The periodicity theorem in the solution to the Arf-Kervaire invariant problem

Harvard-MIT
Summer Seminar
on the Kervaire Invariant

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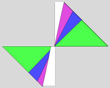
Review of our strategy

Our goal is to prove

Main Theorem

The Arf-Kervaire elements $\theta_j \in \pi_{2^{j+1}-2}(S^0)$ *do not exist for* $j \geq 7$.



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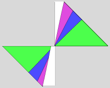

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(iii) It is 256-periodic, meaning $\Sigma^{256}M \cong M$. This is the Periodicity Theorem.

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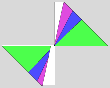

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The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$.

The spectrum M

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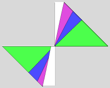

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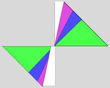

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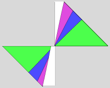

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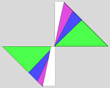

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In order to identify D we need to study the slice spectral sequence in more detail.

The slice spectral sequence

Recall that for $G = C_8$ we have a *slice tower*

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Recall that $\pi_*^G(MU^{(4)})$ is by definition $\pi_*((MU^{(4)})^G)$, the homotopy of the fixed point set.

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In the slice tower for $MU^{(4)}$, every odd slice is contractible and $P_{2n}^{2n} = \hat{W}_n \wedge H\mathbf{Z}$, where $H\mathbf{Z}$ is the integer Eilenberg-Mac Lane spectrum and $\hat{W}_n$ is a certain wedge of the following three types of finite G-spectra:

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The same holds after we invert D, in which case negative values of n can occur.

Slices of the form $S^{m\rho_8} \wedge H\mathbf{Z}$

Here is a picture of some slices $S^{m\rho_8} \wedge H\mathbf{Z}$.

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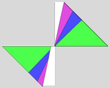

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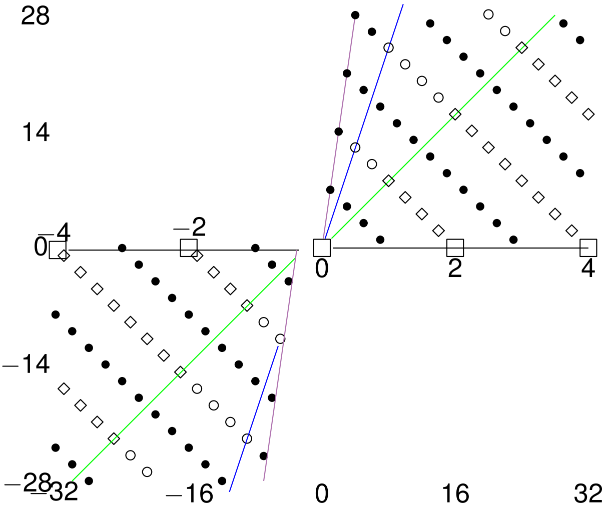

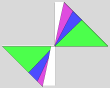

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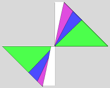

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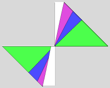

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- A similar picture for $S^{m\rho_4} \wedge H\mathbf{Z}$ would be confined to the regions between the black lines and blue lines with slope 3 and concentrated on diagonals where t is divisible by 4.
- A similar picture for $S^{m\rho_2} \wedge H\mathbf{Z}$ would be confined to the regions between the black lines and green lines with slope 1 and concentrated on diagonals where t is divisible by 2.

Implications for the slice spectral sequence

These calculations imply the following.

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Implications for the slice spectral sequence

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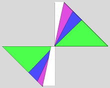

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means that the resulting slice spectral sequence is confined to the regions of the first and third quadrants shown in the picture.

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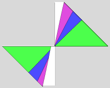

Geometric fixed points

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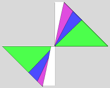

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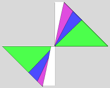

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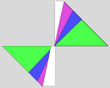

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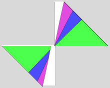

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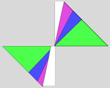

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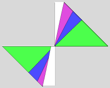

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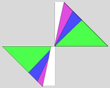

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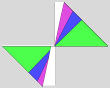

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$$EC_{2+} \to S^0 \to \tilde{E}C_2.$$

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Here $E\mathbf{Z}/2$ is a G-space via the projection $G \to \mathbf{Z}/2$ and S^0 has the trivial action, so $\tilde{E}C_2$ is also a G-space.

Geometric fixed points (continued)

Under this action EC_2^G is empty while for any proper subgroup H of G, $EC_2^H = EC_2$, which is contractible. For an arbitrary finite group G it is possible to construct a G-space with the similar properties.

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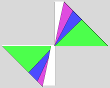

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Definition

For a finite cyclic 2-group G and G-spectrum X, the geometric fixed point spectrum is

$$\Phi^G X = (X \wedge \tilde{E}C_2)^G.$$

Geometric fixed points (continued)

This functor has the following properties:

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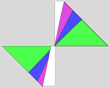

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This functor has the following properties:

- For G-spectra X and Y, $\Phi^G(X \wedge Y) = \Phi^G X \wedge \Phi^G Y$.

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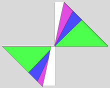

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From the suspension property we can deduce that

$$\Phi^{C_8} MU^{(4)} = MO,$$

the unoriented cobordism spectrum.

Geometric Fixed Point Theorem

Let σ denote the sign representation. Then for any G-spectrum X, $\pi_\star(\tilde{E}C_2 \wedge X) = a_\sigma^{-1}\pi_\star(X)$, where $a_\sigma : S^0 \to S^\sigma$ is the element defined in Hill's lecture.

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Geometric fixed points (continued)

Recall that $\pi_*(MO) = \mathbf{Z}/2[y_i : i > 0, i \neq 2^k - 1]$ where $|y_i| = i$.

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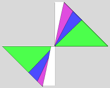
where $|r_i| = 2i$, γ is a generator of G and $\gamma^4(r_i) = (-1)^i r_i$.



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Lemma

The generators r_i and y_i can be chosen so that

$$\Phi^G Nr_i = \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise.} \end{cases}$$

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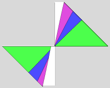

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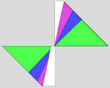

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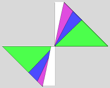

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- It appears in the slice spectral sequence in $E_2^{7i,8i}$, which is on the vanishing line.
- The subring of elements on the vanishing line is the polynomial algebra on the f_i.

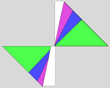
Some slice differentials (continued)

- Under the map

$$\pi_*(MU^{(g/2)}) \to \pi_*(\Phi^G MU^{(g/2)}) = \pi_*(MO)$$

we have

$$f_i \mapsto \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise} \end{cases}$$



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- Any differential landing on the vanishing line must have a target in the ideal $(f_1, f_3, f_7, \ldots)$. A similar statement can be made after smashing with $S^{2^k\sigma}$.

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Some slice differentials (continued)

Recall that for an oriented representation V there is a map $u_V : S^{|V|} \to \Sigma^V H\mathbf{Z}$, which lies in $\pi_{V-|V|}(H\mathbf{Z})$.

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Slice Differentials Theorem

In the slice spectral sequence for $\Sigma^{2^k\sigma} MU^{(4)}$ (for $k > 0$) we have $d_r(u_{2^k\sigma}) = 0$ for $r < 1 + 8(2^k - 1)$, and

$$d_{1+8(2^k-1)}(u_{2^k\sigma}) = a_\sigma^{2^k} f_{2^k-1}.$$

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Inverting a_σ in the slice spectral sequence will make it converge to $\pi_*(MO)$.

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Inverting a_σ in the slice spectral sequence will make it converge to $\pi_*(MO)$. This means each f_{2^k-1} must be killed by some power of a_σ. The only way this can happen is as indicated in the theorem.

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Some slice differentials (continued)

Let

$$\overline{\Delta}_k^{(8)} = N r_{2^k-1} \in \pi_{(2^k-1)\rho_8}(MU^{(4)}).$$

Some slice differentials (continued)

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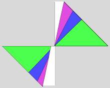
We want to invert this element and study the resulting slice spectral sequence.



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The differential d_r on $u_{2^{k+1}\sigma}$ described in the theorem is the last one possible since its target, $a_\sigma^{2^{k+1}} f_{2^{k+1}-1}$, lies on the vanishing line.

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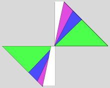

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Some slice differentials (continued)

We have

$$f_{2^{k+1}-1}\overline{\Delta}_k^{(8)} \quad = \quad a_{(2^{k+1}-1)\rho_8} N r_{2^{k+1}-1} N r_{2^k-1}$$

Some slice differentials (continued)

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\begin{aligned}
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&= a_{2^k\rho_8}\overline{\Delta}_{k+1}^{(8)} f_{2^k-1}
\end{aligned}
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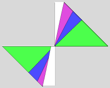
Corollary

In the $RO(G)$-graded slice spectral sequence for $\left(\overline{\Delta}_k^{(8)}\right)^{-1} MU^{(4)}$, the class $u_{2\sigma}^{2^k}$ is a permanent cycle.

The proof of the Periodicity Theorem

The corollary shows that inverting a certain element makes a power of $u_{2\sigma}$ a permanent cycle.



The proof of the Periodicity Theorem

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The proof of the Periodicity Theorem

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We will get this by using the norm property of u, namely that if V is an oriented representation of a subgroup $H \subset G$ with $V^H = 0$ and induced representation V', then the norm functor N_h^g from H-spectra to G-spectra satisfies $N_h^g(u_V)u_{2\rho_{G/H}}^{|V|/2} = u_{V'}$.

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From this we can deduce that $u_{2\rho_8} = u_{8\sigma_3} N_4^8(u_{4\sigma_2}) N_2^8(u_{2\sigma_1})$,

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From this we can deduce that $u_{2\rho_8} = u_{8\sigma_3} N_4^8(u_{4\sigma_2}) N_2^8(u_{2\sigma_1})$, where σ_m denotes the sign representation on C_{2^m}.

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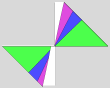
The proof of the Periodicity Theorem (continued)

We have $u_{2\rho_8} = u_{8\sigma_3} N_4^8(u_{4\sigma_2}) N_2^8(u_{2\sigma_1})$.



The proof of the Periodicity Theorem (continued)

We have $u_{2\rho_8} = u_{8\sigma_3} N_4^8(u_{4\sigma_2}) N_2^8(u_{2\sigma_1})$.

By the Corollary we can make a power of each factor a permanent cycle by inverting some $\overline{\Delta}_{k_m}^{(2^m)}$ for $1 \leq m \leq 3$.

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The proof of the Periodicity Theorem (continued)

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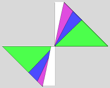

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- Inverting $\overline{\Delta}_4^{(2)}$ makes $u_{32\sigma_1}$ a permanent cycle.
- Inverting $\overline{\Delta}_2^{(4)}$ makes $u_{8\sigma_2}$ a permanent cycle.

The proof of the Periodicity Theorem (continued)

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- Inverting $\overline{\Delta}_1^{(8)}$ makes $u_{4\sigma_3}$ a permanent cycle.
- Inverting the product D of the norms of all three makes $u_{32\rho_8}$ a permanent cycle.

The proof of the Periodicity Theorem (continued)

Let

$$D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)}).$$

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The we define $\tilde{M} = D^{-1} MU^{(4)}$ and $M = \tilde{M}^{C_8}$.

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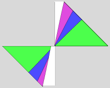
Since the inverted element is represented by a map from $S^{m\rho_8}$, the slice spectral sequence for $\pi_*(M)$ has the usual properties:



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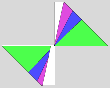
- It is concentrated in the first and third quadrants and confined by vanishing lines of slopes 0 and 7.



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Since the inverted element is represented by a map from $S^{m\rho_8}$, the slice spectral sequence for $\pi_*(M)$ has the usual properties:

- It is concentrated in the first and third quadrants and confined by vanishing lines of slopes 0 and 7.
- It has the gap property, i.e., no homotopy between dimensions -4 and 0.

The proof of the Periodicity Theorem (continued)

Preperiodicity Theorem

Let $\Delta_1^{(8)} = u_{2\rho_8}(\overline{\Delta}_1^{(8)})^2 \in E_2^{16,0}(D^{-1}MU^{(4)})$. *Then* $(\Delta_1^{(8)})^{16}$ *is a permanent cycle.*

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The proof of the Periodicity Theorem (continued)

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To prove this, note that $(\Delta_1^{(8)})^{16} = u_{32\rho_8}\left(\overline{\Delta}_1^{(8)}\right)^{32}$.

The proof of the Periodicity Theorem (continued)

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To prove this, note that $(\Delta_1^{(8)})^{16} = u_{32\rho_8}\left(\overline{\Delta}_1^{(8)}\right)^{32}$. Both $u_{32\rho_8}$ and $\overline{\Delta}_1^{(8)}$ are permanent cycles, so $(\Delta_1^{(8)})^{16}$ is also one.

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The proof of the Periodicity Theorem (continued)

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Thus we have an equivariant map $\Sigma^{256}D^{-1}MU^{(4)} \to D^{-1}MU^{(4)}$ and a similar map on the fixed point set. The latter one is invertible because $u_{2\rho_8}^{32}$ restricts to the identity.

The proof of the Periodicity Theorem (continued)

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Thus we have proved

Periodicity Theorem

Let $M = (D^{-1}MU^{(4)})^{C_8}$. *Then* $\Sigma^{256}M$ *is equivalent to* M.