the following corollary.

**Corollary 2.2.** *Let  $Y$  and  $Z$  denote the finite spectra above. Then:*

- (1)  $MO\langle 8 \rangle_{(3)} \wedge Y$  and  $MU\langle 6 \rangle_{(3)} \wedge Y$  are wedges of suspensions of  $BP$ .
- (2)  $MU\langle 6 \rangle_{(2)} \wedge Z$  is a wedge of suspensions of  $BP$ .
- (3)  $MU\langle 6 \rangle_{(p)}$  and  $MO\langle 8 \rangle_{(p)}$  are wedges of suspensions of  $BP$  when  $p > 3$ .

*In particular, for any  $p$ , the Bousfield class of  $MU\langle 6 \rangle_{(p)}$  and  $MO\langle 8 \rangle_{(p)}$  is the same as that of  $BP$ .*

The Bousfield class part of this corollary was previously conjectured by both of the authors [Hov]. It will be extended to all of the  $MU\langle k \rangle$  and the  $MO\langle k \rangle$  at odd primes in the last section of the paper.

**Theorem 2.3.** *The 3-torsion in  $\pi_*MO\langle 8 \rangle$  and in  $\pi_*MU\langle 6 \rangle$  is all killed by 3 itself. The 2-torsion in  $\pi_*MU\langle 6 \rangle$  is all killed by 16.*

*Proof.* The strategy in both cases is the same, and is due to Mike Hopkins. Let us first start with any ring spectrum  $R$  satisfying the hypotheses of part two of Theorem 2.1. We will show the torsion in  $\pi_*R$  is bounded. We can assume the bottom cell of the finite spectrum  $X$  in that theorem is in dimension 0. The resulting map  $R \rightarrow R \wedge X$  arising from inclusion of the bottom cell must kill all the torsion in  $\pi_*R$ , since  $R \wedge X$  is a wedge of suspensions of  $BP$ . On the other hand, let  $\bar{X} \xrightarrow{j} S^0$  denote the fiber of the inclusion of the bottom cell  $S^0 \rightarrow X$ . Then  $\bar{X}$  has cells in only odd degrees, and therefore  $[\bar{X}, S^0]$  is finite. Thus, there is some  $N$  such that  $p^N j$  is null. We then get a map back  $R \wedge X \rightarrow R$  such that the composite

$$R \rightarrow R \wedge X \rightarrow R$$

is multiplication by  $p^N$ . It follows that  $p^N$  kills all the torsion in  $\pi_*R$ .

To determine specific bounds, we must look at the individual spectrum  $X$ . For the 3-cell complex  $Y$ ,  $\bar{Y} \simeq \Sigma^3C(\alpha_1)$ . It is not too hard to see that

$$[\Sigma^3C(\alpha_1), S^0] = \mathbf{Z}/9$$

generated by a class  $x$  which is  $\alpha_1$  on the bottom cell. This shows that 9 kills the 3-torsion in  $\pi_*MO\langle 8 \rangle$  and in  $\pi_*MU\langle 6 \rangle$ . To see that 3 actually kills the torsion, note that  $3x$  is the composite

$$\Sigma^3C(\alpha_1) \rightarrow S^7 \xrightarrow{\alpha_2} S^0.$$

But we have shown in [Hov] that  $\alpha_2$ , and indeed everything in the image of  $J$  above dimension 3, goes to 0 in both  $\pi_*MU\langle 6 \rangle$  and in  $\pi_*MO\langle 8 \rangle$ . Hence  $3x$  will be 0 upon smashing with either  $MU\langle 6 \rangle$  or  $MO\langle 8 \rangle$ , so 3 will kill the 3-torsion.

Now consider the 2-local 8-cell spectrum  $Z$ . We must find the smallest  $k$  such that

$$\bar{Z} \rightarrow S^0 \xrightarrow{\eta} MU\langle 6 \rangle \xrightarrow{\times 2^k} MU\langle 6 \rangle$$

is null. By Spanier-Whitehead duality,  $[\bar{Z}, MU\langle 6 \rangle] \cong \pi_{11}(MU\langle 6 \rangle \wedge W)$  where  $H^*W$  is  $P(1)$  minus the top class ( $Sq^4Sq^4Sq^4$ ) as a  $P(1)$ -module.

The structure of  $H^*MU\langle 6 \rangle$  through dimension 12 can be computed using the Serre spectral sequences relating  $BSU$  to  $BU\langle 6 \rangle$  and  $BU\langle 6 \rangle$  to  $BU\langle 8 \rangle$ , or by the Hopf ring methods of the preceding section. We find 6 generators over  $\mathbf{F}_2$ : 1 in degree 0,  $x$  in degree 6,  $c_4$  in degree 8,  $y$  in degree 10,  $z$  and  $c_6$  in degree 12. Here  $Sq^8 1 = c_4$ ,  $Sq^2 x = c_4$ ,  $Sq^4 x = y$ ,  $Sq^4 c_4 = c_6$ , and  $Sq^2 y = z$ . All other Steenrod operations follow from these. One can then use Bruner's program [Br] for calculating Ext, or calculate by hand, to determine the  $E_2$ -term of the Adams spectral sequence for  $MU\langle 6 \rangle \wedge W$  through dimension 12. The hand calculation is not terribly difficult: the high point is that the  $Sq^8$  in  $MU\langle 6 \rangle$  means that in  $MU\langle 6 \rangle \wedge W$  one gets all of  $\mathcal{P}$  through dimension 12 except  $Sq^4Sq^4Sq^4$ . This means one must calculate the Ext groups of the extension

$$0 \rightarrow \Sigma^{12}\mathbf{F}_2 \rightarrow \mathcal{P} \rightarrow N \rightarrow 0$$

where the class in dimension 12 goes to  $Sq^4Sq^4Sq^4$ . One finds a  $Z/16$  in dimension 11 in  $Ext(N)$  starting in Adams filtration 1 on a class  $b$ .

This proves the theorem, but one might hope that an Adams differential from the 12-stem could lower the bound to 8. (There are no differentials from the 11-stem to the 10-stem because the 10-stem is all infinite  $h_0$ -towers.) But in fact, there are no differentials from the 12-stem to the 11-stem. Indeed, consider the cofibration

$$MU\langle 6 \rangle \wedge W \rightarrow MU\langle 6 \rangle \wedge X \rightarrow \Sigma^{12}MU\langle 6 \rangle$$

arising from putting in the missing top cell of  $P(1)$ .  $MU\langle 6 \rangle \wedge X$  is a wedge of suspensions of  $BP$ , and it is easy to see what its Adams spectral sequence looks like in dimension 12. It looks precisely the same as that of  $MU\langle 6 \rangle \wedge W$ , except it has a class corresponding to  $v_1^3v_2$  that is not in  $MU\langle 6 \rangle \wedge Z$ , and there is no class

$bh_1$ . The class  $bh_1$  is a permanent cycle since  $b$  is. If any other element supported a differential, we would necessarily have a class in  $\pi_*(MU\langle 6 \rangle \wedge X)$  that is not hit, yet a multiple of it is hit. This would lead to torsion in  $\pi_{12}\Sigma^{12}MU\langle 6 \rangle = \mathbf{Z}$ , so there are no differentials.  $\square$

We conjecture that the correct exponent for the 2-torsion in  $MU\langle 6 \rangle$  is in fact 8, though this method can only give 16.

We now give some similar results for  $v_1$ -torsion. Recall that an element  $x \in \pi_*X$  for  $X$  a  $p$ -local spectrum is said to be  $v_1$ -torsion if  $x$  maps to 0 under the natural map

$$X \rightarrow L_{K(1)}X.$$

(The failure of the telescope conjecture makes this definition problematic for higher  $v_n$ —see [MS].) Recall from [Hov] that an element in a ring spectrum  $R$  is called a  $v_1$ -element if

$$L_{K(1)}R = (v^{-1}R)_p.$$

Here the subscript denotes  $p$ -completion. Another way of saying this is to say that the  $K(1)$ -Hurewicz image of  $v$  is a (possibly fractional) power of  $v_1$ , and that the  $K(n)$ -Hurewicz image of  $v$  is nilpotent for  $n > 1$ . Given such a  $v_1$ -element  $v$ , we say that  $R$  has bounded  $v_1$ -torsion with respect to  $v$  if there is an  $N$  such that  $v^N x = 0$  for all  $v_1$ -torsion elements  $x$ .

Recall from [Hov] that there are  $v_1$ -elements in every  $MU\langle k \rangle$  and  $MO\langle k \rangle$ . There are  $v_1$ -elements in  $\pi_{4k}MO\langle 8 \rangle_{(3)}$  and  $\pi_{4k}MU\langle 6 \rangle_{(3)}$  for all  $k > 1$ . These elements induce the  $k$ th power of the Adams map on  $MO\langle 8 \rangle \wedge M(3)$  and on  $MU\langle 6 \rangle \wedge M(3)$ . There are also  $v_1$ -elements in  $\pi_{8k}MU\langle 6 \rangle_{(2)}$  for  $k > 0$ .

**Theorem 2.4.** *Localize all spectra at  $p = 3$ .*

- (1) *The 3-torsion in  $\pi_*MO\langle 8 \rangle$  coincides with the  $v_1$ -torsion in  $\pi_*MO\langle 8 \rangle$ . Similarly for  $MU\langle 6 \rangle$ .*
- (2) *Let  $v$  denote a  $v_1$ -element in  $\pi_*MO\langle 8 \rangle$  that induces a power of the Adams map on  $MO\langle 8 \rangle \wedge M(3)$ , such as those mentioned above. Then  $x \in \pi_*MO\langle 8 \rangle$  is  $v_1$ -torsion if and only if  $vx = 0$ . Similarly for  $MU\langle 6 \rangle$ .*

*Proof.* We again use the 3-cell complex  $Y$ , and we will just do the  $MO\langle 8 \rangle$  case. Since  $BP$  has no  $v_1$ -torsion, any  $v_1$ -torsion element in  $\pi_*MO\langle 8 \rangle$  must map to 0 under the map  $MO\langle 8 \rangle \rightarrow MO\langle 8 \rangle \wedge Y$ , and therefore must be 3-torsion. Conversely, since  $L_{K(1)}MO\langle 8 \rangle = L_{K(1)}MSO$  [Hov] is torsion-free, any 3-torsion element is also  $v_1$ -torsion. This proves the first part of the theorem.

For the second part, suppose  $x \in \pi_*MO\langle 8 \rangle$  is  $v_1$ -torsion. Then  $x$  is 3-torsion. Note that  $x$ , and all the 3-torsion, will survive the map  $MO\langle 8 \rangle \xrightarrow{i} MO\langle 8 \rangle \wedge M(3)$  since the 3-torsion is killed by 3. Let  $v \in \pi_{4k}MO\langle 8 \rangle$  denote a  $v_1$ -element which induces  $1 \wedge A^k$  on  $MO\langle 8 \rangle \wedge M(3)$ , where  $A$  is the Adams map and  $k$  is necessarily at least 2. It will suffice to prove that  $i(vx) = 0$ , or equivalently,  $(1 \wedge A^k)i(x) = 0$ .

We will show that the composite

$$\Sigma^3 C(\alpha_1) \rightarrow S^0 \rightarrow MO\langle 8 \rangle \wedge M(3) \xrightarrow{1 \wedge A} MO\langle 8 \rangle \wedge \Sigma^{-4} M(3) \xrightarrow{1 \wedge A} MO\langle 8 \rangle \wedge \Sigma^{-8} M(3)$$

is null. We will then get a map

$$MO\langle 8 \rangle \wedge M(3) \wedge Y \xrightarrow{r} MO\langle 8 \rangle \wedge \Sigma^{-8} M(3)$$

such that the composite

$$MO\langle 8 \rangle \wedge M(3) \rightarrow MO\langle 8 \rangle \wedge M(3) \wedge Y \xrightarrow{r} MO\langle 8 \rangle \wedge \Sigma^{-8} M(3)$$

is  $1 \wedge A^2$ . It then follows that  $(1 \wedge A^2)i(x) = 0$ .

Now

$$[\Sigma^3 C(\alpha_1), \Sigma^{-4} M(3)] = \mathbf{Z}/3 \oplus \mathbf{Z}/3.$$

The generators are a map which is  $\alpha_2$  on the bottom cell of  $\Sigma^3 C(\alpha_1)$ , and the composite

$$\Sigma^3 C(\alpha_1) \rightarrow S^7 \xrightarrow{\hat{\beta}_1} \Sigma^{-4} M(3),$$

where  $\hat{\beta}_1$  is  $\beta_1$  on the top cell of the Moore spectrum. Both these generators map to the same class upon smashing with  $MO\langle 8 \rangle$ , since  $\alpha_2$  goes to 0. The composite of  $\hat{\beta}_1$  with the Adams map is an element of  $\pi_{15} M(3)$ , which is generated by  $\alpha_4$ . It will then go to 0 in  $\pi_{15} MO\langle 8 \rangle \wedge M(3)$ , and we are done.  $\square$

We think that the  $v_1$ -torsion in  $\pi_* MU\langle 6 \rangle_{(2)}$  is also bounded, but this method cannot prove that. The 2-torsion and the  $v_1$ -torsion do not coincide in this case, though every  $v_1$ -torsion element is a 2-torsion element.

### 3. ADAMS-NOVIKOV RESOLUTIONS

In this section, we will use the results of the previous section to construct an economical Adams-Novikov resolution for  $MO\langle 8 \rangle_{(3)}$ . We will see that this resolution forces the Adams-Novikov spectral sequence to collapse after at most two differentials. This section is essentially a combination of the Russian approach to  $MSU$ , pioneered by Novikov [Nov] and extended by Botvinnik and Vershinin [Bot, Ver], with the last chapter of [Rav].

Let  $Y$  denote the 3-cell spectrum considered in the previous section. There is a very simple  $Y$ -resolution of the sphere, considered in [Rav, Section 7.4]:

$$\begin{array}{ccccccc} S^0 & \longleftarrow & \Sigma^3 C(\alpha_1) & \longleftarrow & S^{10} & \longleftarrow & \Sigma^{13} C(\alpha_1) & \longleftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ Y & & \Sigma^3 Y & & \Sigma^{10} Y & & \Sigma^{13} Y & & \end{array}$$

Since  $MO\langle 8 \rangle \wedge Y$  is a wedge of suspensions of  $BP$ , when we smash this resolution with  $MO\langle 8 \rangle$  we get an Adams-Novikov resolution of  $MO\langle 8 \rangle$ . We then get the following proposition.

**Proposition 3.1.** *There is a spectral sequence converging to  $\pi_*MO\langle 8\rangle_{(3)}$  which agrees with the Adams-Novikov spectral sequence from  $E_2$  onwards and whose  $E_1$ -term is*

$$\pi_*(MO\langle 8\rangle \wedge Y) \otimes E(\alpha) \otimes P(\beta).$$

Here  $\pi_*(R \wedge Y)$  is in filtration 0,  $\alpha$  is in bidegree  $(1, 4)$ , and  $\beta$  is in bidegree  $(2, 10)$ . The tensor product is taken over  $\mathbf{Z}_{(3)}$ , and  $\alpha$  and  $\beta$  have infinite additive order in  $E_1$ .

Note that in  $E_2$ ,  $\alpha$  corresponds to the usual  $\alpha_1$  and  $\beta$  corresponds to  $\beta_1$ . (Both  $\alpha$  and  $\beta$  are permanent cycles for dimensional reasons, and easy low dimensional computation shows that they survive to  $E_\infty$ .) In particular, in  $E_2$ , they both have order 3. This is not true yet in  $E_1$ . This means there must be a  $d_1$  starting on the 1-line that kills  $3\beta$ . In particular,  $d_1$  cannot be a derivation. One expects a multiplicative formula for  $d_1$  similar to a Bockstein spectral sequence, but we do not yet know if such a formula exists. We do not even know if  $MO\langle 8\rangle \wedge Y$  is a ring spectrum.

Note that we have the usual sparseness phenomenon in this Adams-Novikov spectral sequence, so that  $d_i$  is 0 unless  $i = 4k + 1$  for some  $k$ .

The proposition above has the following corollary.

**Corollary 3.2.** *In the Adams-Novikov spectral sequence for  $\pi_*MO\langle 8\rangle_{(3)}$ ,  $\times\beta : E_2^s \rightarrow E_2^{s+2}$  is an isomorphism if  $s > 0$  and is surjective if  $s = 0$ .*

*Proof.* Inspect the chain complex whose homology is  $E_2$ . That chain complex is

$$\pi_*(R \wedge X) \xrightarrow{f} \pi_*(R \wedge \Sigma^q X) \xrightarrow{g} \pi_*(R \wedge \Sigma^{pq} X) \xrightarrow{f} \pi_*(R \wedge \Sigma^{(p+1)q} X) \xrightarrow{g} \dots \quad \square.$$

This property continues to be true, in a weaker sense, as we get farther along in the spectral sequence.

**Lemma 3.3.** *In the Adams-Novikov spectral sequence for  $\pi_*MO\langle 8\rangle_{(3)}$ , the map*

$$\times\beta : E_k^s \rightarrow E_k^{s+2}$$

*is surjective and is injective if  $s \geq k - 1$ .*

*Proof.* We proceed by induction. The cases  $k = 1, 2$  have already been discussed. So suppose  $k > 2$ . Suppose  $\bar{x} \in E_k^{s+2}$ , where  $x \in E_{k-1}^{s+2}$  has  $d_{k-1}x = 0$ . Then there is a  $y \in E_{k-1}^s$  such that  $x = y\beta$ . Thus

$$0 = d_{k-1}(y\beta) = d_{k-1}(y)\beta$$

since  $d_{k-1}$  is a derivation. But  $d_{k-1}(y)$  is of high enough filtration for this to mean that  $d_{k-1}(y) = 0$ . Thus  $y$  survives to a class  $\bar{y} \in E_k^s$  with  $\bar{y}\beta = \bar{x}$ . This shows that multiplication by  $\beta$  is surjective.

Now suppose  $\bar{x}\beta = 0$ , where  $\bar{x} \in E_k^s$  and  $s \geq k - 1$ . Then there is a  $y \in E_{k-1}^{s+3-k}$  such that  $d_{k-1}y = x\beta$ . Since  $s \geq k - 1$ , we have  $s + 3 - k \geq 2$ , so there is a  $z \in E_{k-1}^{s+1-k}$  such that  $y = z\beta$ . Then

$$d_{k-1}(z)\beta = x\beta.$$

But these elements are in filtration high enough so that multiplication by  $\beta$  is injective, by induction. So  $d_{k-1}z = x$ , and so  $\bar{x} = 0$ .  $\square$

Note that this proof fails, and in fact the lemma is false, for the spectral sequence used in [Rav, Section 7.4], because in that case none of the differentials are derivations. It is the fact that this spectral sequence is really the Adams-Novikov spectral sequence, so that the differentials are derivations, that makes the lemma work.

Now the homotopy class corresponding to  $\beta$  must be nilpotent. Since in this case  $\beta$  is just  $\beta_1$ , there are in fact specific bounds, but in general in class of positive Novikov filtration in a ring spectrum is nilpotent by the nilpotence theorem of [DHS]. So some power of  $\beta$  must be killed by a differential. Choose the least  $k$  such that  $\beta^k$  does not survive the spectral sequence.

**Lemma 3.4.** *There is an  $x \in E_{2k-1}^1$  such that  $d_{2k-1}(x) = \beta^k$ .*

*Proof.* There is some  $i$  and some class  $x \in E_i^{2k-i}$  such that  $d_i(x) = \beta^k$ . If  $2k - i > 1$ , write  $x = y\beta$ , for some  $y \in E_i^{2k-i-2}$ . Then  $d_i(y)\beta = \beta^k$ , and  $d_i(y)$  has filtration  $\geq i - 1$ . Thus  $d_i(y) = \beta^{k-1}$ , which is impossible.  $\square$

**Theorem 3.5.** *The Adams-Novikov spectral sequence for  $\pi_*MO\langle 8 \rangle_{(3)}$  collapses at  $E_{2k}$ , and  $E_\infty^s = E_{2k}^s = 0$  for  $s > 2k - 2$ .*

*Proof.* Suppose we have a class  $z \in E_{2k-1}^s$ , where  $s \geq 2k$  and  $d_{2k-1}(z) = 0$ . Then we can write  $z = y\beta^k$  for some  $y \in E_{2k-1}^{s-2k}$ . Thus  $d_{2k-1}(y)\beta^k = 0$ . But  $d_{2k-1}(y)$  is in high enough filtration for this to mean that  $d_{2k-1}(y) = 0$ .

Now from the preceding lemma, there is a class  $x \in E_{2k-1}^1$  such that  $d_{2k-1}(x) = \beta^k$ . Thus  $d_{2k-1}(xy) = y\beta^k = z$ . Thus  $E_{2k}^s = 0$  is  $s \geq 2k$ . The spectral sequence therefore collapses at  $E_{2k}$ . To see that  $E_{2k}^{2k-1} = 0$ , note that  $\times\beta : E_{2k}^{2k-1} \rightarrow E_{2k}^{2k+1}$  is injective.  $\square$

Note that the class  $\alpha\beta^3 \in E_2^{7,40}$  cannot survive the spectral sequence. Indeed, it is in the image of the  $E_2$  term of the sphere, and does not survive that spectral sequence, as is well-known. One can consult the charts in [Rav] to see how it dies in the sphere. In  $MO\langle 8 \rangle$ ,  $\alpha\beta^3$  is a permanent cycle, and can only be hit by a  $d_5$ . But then Corollary 3.2 shows that there must be a class  $w \in E_2^{0,24}$  with  $d_5w = \alpha\beta^2$ .

This in turn implies, using the standard Toda bracket formulas for differentials, and recalling that  $\beta_1 = \langle \alpha_1, \alpha_1, \alpha_1 \rangle$ , that  $d_9(w^2\alpha) = \beta^5$ . Of course, we don't know that this differential really occurs, because  $\beta^5$  could conceivably be hit by a  $d_5$ . But if  $\beta^5$  is hit by a  $d_5$ , Lemma 3.3 shows  $\beta^3$  must also be hit. We have proved the following theorem.

**Theorem 3.6.** *The Adams-Novikov spectral sequence for  $\pi_*MO\langle 8 \rangle_{(3)}$  has  $E_{10} = E_\infty$  and  $E_\infty^s = 0$  for  $s > 8$ .*

Note that the same theorem is true for  $EO_2$  at the prime 3 [HM]. We also point out that this method may be applicable to  $MU\langle 6 \rangle$  at  $p = 2$  as well. There one would need a  $Z$ -resolution of  $S^0$ , where  $Z$  is one of the models for the double of  $A(1)$ . We do not know if a compact such resolution exists, but if so one would get an  $E_1$ -term for the Adams-Novikov spectral sequence for  $MU\langle 6 \rangle$  that would look something like

$$\pi_*(MU\langle 6 \rangle \wedge Z) \otimes P(\eta, \nu, w, \kappa) / (\nu^3, \eta\nu, w\nu, w^2 - \eta^2\kappa).$$

We have given these classes the names they normally have in  $\pi_*S^0$ :  $\eta$  is the degree 1 Hopf map,  $\nu$  the degree 3 Hopf map, and  $\kappa$  a class in bidegree  $(4, 24)$ . The class  $w$  in bidegree  $(3, 14)$  is not in the homotopy of the sphere, and does not seem to get involved in  $MU\langle 6 \rangle$  either. The ring  $P(\eta, \nu, w, \kappa) / \sim$  is basically  $Ext_{P(1)}(\mathbf{F}_2, \mathbf{F}_2)$ , except that all of the classes in those  $Ext$ -groups have order 2, and here they have infinite order. Unfortunately, one does not get such a nice periodicity result here: there are two classes with infinite multiplicative order,  $\eta$  and  $\kappa$ . But  $\eta^4$  should be killed by a  $d_3$ , so what should happen is that  $\times\kappa : E_4^s \rightarrow E_4^{s+4}$  is surjective, and, if  $s > 0$ , is an isomorphism. If that were so, the whole spectral sequence would collapse with a flat vanishing line as soon as a power of  $\kappa$  is killed. This should happen by a  $d_{23}$  to kill  $\kappa^6$ , as it does in  $EO_2$  at the prime 2 [HM]. So we expect, in the Adams-Novikov spectral sequence for  $MU\langle 6 \rangle$  at  $p = 2$ , that  $E_{24} = E_\infty$  and that  $E_\infty^s = 0$  for  $s > 20$ .

#### 4. CALCULATIONS

In this section, we calculate enough of the Adams spectral sequence for  $MO\langle 8 \rangle$  at  $p = 3$  to see that  $\beta_1^3$  is non-zero, so that the Adams-Novikov spectral sequence really collapses at  $E_{10}$  and not at  $E_6$ . The calculations suggest some conjectures about this Adams spectral sequence and about the homology of  $MO\langle 8 \rangle$ .

We first calculate  $H_*(BO\langle 8 \rangle; \mathbf{F}_3) = H_*(\mathbf{BP}\langle 1 \rangle_8; \mathbf{F}_3)$  through dimension 32. Recall we have the elements  $b_i \in H_{2i}\mathbf{BP}\langle 1 \rangle_2$  and  $[v_1] \in H_0\mathbf{BP}\langle 1 \rangle_{-4}$ . The elements  $b_i$  are the image of the corresponding elements  $\beta_i$  in  $H_{2i}CP^\infty$  dual to the powers of the generator, under the complex orientation  $CP^\infty \rightarrow \mathbf{BP}\langle 1 \rangle_2$ . Through this range, the homology is generated multiplicatively by the classes in the following table, together with the class 1 in dimension 0.

<i>Dim</i>	<i>Gen</i>	<i>Dim</i>	<i>Gen</i>
8	$x_8 = b_1^{\circ 4}$	24	$x_{24} = b_1^{\circ 3} \circ b_9$
12	$x_{12} = b_1^{\circ 3} \circ b_3$	24	$y_{24} = b_3^{\circ 4}$
16	$x_{16} = b_1^{\circ 2} \circ b_3^{\circ 2}$	28	$x_{28} = b_1^{\circ 2} \circ b_3 \circ b_9$
20	$x_{20} = b_1 \circ b_3^{\circ 3}$	32	$x_{32} = b_1 \circ b_3^{\circ 2} \circ b_9$

In general, to find the generators in dimension  $4k$ , one takes the 3-adic expansion of  $2k$ :

$$2k = \sum a_i 3^i.$$

If the sum of the  $a_i$  is greater than 2, one has a single generator  $x_{4k}$  which is the circle product of the  $b_{3^i}^{\circ a_i}$  and an appropriate power of  $[v_1]$ . If  $2k = 3^i + 3^j$  with  $0 < i < j$  both positive and distinct, there are 2 generators, namely  $x_{4k} = b_{3^{i-1}}^{\circ 3} \circ b_{3^j}$  and  $y_{4k} = b_{3^i} \circ b_{3^{j-1}}^{\circ 3}$ . Otherwise, there is just one generator  $x_{4k} = b_{3^i} \circ b_{3^{j-1}}^{\circ 3}$ .

Now  $H_*BO\langle 8 \rangle$  is not a polynomial ring. In fact, we know from [Sin] that it has some generators which are truncated at height 3, one in each dimension of the form  $3^i + 3^j$  where  $i < j$ . In particular,  $x_8$ , as the only element in dimension 8, must have cube 0. There are no other multiplicative relations in our range, as can be easily checked by counting dimensions. But there are multiplicative relations in higher dimensions. For example, there has to be a class in dimension 20 whose cube is 0. Hopf ring relations tell us that  $x_{20}^3$  is congruent to 0 modulo 4-fold star-decomposables. But it is probably not 0 on the nose, and will have to be modified by adding a multiple of  $x_8 x_{12}$  to produce a class whose cube is actually 0.

We now must compute the Steenrod algebra coaction through our range. This is completely mechanical: the  $b_i$  come from the homology of  $CP^\infty$  where the coaction is completely known. Indeed, we have

$$\begin{aligned} \psi(b_1) &= 1 \otimes b_1. \\ \psi(b_3) &= 1 \otimes b_3 + 2\zeta_1 \otimes b_1. \\ \psi(b_9) &= 1 \otimes b_9 + 2\zeta_1 \otimes b_7 + \zeta_1^2 \otimes b_5 + 2\zeta_1^3 \otimes b_3. \end{aligned}$$

Here we have used  $\zeta_i$  instead of  $\xi_i$  as is frequently convenient in dealing with cobordism theories. The elements  $b_7$  and  $b_5$  are  $*$ -decomposable. In our range,  $b_9$  always occurs in a circle product with  $b_1$ . As we saw in the first section,  $b_1 \circ (x * y) = 0$  if  $x, y$  are both in the augmentation ideal. So we can ignore  $b_7$  and  $b_5$ . We then get the following formulae for the coaction in  $H_*BO\langle 8 \rangle$  through dimension 32.

$$\begin{aligned}
\psi x_8 &= 1 \otimes x_8. \\
\psi x_{12} &= 1 \otimes x_{12} + 2\zeta_1 \otimes x_8. \\
\psi x_{16} &= 1 \otimes x_{16} + \zeta_1 \otimes x_{12} + \zeta_1^2 \otimes x_8. \\
\psi x_{20} &= 1 \otimes x_{20} + 2\zeta_1^3 \otimes x_8. \\
\psi x_{24} &= 1 \otimes x_{24} + 2\zeta_1^3 \otimes x_{12} + (2\zeta_2 + \zeta_1^4) \otimes x_8. \\
\psi y_{24} &= 1 \otimes y_{24} + 2\zeta_1 \otimes x_{20} + 2\zeta_1^3 \otimes x_{12} + \zeta_1^4 \otimes x_8. \\
\psi x_{28} &= 1 \otimes x_{28} + 2\zeta_1 \otimes x_{24} + 2\zeta_1^3 \otimes x_{16} + (2\zeta_2 + 2\zeta_1^4) \otimes x_{12} + (\zeta_1\zeta_2 + 2\zeta_1^5) \otimes x_8. \\
\psi x_{32} &= 1 \otimes x_{32} + \zeta_1 \otimes x_{28} + \zeta_1^2 \otimes x_{24} + 2\zeta_1^3 \otimes x_{20} + \\
&\quad 2\zeta_2 \otimes x_{16} + 2\zeta_1\zeta_2 \otimes x_{12} + (2\zeta_1^2\zeta_2 + \zeta_1^6) \otimes x_8.
\end{aligned}$$

To get the coaction on the homology of  $MO\langle 8 \rangle$ , we apply the Thom isomorphism. The ring structure in homology is preserved by the Thom isomorphism, but the coaction is the composite below, dual to the usual description of the action of  $\mathcal{A}$  on a Thom spectrum.

$$H_*BO\langle 8 \rangle \xrightarrow{\Delta} H_*BO\langle 8 \rangle \otimes H_*BO\langle 8 \rangle \xrightarrow{f \otimes \psi} \mathcal{P}_* \otimes \mathcal{P}_* \otimes H_*BO\langle 8 \rangle \xrightarrow{\mu \otimes 1} \mathcal{P}_* \otimes H_*BO\langle 8 \rangle.$$

Here we have identified  $H_*MO\langle 8 \rangle$  and  $H_*BO\langle 8 \rangle$ , and  $f$  is the map considered in the first section. So we must explicitly compute both the diagonal and  $f$ . The diagonal in  $H_*CP^\infty$  is easy. In particular, we have

$$\Delta b_1 = [0_2] \otimes b_1 + b_1 \otimes [0_2].$$

Since  $[0_2] \circ x = 0$  for  $x$  in the augmentation ideal, this means that for all the generators except  $b_3^{\circ 4} = y_{24}$  in our range, we have

$$\Delta x = [0_8] \otimes x + x \otimes [0_8].$$

To compute  $\Delta b_3^{\circ 4}$ , we have

$$\Delta b_3 = [0_2] \otimes b_3 + b_1 \otimes b_2 + b_2 \otimes b_1 + b_3 \otimes [0_2].$$

One can check that  $b_2 = 2b_1^{*2}$ . Recall the Hopf ring relations

$$\begin{aligned}
b_1 \circ (x * y) &= 0. \\
b_1^{*2} \circ (x * y) &= 2(b_1 \circ x) * (b_1 \circ y).
\end{aligned}$$

for  $x, y$  in the augmentation ideal. These give

$$\Delta y_{24} = 1 \otimes y_{24} + 2x_8 \otimes x_8^2 + 2x_8^2 \otimes x_8 + y_{24} \otimes 1.$$

Recall from the second section that the image of  $f$  is  $P(\zeta_1^3, \zeta_2, \zeta_3, \dots)$ . In particular  $f(x_8) = 0$ , so since  $f$  is an algebra map,  $f(x_8^2) = 0$  as well. Thus, if we denote the

coaction in  $H_*MO\langle 8 \rangle$  by  $\psi$  and the coaction in  $H_*BO\langle 8 \rangle$  by  $\psi'$ , we have, in our range,

$$\psi(x) = \psi'(x) + f(x) \otimes 1.$$

To get explicit formulas for  $f$ , one can work one's way up dimension by dimension, using coassociativity to determine  $f$  at each stage. Of course  $f(x_8) = f(x_{20}) = 0$ , and one has two possibilities for  $f(x_{12})$ , namely  $\zeta_1^3$  and  $2\zeta_1^3$ . By changing the generator  $x \in H^2CP^\infty$ , one can assume  $f(x_{12}) = \zeta_1^3$ . The resulting formulas for  $f$  are given below.

$$\begin{aligned} f(x_8) &= 0. & f(x_{12}) &= \zeta_1^3. \\ f(x_{16}) &= \zeta_2 & f(x_{20}) &= 0. \\ f(x_{24}) &= 2\zeta_1^6. & f(y_{24}) &= 2\zeta_1^6. \\ f(x_{28}) &= \zeta_1^3\zeta_2. & f(x_{32}) &= 2\zeta_2^2. \end{aligned}$$

It is then straightforward, though tedious, to analyze the comodule structure of  $H_*MO\langle 8 \rangle$  in this range. We get a splitting

$$H_*MO\langle 8 \rangle \cong M \oplus \Sigma^{16}\mathcal{P}_* \oplus \Sigma^{24}\mathcal{P}_* \oplus \Sigma^{28}\mathcal{P}_* \oplus \Sigma^{32}\mathcal{P}_*.$$

Here  $M$  is an extension

$$0 \rightarrow \Sigma^8 N \rightarrow M \rightarrow N \rightarrow 0$$

where

$$N = (\mathcal{P}/P(0))_* = P(\zeta_1^3, \zeta_2, \zeta_3, \dots).$$

It is easy to verify that such an extension is given by  $P^1\zeta_1^3$ , where  $P^1$  is thought of as acting downward, and in this case it is  $\Sigma^8 1$ .

We would then like to compute  $Ext_{\mathcal{A}_*}(\mathbf{F}_3, H_*MO\langle 8 \rangle)$ , the  $E_2$ -term of the Adams spectral sequence. Of course,

$$Ext_{\mathcal{A}_*}(\mathbf{F}_3, \mathcal{P}_*) = P(a_0, a_1, \dots)$$

where the  $a_i$  are in bidegree  $(1, 2 \cdot 3^i - 1)$ . To compute  $Ext(N)$ , we use the isomorphisms

$$Ext_{\mathcal{A}_*}(N) = Ext_{\mathcal{P}_*}(N \otimes P(a_0, a_1, \dots)) = Ext_{P(0)_*}(P(a_0, a_1, \dots)).$$

The first isomorphism can be found for example in [Rav, Theorem 4.4.3], where one sees that the  $P(0)_*$ -comodule structure on  $P(a_0, a_1, \dots)$  is determined by

$$\psi a_1 = 1 \otimes a_1 + \xi_1 \otimes a_0$$

with the other  $a_i$  being primitive.

It is then easy to calculate

$$Ext(N) = P(a_1^3, a_2, \dots) \otimes P(a_0, \alpha_1, \alpha_2, \beta_1)/R,$$

where  $\alpha_1 \in Ext^{1,4}$ ,  $\alpha_2 \in Ext^{2,9}$  and  $\beta_1 \in Ext^{2,12}$ . The relations  $R$  are generated by

$$\alpha_1^2, \alpha_2^2, a_0\alpha_1, a_0\alpha_2, \alpha_1\alpha_2 - a_0\beta_1.$$

Of course  $a_0, \alpha_1, \alpha_2, \beta_1$  are all in the image of the map  $Ext_{\mathcal{A}_*}(\mathbf{F}_3) \rightarrow Ext_{\mathcal{A}_*}(N)$ .

To calculate  $Ext(M)$ , we must then calculate the coboundary map

$$\delta : Ext^{s,t}(N) \rightarrow Ext^{s+1,t}(\Sigma^8 N).$$

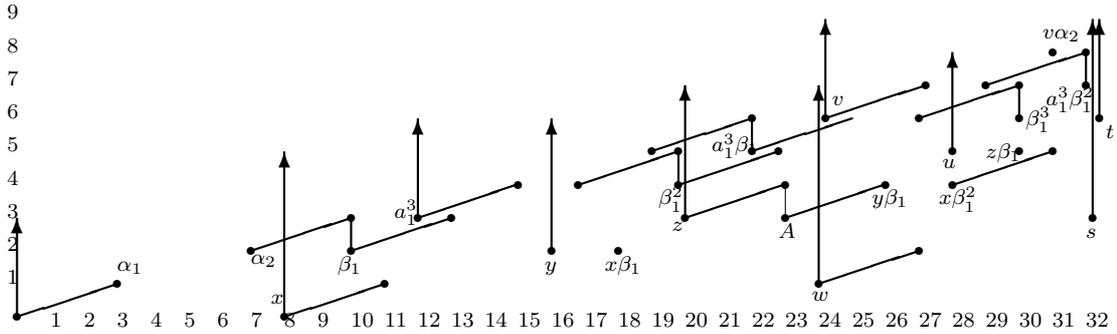
The coboundary map preserves multiplication by elements in  $Ext_{\mathcal{A}_*}(\mathbf{F}_3)$ . It is 0 on  $a_1^3$  and  $a_1^6$  for dimensional reasons. We still must determine it on the classes  $a_2, a_1^3 a_2$ , and  $a_2^2$ . A cobar representative for  $a_2$  is

$$[\bar{\tau}_2]\zeta_2 + [\bar{\tau}_1]\zeta_1^3 + [\bar{\tau}_2]1.$$

Here  $\bar{\tau}_i$  denotes the conjugate of  $\tau_i$ . It follows that  $\delta a_2 = \Sigma^8 \alpha_2$ . If one knew that  $\delta$  were a derivation it would follow easily that

$$\delta(a_1^3 a_2) = \Sigma^8 a_1^3 \alpha_2 \delta(a_2^2) = 2\Sigma^8 a_2 \alpha_2.$$

We do not know how to see that  $\delta$  is a derivation, so one must check these by hand. But that can be done. The resulting Adams  $E_2$ -term is displayed below, without the  $BP$  summands.



There are some extensions and multiplicative behavior in the  $E_2$  term that we need to determine. The extensions are  $a_0 A = z\alpha_1$ , shown on the chart by a thin line, and  $y\alpha_2 = z\alpha_1$ . The multiplicative relations we need are:

$$\begin{aligned} z &= xa_1^3. \\ v &= (a_1^3)^2. \\ u &= ya_1^3. \\ t &= xv. \end{aligned}$$

One must remember in these relations that most of these elements are not really well-defined. For example  $t$  is just some class in filtration 6 in the 32-stem which is not divisible by  $a_0$ . There are many such elements, because of the  $BP$  summands not shown on the chart. What we really mean is that we can choose  $t$  and  $v$  so that  $t = xv$ . Or, equivalently, that any choice for  $t$  and  $v$  will give  $t \equiv xv$  modulo  $a_0$ .

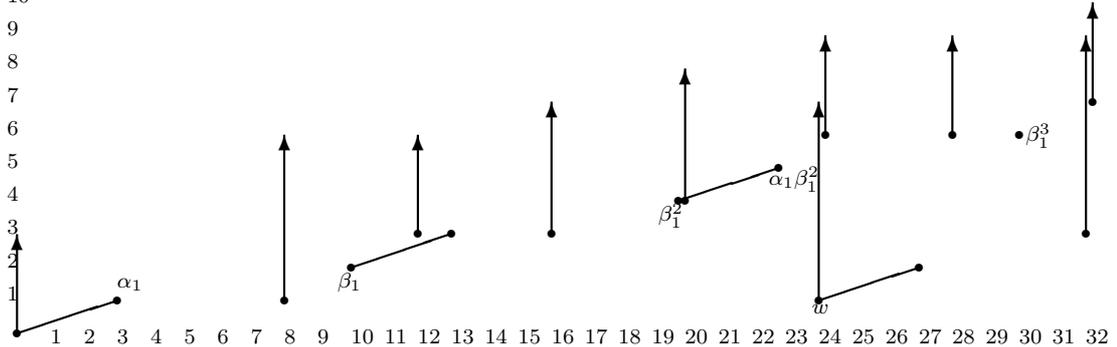
The easiest way to get these extensions is to consider the differential  $d_2$  simultaneously. The reader is encouraged to do this herself, as this kind of argument is easier to think of than to follow. The facts we need are that the torsion is all killed by 3 and any well-behaved  $v_1$ -element, that no  $v_1$ -element can be divisible by 3, that the product of 2  $v_1$ -elements is again a  $v_1$ -element, and there have to be well-behaved  $v_1$ -elements in every dimension  $4k$ , where  $k > 1$ .

Note that  $a_0\beta_1$  cannot survive the spectral sequence, since we know that  $\beta_1$ , and indeed all of the torsion in  $\pi_*MO\langle 8 \rangle$  is killed by 3. Thus  $d_2(x\alpha_1) = a_0\beta_1$ , which implies  $d_2x = \alpha_2$ . The same argument implies that  $d_2(z\alpha_1) = a_0a_1^3\beta_1$ , which implies that  $d_2(z) = a_1^3\alpha_2$ . It follows that  $z = xa_1^3$ , at least modulo  $a_0^2$ , which is all we need.

Now consider the class  $a_1^3\alpha_1\beta_1$ . This class cannot survive the spectral sequence, since  $a_1^3$  is a  $v_1$ -element. Thus we must have  $d_2(y\beta_1) = a_1^3\alpha_1\beta_1$ . It follows that  $d_2y = a_1^3\alpha_1$ . It also follows that  $d_2(A) = a_1^3\beta_1$ , from which we get the extensions  $a_0A = z\alpha_1$  and  $y\alpha_2 = z\alpha_1$ .

Whatever  $(a_1^3)^2$  is, it must be a  $v_1$ -element of filtration at least 6. It follows that it must be  $v$ , again modulo  $a_0$ . Then  $d_2(ya_1^3) = v\alpha_1$ , so  $ya_1^3 = u$ , modulo  $a_0$ . Finally,  $d_2(xv) = v\alpha_2$ , so we must have  $t = xv$  modulo  $a_0$ .

These extensions and the  $d_2$ 's calculated above determine all the possible  $d_2$ 's in this range, and the resulting  $E_3$  term is shown below.



In this range, there is only one higher differential,  $d_4(w) = \alpha_1\beta_1^2$ . This differential gives rise to an extension  $\alpha_1w\alpha_1 = \beta_1^3$ , just as one gets the corresponding  $\alpha_1$  extension to  $\beta_1^4$  in  $\pi_*S^0$ . In any case,  $\beta_1^3$  certainly survives the spectral sequence, so the Adams-Novikov spectral sequence cannot collapse at  $E_6$ .

At this point, we present some speculations about the behavior of this Adams spectral sequence in larger dimensions. These conjectures are based on speculative calculations.

- Conjecture 4.1.** (1)  $H_*MO\langle 8 \rangle$  splits into a direct sum of suspensions of  $M$ ,  $N = M \otimes \mathbf{F}_3[\zeta_1]/(\zeta_1^2)$ , and  $\mathcal{P}_*$ . The first  $N$  summand begins in dimension 28.  
 (2) There are differentials  $d_3a_2^3 = w\alpha_1\beta_1^2$ , and  $d_6a_2^3\alpha_1 = \beta_1^5$ .  
 (3) The Adams spectral sequence collapses after  $d_2, d_3, d_4$ , and  $d_6$ .

- (4) *The algebraic Novikov spectral sequence [Rav, Theorem 4.4.4] collapses after  $d_1$ , which corresponds to the Adams  $d_2$ , so that the Adams  $E_3$ -term is a reindexed form of the Adams-Novikov  $E_2$ -term.*
- (5) *The Adams-Novikov  $d_5$  corresponds to the Adams  $d_3$  and  $d_4$ , and the Adams-Novikov  $d_9$  corresponds to the Adams  $d_6$ .*

Of course, there are many other questions to ask and answer here. For example,  $a_2^9$  should survive the spectral sequence and represent a  $v_2$ -element. One would expect, in analogy to  $MSU$  at  $p = 2$ , that the higher  $a_i$  survive as well and represent  $v_i$ -elements. One would also expect a splitting of  $MO\langle 8 \rangle$ , in analogy to the splitting of  $MSU$ . And certainly one should determine the image of the unit in homotopy. In our range, the only classes in that image are  $\alpha_1, \beta_1, \alpha_1\beta_1, \beta_1^2$ , and  $\beta_1^3$ . However, just outside our range, the class  $w\beta_1$  is also in the image. It is usually denoted  $\epsilon$ .

## 5. GENERALIZATIONS

In this section, we point out that our methods do apply to more spectra than  $MO\langle 8 \rangle$  at  $p = 3$  and  $MU\langle 6 \rangle$  at  $p = 2$  and  $p = 3$ , though they do not apply to other  $MO\langle k \rangle$ . Recall that we identified  $BO\langle 8 \rangle$  at  $p = 3$  with  $\mathbf{BP}\langle \mathbf{1} \rangle_8$  and used knowledge of Hopf rings. The problem is that for  $n > 2p + 2$ , the map

$$\mathbf{BP}_n \rightarrow \mathbf{BP}\langle \mathbf{1} \rangle_n$$

is not onto in homology, so we don't know that the homology of  $\mathbf{BP}\langle \mathbf{1} \rangle_n$  is evenly graded, nor do we know how to compute the coaction of the Steenrod algebra.

However, as long as  $n \leq 2(p^r + p^{r-1} + \cdots + p + 1)$ , the map

$$\mathbf{BP}_n \rightarrow \mathbf{BP}\langle \mathbf{r} \rangle_n$$

will be onto in homology [Wil]. So we should build Thom spectra over these spaces. We must assume that  $n$  is even, and that  $n$  is divisible by 4 if  $p$  is odd. At  $p = 2$ , write  $n = 2t + 2$ . For  $p$  odd, write  $n = tq + s$ , where  $4 \leq s \leq q$ . If  $p$  is odd, consider the composite of infinite loop maps

$$\mathbf{BP}\langle \mathbf{r} \rangle_n \rightarrow \mathbf{BP}\langle \mathbf{1} \rangle_n \xrightarrow{[v_1^t]} \mathbf{BP}\langle \mathbf{1} \rangle_s \rightarrow BO_{(p)}.$$

For  $p = 2$ , consider the analogous map

$$\mathbf{BP}\langle \mathbf{r} \rangle_n \rightarrow \mathbf{BP}\langle \mathbf{1} \rangle_n \xrightarrow{[v_1^t]} \mathbf{BP}\langle \mathbf{1} \rangle_2 \rightarrow BU_{(2)}.$$

In both cases, the last map comes from the splitting of infinite loop spaces of Corollary 1.5 .

These maps do not give us vector bundles over  $\mathbf{BP}\langle \mathbf{r} \rangle_n$ , because they are  $p$ -local. So we must build Thom spectra over maps  $X \rightarrow BO_{(p)}$ . We adopt a naive approach to this problem: surely there is a better way that would involve equivariant homotopy theory. In our cases,  $X$  is always simply connected, and the map actually comes

from a map to  $BSO_{(p)}$ . We build Thom spectra by putting together Thom spaces as usual. So assume we have a map  $Y \xrightarrow{f} BSO(r)_{(p)}$ , where  $Y$  is simply connected. We can assume  $r$  is at least 3, since we are really only interested in spectra. These assumptions get rid of the fundamental group, which can be a problem in localizations. What one would normally do is to pull back the universal bundle to  $Y$ , take its disk bundle, and mod out by its sphere bundle. But  $p$ -localization is really only defined on the homotopy category of spaces, where one cannot see the  $SO(r)$ -action very well. But we can localize the universal sphere bundle  $S(\xi_r)$  over  $BSO(r)$  to get a fibration  $S(\xi_r)_{(p)} \rightarrow BSO(r)_{(p)}$  with fiber  $S_{(p)}^{r-1}$ . Then we can define  $S(f)$  to be the induced fibration over  $Y$ . Then define the Thom space  $T(f)$  to be the cofiber of  $S(f) \rightarrow Y$ . We will leave it to the reader to verify the usual properties of Thom spaces and Thom spectra, which all hold.

Let us denote the resulting Thom spectrum over  $\mathbf{BP}\langle r \rangle_n$  by  $MBP(r, n)$ . The  $B$  is still there because it refers to Brown, not to classifying space. The  $MBP(r, n)$  are only defined when  $n$  is divisible by 4 if  $p$  is odd and when  $n$  is even if  $p = 2$ . They are commutative ring spectra. We will concentrate on the case  $n = tq$ . Corollary 1.5 shows that  $MBP(r, tq)$  admits a ring spectrum map to  $MO\langle k \rangle$  and to  $MU\langle k \rangle$  when  $k \leq tq$ . The proof of Rosen's theorem applies without change to the  $MBP(r, tq)$ , since the proof actually showed that the image of  $H_*\mathbf{BP}_{tq}$  in  $H_*BP$  is at least as large as  $(P//P(t-2))_*$ , and that the image of  $H_*MO\langle k \rangle$  is at least as small as  $(P//P(t-2))_*$  when  $k \geq (t-1)q + 2$ . Thus we get the following theorem.

**Theorem 5.1.** *Let  $f$  denote the map*

$$MBP(r, tq) \rightarrow MSO_{(p)} \rightarrow BP.$$

*Then the kernel of  $H^*f$  is the left ideal generated by the augmentation ideal of  $P(t-2)$ .*

We then get the following corollary.

**Corollary 5.2.** *Let  $X$  denote a  $p$ -local finite spectrum whose cohomology is evenly graded and free over  $P(t-2)$ . Then, if  $tq \leq 2(p^r + \cdots + p + 1)$ ,  $MBP(r, tq) \wedge X$  is a wedge of suspensions of  $BP$ .*

Then methods of Section 2 then apply, and we get

**Theorem 5.3.** *Suppose  $tq \leq 2(p^r + \cdots + p + 1)$ . Then the torsion in  $\pi_*MBP(r, tq)$  is bounded and the Bousfield class of  $MBP(r, tq)$  coincides with the Bousfield class of  $BP$ .*

This theorem does allow us to deduce the Bousfield class of the  $MU\langle k \rangle$ .

**Corollary 5.4.** (1) *For any  $k$ , the Bousfield class of  $MU\langle k \rangle_{(p)}$  is the same as that of  $BP$ . If  $p$  is odd, the same is true for the Bousfield class of  $MO\langle k \rangle_{(p)}$ .*  
 (2) *At  $p = 2$ , the Bousfield class of  $MO\langle \phi(r) \rangle$  is less than or equal to that of  $BP\langle r-1 \rangle$ .*

Here  $\phi(r)$  is the dimension of the  $r$ th nonzero homotopy group of  $BSO$  as before.

*Proof.* if  $r$  and  $t$  are large enough, there is an orientation  $MBP(r, tq) \rightarrow MU\langle k \rangle$ , so the Bousfield classes of  $MU\langle k \rangle$  and  $MO\langle k \rangle$  are bounded above by that of  $BP$ . On the other hand, there is an orientation  $MU\langle k \rangle \rightarrow BP$  and if  $p$  is odd, an orientation  $MO\langle k \rangle \rightarrow BP$ . Thus the Bousfield classes of  $MU\langle k \rangle$  and, if  $p$  is odd,  $MO\langle k \rangle$ , are also bounded below by that of  $BP$ .

To examine the Bousfield class of  $MO\langle \phi(r) \rangle$  at  $p = 2$ , we must recall the usual notation for Bousfield class that we have been avoiding,  $\langle X \rangle$ . The results of Section 2 together with the work of Bahri and Mahowald previously mentioned [BM] imply that  $MO\langle \phi(r) \rangle \wedge X$  is a wedge of suspensions of  $H\mathbf{F}_2$  if  $X$  is a finite spectrum whose homology which is free over  $A(r-1)$ . Any such spectrum will have type  $r$ . We now apply the fundamental Bousfield class decomposition used in [Hov1]:

$$\langle S^0 \rangle = \langle Tel(0) \rangle \vee \cdots \vee \langle Tel(r-1) \rangle \vee \langle F(r) \rangle.$$

Here  $Tel(i)$  is the telescope of a  $v_i$ -self map on a type  $i$  finite spectrum, and  $F(r)$  is a finite spectrum of type  $r$ . The Bousfield classes are independent of the specific spectra chosen.

If we smash this decomposition with  $MO\langle \phi(r) \rangle$ , the last term will be  $\langle H\mathbf{F}_2 \rangle$ . Also, since  $MO\langle \phi(r) \rangle$  is a module over  $MU\langle \phi(r) \rangle$ , which has the same Bousfield class as  $BP$ , we have  $\langle MO\langle \phi(r) \rangle \wedge Tel(i) \rangle = \langle MO\langle \phi(r) \rangle \wedge K(i) \rangle$ . Thus

$$\langle MO\langle \phi(r) \rangle \rangle \leq \langle K(0) \rangle \vee \cdots \vee \langle K(r-1) \rangle \vee \langle H\mathbf{F}_2 \rangle = \langle BP\langle r-1 \rangle \rangle,$$

as required.  $\square$

It is natural to conjecture that the Bousfield class of  $MO\langle \phi(r) \rangle$  at  $p = 2$  is precisely that of  $BP\langle r-1 \rangle$ . To prove this, it would suffice to show that  $K(i)_*MO\langle \phi(r) \rangle \neq 0$  for  $i < r$ .

Now we will show that, just as the 3-torsion in  $\pi_*MO\langle 8 \rangle_{(3)} = \pi_*MBP(1, 2q)$  is all killed by 3, the  $p$ -torsion in  $\pi_*MBP(2, 2q)$  is all killed by  $p$ . To do this we first must examine how the image of  $J$  behaves in  $\pi_*MBP(r, tq)$ .

**Lemma 5.5.** *If  $p$  is an odd prime, the composite*

$$Im J \rightarrow \pi_*S^0 \rightarrow \pi_*MBP(r, tq)$$

*is injective in dimensions  $\leq tq - 2$  and 0 in dimensions  $\geq tq - 1$ . At  $p = 2$  the same theorem is true for the image of the complex  $J$  homomorphism.*

*Proof.* The proof is just like the proof of the analogous theorem for  $MO\langle k \rangle$  and  $MU\langle k \rangle$  in [Hov]. First assume  $p$  is odd. Suppose  $x \in Im J$  in dimension  $k$ . Then the mapping cone on  $x$ ,  $C(x)$ , is the Thom complex of a map  $S^{k+1} \rightarrow BO$ . Because the image of  $J$  is concentrated in dimensions congruent to  $-1 \pmod q$ , this map will lift to  $S^{k+1} \rightarrow \mathbf{BP}\langle 1 \rangle_q$ . Assume  $k > tq - 2$ . Then it will lift further to  $S^{k+1} \rightarrow \mathbf{BP}\langle 1 \rangle_{tq}$ . Since the map of spectra  $BP\langle r \rangle \rightarrow BP\langle 1 \rangle$  is onto on homotopy groups, it will lift

even further to  $S^{k+1} \rightarrow \mathbf{BP}\langle \mathbf{r} \rangle_{tq}$ . Hence  $x$  maps to 0 in  $\pi_*MBP(r, tq)$ . Injectivity for smaller values of  $k$  just follows from the high connectivity of  $\mathbf{BP}\langle \mathbf{1} \rangle_{tq}$ . The case  $p = 2$  is similar.  $\square$

**Theorem 5.6.** *The torsion in  $\pi_*MBP(2, 2q)$  is all killed by  $p$ .*

*Proof.* We follow the same outline as the proof of the analogous theorem for  $\pi_*MO\langle 8 \rangle$ . We know that  $MBP(2, 2q) \wedge X$  is a wedge of suspensions of  $BP$  for an evenly graded  $X$  whose cohomology is a free  $P^1$ -module. So we can take  $X$  to be the  $(p-1)q$  skeleton of  $BP$ , a finite spectrum with  $p$  cells. If we denote by  $\overline{X}$  the fiber of the inclusion of the bottom cell  $S^0 \rightarrow X$ , we must calculate the image of  $[\overline{X}, S^0]$  in  $[\overline{X}, MBP(2, 2q)]$ . As before,  $[\overline{X}, S^0] \cong \mathbf{Z}/p^{p-1}$ , with generator  $\alpha_1$  on the bottom cell. The  $\alpha_i$  for  $1 \leq i \leq p-1$  all pile up to make the  $\mathbf{Z}/p^{p-1}$ , in the sense that  $p\alpha_1$  is 0 on the bottom cell but  $\alpha_2$  on the second cell, etc. But all of these  $\alpha_i$  go to 0 in  $MBP(2, 2q)$  except  $\alpha_1$ . So the image in  $[\overline{X}, MBP(2, 2q)]$  has order  $p$ .  $\square$

One can also find  $v_1$ -elements in  $\pi_*MBP(r, tq)$ , show that the  $v_1$ -torsion and the  $p$ -torsion in  $\pi_*MBP(2, 2q)$  coincide if  $p$  is odd, and show that any such torsion element is killed by any well-behaved  $v_1$ -element. The proofs are all analogous to the corresponding theorems for  $MO\langle 8 \rangle$ .

One also gets results analogous to those of Section 3 for  $MBP(2, 2q)$ . That is, the Adams-Novikov spectral sequence collapses with a flat vanishing line as soon as a power of  $\beta_1$  is killed. Note that sparseness tells us the only non-zero differentials are  $d_{kq+1}$ , since  $H_*MBP(2, 2q)$  is concentrated in degrees congruent to 0 mod  $q$ . Again  $\alpha_1\beta_1^p$  cannot survive the spectral sequence, since it does not survive the Adams-Novikov spectral sequence for the sphere. (It is killed by the Toda differential [Rav].) Once again, the only way this can happen is if there is a  $w \in E_2^{0, pq(p-1)}$  with  $d_{q+1}w = \alpha_1\beta_1^{p-1}$ . The fact that  $\beta_1$  is the  $p$ -fold Toda bracket of  $\alpha_1$  then induces a differential to kill  $\beta_1^{p^2-2p+2}$ . So the Adams-Novikov spectral sequence for  $MBP(2, 2q)$  collapses after at most 2 differentials. This differential may of course be pre-empted by a shorter one, though we would be very surprised if that happened.

This behavior is of course just like the Adams-Novikov spectral sequence of  $EO_{p-1}$ . The first author has shown [Hov] that no  $MO\langle k \rangle$  can admit an orientation to  $EO_{p-1}$  if  $p > 3$ . That proof does not apply to the  $MBP(2, 2q)$ , so it is possible that they do in fact admit orientations to  $EO_{p-1}$ .

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