Global Methods in Homotopy Theory Seminar

Hopes and dreams about Artin-Schreier curves

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1. Recollections about Artin-Schreier curves

We will use the following notation throughout. Fix a prime p and positive integer f. Then let

$$\begin{array}{rclcrcl} e & = & p^f - 1 & & q & = & p - 1 \\ h & = & qf & & m & = & qe. \end{array}$$

Theorem 1 (2002). *Let $C(p, f)$ be the Artin-Schreier curve over $\mathbf{F}_p$ defined by the affine equation*

$$y^e = x^p - x.$$

(Assume that $(p, f) \neq (2, 1)$.) Then its Jacobian $J(C(p, f))$ has a 1-dimensional formal summand of height h.

Properties of $C(p, f)$:

- Its genus is $q(e-1)/2$, eg it is 0 in the excluded case, and 1 in the cases $(p, f) = (2, 2)$ and $(3, 1)$. In these cases C is an elliptic curve whose formal group law has height 2.
- Over $\mathbf{F}_{p^h}$ it has an action by the group
$$G = \mathbf{F}_p \rtimes \mu_m$$
given by
$$(x, y) \mapsto (\zeta^e x + a, \zeta y)$$
for $a \in \mathbf{F}_p$ and $\zeta \in \mu_m$.

Remarks

- Let $\mathbf{G}_n$ denote the extension of the Morava stabilizer group S_n by the Galois group C_n. Given a finite subgroup $G \subset \mathbf{G}_n$, Hopkins-Miller can construct a "homotopy fixed point spectrum" E_n^{hG}. The group G above was shown by Hewett to be a maximal finite subgroup of $\mathbf{G}_h$. It acts on the 1-dimensional summand of $\widehat{J}(C(p, f))$ in the appropriate way.
- The curve above does not lead to a Landweber exact functor and cohomology theory. In order to get on we need to lift the curve to characteristic 0 in the right way. We will describe such a lifting below.
- Gorbunov-Mahowald studied this curve for $f = 1$. They found a lifting of the curve to characteristic zero associated with the Lubin-Tate lift of the formal group law of height $p - 1$.

2. Deforming the Artin-Schreier curve

We want a lifting of $C(p, f)$ that admits a coordinate change similar to the one for the Weierstrass curve used in the construction of tmf. The equation will have the form

$$y^e = x^p + \cdots$$

with (nonaffine) coordinate change

$$\begin{aligned} x &\mapsto x + \tilde{t} \qquad \text{where } \tilde{t} = \sum_{i=1}^{f} t_i y^{(p^f - p^j)/p} \\ y &\mapsto y \end{aligned}$$

The t_i above are related to the generators of the same name in $BP_*(BP)$.

In order to state this precisely we need some notation. Let

$$I = (i_1, \ldots, i_f)$$

be an f-tuple of nonnegative integers and define

$$\begin{aligned} |I| &= \textstyle\sum_k i_k & ||I|| &= \textstyle\sum_k (p^k - 1) i_k \\ t^I &= \textstyle\prod_k t_k^{i_k} & I! &= \textstyle\prod_k i_k! \end{aligned}$$

The coefficients in our equation will be formal variables a_I with $|I| \leq p$ (where $a_0 = p!$) with topological dimension $2||I||$. We will sometimes write a_I as $a_{||I||}$. For $|I| \leq p$, I is uniquely determined by its norm $||I||$. The number of indices I with $0 < |I| \leq p$ is $\binom{p+f}{f} - 1$.

Then the equation for our curve is

$$\begin{aligned} y^e &= \sum_{i=0}^{p} \frac{x^{p-i}}{(p-i)!} \sum_{|I|=i} a_I y^{(ei-||I||)/p} \\ &= x^p + a_m x + \cdots \end{aligned}$$

(recall that $e = p^f - 1$) and the effect of the coordinate change on the coefficients a_I is given by

$$a_I \mapsto \sum_{J+K=I} a_J \frac{t^K}{K!}.$$

For $f = 1$ the equation simplifies to the Gorbunov-Mahowald equation

$$y^{p-1} = x^p + \sum_{i=1}^{p} \frac{a_{qi} x^{p-i}}{(p-i)!}$$

with coordinate change

$$a_{qi} \mapsto a_{qi} + \sum_{0<j<i} \frac{a_{qj} t_1^{i-j}}{(i-j)!} + \frac{p! t_1^i}{i!}.$$

Theorem 2 (2004). *Let*

$$\begin{aligned} A &= \mathbf{Z}_p[a_I\colon 0 < |I| \leq p] \\ \overline{A} &= A/(a_m - 1) \\ \overline{A} \supset J &= (a_i : i \neq m,) \end{aligned}$$

Then the Jacobian of curve above defined above over the ring $\overline{A}/J^2$ *has a 1-dimensional formal summand of height* h*. The corresponding formal group law has Landweber exact liftings to* $\overline{A}$ *and* $a_m^{-1}A$ *with the former given by*

$$v_r = \begin{cases} pa_{m+p^r-1} + a_{p^r-1} & \textit{if } 1 \leq r \leq \min(f, h-1) \\ a_{se+p^i-1} & \textit{if } f < r < h \textit{ and } p > 2 \\ m - 2a_{2e} & \textit{if } r = h \textit{ and } p = 2 \\ 1 & \textit{if } r = h \textit{ and } p > 2; \end{cases}$$

up to unit scalar, where $r = sf + i$ *with* $1 \leq i \leq f$*.*

There is an associated Hopf algebroid

$$\Gamma = A[t_1, \dots, t_f]$$

where each t_i is primitive and the right unit given by the coordinate change formula above.

Fantasy 3. *For each* (p, f) *as above there is a spectrum generalizing* tmf *whose homotopy can be computed by an Adams-Novikov type spectral sequence with*

$$E_2 = \mathrm{Ext}_\Gamma(A, A).$$

Remarks

(i) This fantasy is not likely to be true for $f > 1$ because the ring A is too large. Ideally its Krull dimension should be pf, the sum of the height of the formal group law and the number of coordinate change parameters.

Replace the equation above with

$$y^e = \prod_{j=1}^{p}(x + \tilde{r}_j)$$

with

$$\tilde{r}_j = \sum_{i=1}^{f} r_{j,i} y^{(p^f - p^i)/p} \qquad \text{and} \qquad |r_{j,i}| = 2(p^i - 1).$$

Thus we get a curve defined over the ring

$$R = \mathbf{Z}_p[r_{j,i} : 1 \leq j \leq p,\ 1 \leq i \leq f],$$

which has the desired Krull dimension.

However it leads to an uninteresting Ext group. The coordinate change above induces

$$r_{j,i} \mapsto r_{j,i} + t_i$$

and

$$\mathrm{Ext}_\Gamma^s(R) = \begin{cases} \mathbf{Z}_p[r_{j,i} - r_{p,i}] & \text{for } s = 0 \\ 0 & \text{for } s > 0. \end{cases}$$

The equation for the curve is actually defined over the subring

$$B = R^{\Sigma_p},$$

where Σ_p acts on R via the second subscript. This ring is a quotient of A, but its structure is unknown for $f > 1$ except for $(p, f) = (2, 2)$. It is clearly a module (presumably free of rank $p!^{f-1}$) over the subring

$$C = R^{\Sigma_p^f}$$

where the f copies of Σ_p act independently on the f sets of p generators of R. Its structure is well known, namely

$$C = \mathbf{Z}_p[\sigma_{k,i} : 1 \leq i \leq f,\ 1 \leq k \leq p]$$

where $\sigma_{k,i}$ is the kth elementary symmetric function in the variables $r_{1,i}, \ldots, r_{p,i}$. It is the image of $a_{k(p^i-1)}/(p-k)!$.

(ii) Relation to tmf. The case $(p, f) = (3, 1)$ leads to eo_2. We will say more about the Ext computation below.

For $(p, f) = (2, 2)$ our equation reads

$$y^3 = x^2 + (a_1 y + a_3)x + a_2 y^2 + a_4 y + a_6,$$

so our a_is are the Weierstrass a_is up to sign. In the ring B there is a relation

$$a_4^2 - a_1 a_3 a_4 = 4a_2 a_6 - a_2 a_3^2 - a_1^2 a_6,$$

which makes it a free module on $\{1, a_4\}$ over

$$C = \mathbf{Z}_2[a_1, a_2, a_3, a_6].$$

Our coordinate change is

$$y \mapsto y \qquad \text{and} \qquad x \mapsto x + t_1 y + t_2,$$

while in tmf it is

$$y \mapsto y + r \qquad \text{and} \qquad x \mapsto x + sy + t.$$

It seems likely that our fantasy (with A replaced by B) would lead to the spectrum

$$tmf \wedge \left(S^0 \cup_\nu e^4\right).$$

Our right unit formula is

$$\begin{aligned}
a_{(0,2)} = a_6 &\mapsto a_6 + a_3\, t_2 + {t_2}^2 \\
a_{(1,1)} = a_4 &\mapsto a_4 + a_3\, t_1 + a_1 t_2 + 2\, t_1\, t_2 \\
a_{(0,1)} = a_3 &\mapsto a_3 + 2\, t_2 \\
a_{(2,0)} = a_2 &\mapsto a_2 + a_1\, t_1 + {t_1}^2 \\
a_{(1,0)} = a_1 &\mapsto a_1 + 2\, t_1,
\end{aligned}$$

while in tmf it is

$$\begin{aligned}
a_6 &\mapsto a_6 + a_4\, r + a_3\, t + a_2\, r^2 \\
&\qquad + a_1\, r\, t + t^2 - r^3 \\
a_4 &\mapsto a_4 + a_3\, s + 2\, a_2\, r \\
&\qquad + a_1(r\, s + t) + 2\, s\, t - 3\, r^2 \\
a_3 &\mapsto a_3 + a_1\, r + 2\, t \\
a_2 &\mapsto a_2 + a_1\, s - 3\, r + s^2 \\
a_1 &\mapsto a_1 + 2\, s.
\end{aligned}$$

The former can be obtained from the latter by

$$\begin{aligned}
r &\mapsto 0 \\
s &\mapsto t_1 \\
t &\mapsto t_2
\end{aligned}$$

3. Some Ext calculations

Recall our right unit formula

$$\eta_R(a_I) = \sum_{J+K=I} a_J \frac{t^K}{K!}.$$

In particular

$$\eta_R(a_{p(p^i-1)}) = a_{p(p^i-1)} + \sum_{0<j<p} a_{j(p^i-1)} \frac{t_i^{p-j}}{(p-j)!} + t_i^p.$$

This leads to a change-of-rings isomorphism

$$\mathrm{Ext}_\Gamma(A, A) = \mathrm{Ext}_{\Gamma'}(A', A')$$

where

$$\begin{aligned} A' &= A/(a_{p\Delta_1}, \ldots, a_{p\Delta_f}) \\ \text{and} \qquad \Gamma' &= A'[t_1, \ldots, t_f]/(\eta_R(a_{p(p^i-1)}) - a_{p(p^i-1)}). \end{aligned}$$

Note that Γ' is a free A'-module of rank p^f.

Next it is convenient to filter by powers of the maximal ideal J in A'. We get

$$\begin{aligned} E_0A' &= \mathbf{Z}/(p)[a_I : 0 \le |I| \le p, I \ne p\Delta_i] \\ &= SM \qquad \text{where } M = J/J^2 \\ E_0\Gamma' &= E_0A' \otimes P \\ \text{where} \qquad P &= \mathbf{Z}/(p)[t_i]/(t_i^p) \end{aligned}$$

The P-comodule M is a vector space of rank

$$\binom{p+f}{f} - f.$$

For $f = 1$, M has basis

$$\{a_0, a_q, \ldots, a_{(p-1)q}\}$$

and is a free P-comodule. Its symmetric algebra is stably equivalent to

$$\mathbf{Z}/(p)[a_{(p-1)q}^p],$$

so above the 0-line we have

$$\mathrm{Ext}_P(SM) = \mathbf{Z}/(p)[\Delta] \otimes E(h_{1,0}) \otimes P(b_{1,0}).$$

where $\Delta = a_{(p-1)q}^p$. In the spectral sequence there are differentials

$$\begin{aligned} d_{2q+1}(\Delta) &= h_{1,0}b_{1,0}^q \\ d_{2q^2+1}(h_{1,0}\Delta^{p-1}) &= b_{1,0}^{q^2+1} \end{aligned}$$

We now turn to $(p, f) = (3, 2)$. The following is a picture of M.

$$\begin{array}{ccccc} a_{16} & \longleftarrow & a_{18} & & \\ \downarrow & & \downarrow & & \\ a_8 & \longleftarrow & a_{10} & \longleftarrow & a_{12} \\ \downarrow & & \downarrow & & \downarrow \\ a_0 & \longleftarrow & a_2 & \longleftarrow & a_4 \end{array}$$

Horizontal and vertcial arrows represent “Quillen operations” dual to t_1 and t_2 respectively. This comodule is is dual to unit coideal I.

The following 2-variable Poincaré series describes SM up to stable equivalance.

$$SM = \left(\frac{1}{1 - s^3t^{24}}\right)\left(\frac{1}{1 - s^3t^{72}}\right)\left(\frac{1 + \Sigma^{40}I^{-1}}{1 - s^3I^4}\right).$$

Without the term involving I^{-1}, the Ext group in positive filtrations is contained in

$$P(a_{12}^3, a_{18}^3, z) \otimes E(h_{1,0},\ h_{2,0}) \otimes P(b_{1,0},\ b_{2,0})$$

where $z \in \mathrm{Ext}^{-4,0}$. In particular,

$$\begin{aligned} a_4^3 &= z b_{1,0}^2 \\ a_{16}^3 &= z b_{2,0}^2 \end{aligned}$$

Tensoring with $1+\Sigma^{40} I^{-1}$ corresponds to tensoring the Ext group with $E(u)$ with $u \in \mathrm{Ext}^{1,40}$.

It is likely that there are virtual Adams differentials

$$\begin{aligned} d_5(z) &= h_{1,0} \\ d_9(z^2 h_{1,0}) &= b_{1,0} \\ d_5(a_{18}^3) &= h_{2,0} b_{2,0}^2 \\ d_9(h_{2,0} a_{18}^6) &= b_{2,0}^5 \end{aligned}$$

To get the 2-variable Poincaré series above:

Over $T(t_i)$, let x_i denote the class of the comodule which is the desuspension of the unit coideal I_i centered in dimension 0, so that $x_i^2 = 1$. We know that

$$\begin{aligned} S(\Sigma^n T(t_i)) &= \frac{1}{1 - s^3 t^{3n+6|v_i|}} \\ S(\Sigma^n x_i) &= \frac{1 + \Sigma^n x_i}{1 - s^3 t^{3n+3|v_i|/2}} \end{aligned}$$

As a stable comodule over $E(t_i)$, we have

$$I^n = \Sigma^{3n|v_i|/2} x_i^n.$$

Now over $T(t_1)$ we have

$$M = T(t_1) \oplus \Sigma^{16} T(t_1) \oplus \Sigma^{34} x_1$$

so

$$SM = \left(\frac{1}{1 - s^3 t^{24}}\right)\left(\frac{1}{1 - s^3 t^{72}}\right)\left(\frac{1 + s^{34} x_1}{1 - s^3 t^{108}}\right).$$

Similarly over $T(t_2)$ we have

$$M = T(t_2) \oplus \Sigma^{4} T(t_2) \oplus \Sigma^{16} x_2$$

so

$$SM = \left(\frac{1}{1 - s^3 t^{96}}\right)\left(\frac{1}{1 - s^3 t^{108}}\right)\left(\frac{1 + s^{16} x_2}{1 - s^3 t^{72}}\right).$$

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