# MIT Topology Seminar

# Toward higher chromatic analogs of tmf
or
Why I am spending this semester in Cambridge


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September 26, 2005

## 1. Introduction

The theory of topological modular forms ($tmf$) began with a calculation described in Hopkins–Mahowald's paper *From elliptic curves to homotopy theory* of 1995. Consider the elliptic curve is defined by the Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Under the affine coordinate change

$$y \mapsto y + r \qquad \text{and} \qquad x \mapsto x + sy + t$$

we get

$$\begin{aligned}
a_6 &\mapsto a_6 + a_4\,r + a_3\,t + a_2\,r^2 \\
&\qquad\quad +a_1\,r\,t + t^2 - r^3 \\
a_4 &\mapsto a_4 + a_3\,s + 2\,a_2\,r \\
&\qquad\quad +a_1(r\,s + t) + 2\,s\,t - 3\,r^2 \\
a_3 &\mapsto a_3 + a_1\,r + 2\,t \\
a_2 &\mapsto a_2 + a_1\,s - 3\,r + s^2 \\
a_1 &\mapsto a_1 + 2\,s.
\end{aligned}$$

This can be used to define a Hopf algrebroid $(A,\,\Gamma)$ with

$$\begin{aligned}
A &= \mathbf{Z}[a_1, a_2, a_3, a_4, a_6] \\
\Gamma &= A[r, s, t]
\end{aligned}$$

and right unit $\eta_R : A \to \Gamma$ given by the formulas above. Its Ext group is the $E_2$-term of a spectral sequence converging to $\pi_*(\mathrm{tmf})$. Tilman Bauer has written a nice account of this calculation.

[Show Hopkins-Mahowald illustration.]

**Theorem 1.** *[Hopkins et al]*
*The Ext group* $\mathrm{Ext}_\Gamma(A, A)$ *defined above is the* $E_2$*-term of an spectral sequence converging to the homotopy of an* $E_\infty$*-ring spectrum* $tmf$*.*

There are two steps in this construction.

(i) A 1-dimensional formal group law over and ring $R$ leads to a homomorphism (called a genus)

$$\varphi : \pi_*(MU) \to R$$

by Quillen's theorem. The functor

$$X \mapsto MU_*(X) \otimes_\varphi R$$

is a homology theory if certain algebraic conditions on $\varphi$ are satisfied; this is the Landweber Exact Functor Theorem.

Suppose $E$ is an elliptic curve defined over $R$. It is a 1-dimensional algebraic group, and choosing a local paramater at the identity leads to a formal group law $\widehat{E}$, the formal completion of $E$. Thus we can apply the machinery above and get an $R$-valued genus.

Here are two examples:

(a) The *Jacobi quartic*, defined by the equation

$$y^2 = 1 - 2\delta x^2 + \epsilon x^4,$$

is an elliptic curve over the ring

$$R = \mathbf{Z}[1/2, \delta, \epsilon].$$

The resulting formal group law is

$$F(x,y) = \frac{x\,R(y) + y\,R(x)}{1 - \epsilon x^2 y^2},$$

where

$$R(t) = \sqrt{1 - 2\delta t^2 + \epsilon t^4};$$

this calculation is originally due to Euler. The resulting genus is known to satisfy Landweber's conditions (Landweber-Ravenel-Stong), and this leads to one definition of elliptic cohomology theory.

(b) The Weierstrass curve defined above over $A[\Delta^{-1}] = \mathbf{Z}[a_1, a_2, a_3, a_4, a_6][\Delta^{-1}]$, where the discriminant $\Delta$ is a certain polynomial in the $a_i$, leads to a formal group law satisfying Landweber's conditions.

(ii) The spectrum $tmf$ is derived from elliptic cohomology as a certain homotopy inverse limit defined in terms of a sheaf of $E_\infty$-ring spectra over the moduli stack of elliptic curves. The details of this theory will appear soon.

## 2. Program to generalize $tmf$:

(i) Find a curve $C$ of genus $g$ whose Jacobian $J(C)$ (a $g$-dimensional abelian variety) has a formal completion $\widehat{J}(G)$ (a $g$-dimensional formal group law) has a 1-dimensional formal summand of height $n$. For $n > 2$, this means that $g \geq n$.

(ii) "Deform" this curve into one with similar properties defined over a Landweber exact ring. This will give a higher chromatic analog of elliptic cohomology.

(iii) Find a groups of coordinate transformations and compute the resulting Ext group. Use Lurie's machinery to prove an analog of Theorem 1.

## 3. Artin-Schreier curves: the first step in the program

**Theorem 2** (2002)**.** *Let $C(p, f)$ be the Artin-Schreier curve over $\mathbf{F}_p$ defined by the affine equation*

$$y^e = x^p - x \qquad \text{where } e = p^f - 1.$$

*(Assume that $(p, f) \neq (2, 1)$.) Then its Jacobian $J(C(p, f))$ has a 1-dimensional formal summand of height $h = (p - 1)f$.*

Properties of $C(p, f)$:

- Its genus is $(p - 1)(e - 1)/2$, eg it is 0 in the excluded case, and 1 in the cases $(p, f) = (2, 2)$ and $(3, 1)$. In these cases $C$ is an elliptic curve whose formal group law has height 2.
- Over $\mathbf{F}_{p^h}$ it has an action by the group

  $$G = \mathbf{F}_p \rtimes \mu_{(p-1)e}$$

  given by

  $$(x, y) \mapsto (\zeta^e x + a, \zeta y)$$

  for $a \in \mathbf{F}_p$ and $\zeta \in \mu_{(p-1)e}$.

  Let $\mathbf{G}_n$ denote the extension of the Morava stabilizer group $S_n$ by the Galois group $C_n$. Given a finite subgroup $G \subset \mathbf{G}_n$, Hopkins-Miller can construct a "homotopy fixed point spectrum" $E_n^{hG}$. The group $G$ above was shown by Hewett to be a maximal finite subgroup of $\mathbf{G}_n$ for $n = h = (p-1)f$. It acts on the 1-dimensional summand of $\widehat{J}(C(p, f))$ in the appropriate way.

Remarks

- This result was known to and cited by Manin in 1963. Most of what is needed for the proof can be found in Katz's 1979 Bombay Colloquium paper and in Koblitz' Hanoi notes.
- The original proof rests on the determination of the zeta function of the curve by Davenport-Hasse in 1934, and on some properties of Gauss sums proved by Stickelberger in 1890. The method leads to complete determination of $\widehat{J}(C(p,f))$ up to isogeny.
- We have a simpler proof based on Honda's theory of commutative formal group laws developed in the early '70s. It does not rely on knowledge of the zeta function and is therefore more flexible. The starting point for it is the following consequence of the Lagrange inversion formula. If $z = x - x^p$, then $x$ has the power series expansion

$$\begin{aligned} x &= \sum_{i \geq 0} \frac{1}{pi+1} \binom{pi+1}{i} z^{1+(p-1)i} \\ &= z + z^p + pz^{2p-1} + \frac{p(3p-1)}{2} z^{3p-2} + \cdots \end{aligned}$$

## More remarks

- The curve above does not lead to a Landweber exact functor and cohomology theory. In order to get on we need to lift the curve to characteristic 0 in the right way. We will describe such a lifting below.
- Gorbunov-Mahowald studied this curve for $f = 1$. They found a lifting of the curve to characteristic zero associated with the Lubin-Tate lift of the formal group law of height $p-1$.

## 4. Sketch of the Honda theoretic proof of Theorem 2 .

**Notation:**

- Let $A$ be a torsion free local ring with maximal ideal $\mathfrak{m}$ and residue field of characteristic $p$ with an automorphism $a \mapsto a^\sigma$ which reduces the Frobenius (or $p$th power) automorphism modulo $\mathfrak{m}$.
- Let $A_\sigma\langle\langle T\rangle\rangle$ be the ring of noncommutative power series in $T$ over $A$ subject to the rule $Ta = a^\sigma T$.
- Let $M_d(A)$ denote the ring of ring of $d \times d$-matrices over $A$, and define $M_d(A)_\sigma\langle\langle T\rangle\rangle$ in a similar way.
- Suppose we have a $d$-dimensional formal group law $F$ over the ring $A$. $F$ is characterized by its logarithm $f$, which is a vector of $d$ power series in $d$ variables over the field $A \otimes \mathbf{Q}$. Given such an $f$, let $f^{\sigma^i}$ be the vector of power series obtained from $f$ by applying $\sigma^i$ to each coefficient. Given a matrix $H = \sum_i C_i T^i$ in $M_d(A)_\sigma\langle\langle T\rangle\rangle$, define

$$(H * f)(x_1, \ldots, x_d) = \sum_i C_i f^{\sigma^i}(x_1^{p^i}, \ldots, x_d^{p^i}).$$

**Definition 3.** *We say that $H$ is a* **Honda matrix** *for $F$ (or for the vector $f$) and that $F$* is of type *$H$, if $H \equiv pI_d$ modulo $T$ ($I_d$ is the $d \times d$ identity matrix) and $(H * f)(x) \equiv 0$ modulo $(p)$.*

*Two such matrices are said to be* **equivalent** *if they differ by unit multiplication on the left.*

**Examples of Honda matrices:**

- For $d = 1$ and $A = \mathbf{Z}_p$, let $H$ be the $1 \times 1$ matrix with entry $h = p - T^n$ for a positive integer $n$. Then
$$f(x) = \sum_{i \geq 0} \frac{x^{p^{ni}}}{p^i}$$
and $F$ is the formal group law for the Morava K-theory $K(n)_*$.
- Let $A = \mathbf{Z}_p[[u_1, u_2, \ldots u_{n-1}]]$ for a positive integer $m$, and let $u_i^\sigma = u_i^p$. Let $H$ be the $1 \times 1$ matrix with entry
$$h = p - T^n - \sum_{0<i<n} u_i T^i.$$
Then $f(x)$ is the logarithm for the Lubin-Tate lifting of the formal group law above.

**Theorem 4** (Honda, 1970)**.** *For $A$ as above and $\mathfrak{m}$-adically complete, the strict isomorphism classes of d-dimensional formal group laws over $A$ correspond bijectively to the equivalence classes of matrices*

$$H \in M_d(\mathbf{Z}_p)_\sigma\langle\langle T\rangle\rangle$$

*congruent to $pI_d$ modulo degree 1. $H$ and $f$ are related by the formula*

$$f(x) = (H^{-1} * p)(x).$$

**Remarks:**

- Under suitable hypotheses, the determinant of the Honda matrix is he charctaeristic polynomial of the Frobenius endomorphism ($x \mapsto x^p$) of the mod $\mathfrak{m}$ reduction of the formal group law.

**Question:** How can we find the Honda matrix for the formal completion of the Jacobian of an algebraic curve?

**Theorem 5** (Honda, 1973)**.** *Let $C$ be a curve of genus $g$ over $A$ with smooth reduction modulo $\mathfrak{m}$, and let*

$$\{\omega_1, \ldots, \omega_g\}$$

*be a basis for the space of holomorphic 1-forms of $C$ written as power series in a local parameter $y$, and let*

$$\psi_i = \int_0^y \omega_i.$$

*Then if $H$ is a Honda matrix for the vector $(\psi_1, \ldots, \psi_g)$, it is also one for $\widehat{J}(C)$, the formal completion of the Jacobian $J(C)$.*

Note that $\psi$ above is a vector of power series in one variable over $A \otimes \mathbf{Q}$, while the logarithm of $\widehat{J}(C)$ is a vector of power seires in $g$ variables. The theorem asserts that they have the same Honda matrix.

**Theorem 6** (Tate, 1966)**.** *The determinant of the Honda matrix for the curve $C$ of genus $g$ above is a polynomial of the form*

$$T^{2g} + \cdots + p^g.$$

Recall that our curve $C(p,f)$ is defined by the affine equation

$$y^e = x^p - x \qquad \text{where } e = p^f - 1.$$

A basis for the holomorphic 1-forms for $C(p,f)$ is

$$\{\omega_{i,j} \colon ei + pj < (e-1)(p-1) - 1\},$$

where

$$\omega_{i,j} = \frac{x^i y^j \mathrm{d}x}{y^{e-1}}.$$

We denote the integral of its expansion in terms of $y$ by $\psi_{ei+j+1}$, which has a power series expansion of the form

$$y^{ei+j+1} \sum_{k \geq 0} c_{ei+j+1,k} y^{mk} \qquad \text{where } m = (p-1)e.$$

We have explicit formulas for thses coefficients.

**Examples of Honda matrices of the curves $C(p,f)$:**

- For $C(2,3)$ (where $g = 3$ and $m = 7$), the integrals have the form

$$\begin{aligned} \psi_1 &\in y\mathbf{Q}_2[[y^7]] \\ \psi_2 &\in y^2\mathbf{Q}_2[[y^7]] \\ \psi_3 &\in y^3\mathbf{Q}_2[[y^7]] \end{aligned}$$

This means that

$$\begin{array}{lll} T\psi_1 \in y^2\mathbf{Q}_2[[y^7]] & T^2\psi_1 \in y^4\mathbf{Q}_2[[y^7]] & \\ T\psi_2 \in y^4\mathbf{Q}_2[[y^7]] & T^2\psi_2 \in y^8\mathbf{Q}_2[[y^7]] & \subset y\mathbf{Q}_2[[y^7]] \\ T\psi_3 \in y^6\mathbf{Q}_2[[y^7]] & T^2\psi_3 \in y^{12}\mathbf{Q}_2[[y^7]] & \subset y^5\mathbf{Q}_2[[y^7]] \end{array}$$

This implies that the Honda matrix has the form

$$H = \begin{bmatrix} h_{1,1}(T^3) & T^2h_{1,2}(T^3) & 0 \\ Th_{2,1}(T^3) & h_{2,2}(T^3) & 0 \\ 0 & 0 & h_{3,3}(T^3) \end{bmatrix}$$

where

$$h_{i,j}(T^3) = \sum_{k\geq 0} h_{i,j,k}T^{3k}$$

with $h_{i,i,0} = 2$. Thus we have

$$\begin{aligned} \det H &= (h_{1,1}(T^3)h_{2,2}(T^3) - h_{2,1}(T^3)Th_{1,2}(T^3)T^2)h_{3,3}(T^3) \\ &\equiv h_{2,1,0}h^{\sigma}_{1,2,0}h_{3,3,1}T^6 \qquad \bmod (2, T^7) \\ &= T^6 + \cdots + 8 \qquad \text{by Theorem 6.} \end{aligned}$$

This means that $h_{3,3,1}$ is a unit, which gives us a 1-dimensional summand of $F$ of height 3 as desired.

- For $C(3,2)$ (where $g = 7$ and $m = 16$), we get integrals $\psi_i$ for

$$i \in S = \{1,\ 2,\ 3,\ 4,\ 5,\ 9,\ 10\}\,.$$

A similar computation shows that $\psi_5$ corresponds to a 1-dimensional formal summand of height 4. The argument boils down to seeing how the orbits of $\mathbf{Z}/(16)$ under multiplication by 3 intersect the set $S$ above. One such orbit is $\{5, 15, 13, 7\}$, whose intersection with $S$ is the singleton $\{5\}$.

## 5. Deforming the Artin-Schreier curve

We want a lifting of $C(p,f)$ that admits a coordinate change similar to the one for the Weierstrass curve. The equation will have the form

$$y^e = x^p + \cdots$$

with (nonaffine) coordinate change

$$\begin{aligned} x &\mapsto x + \tilde{t} \qquad \text{where } \tilde{t} = \sum_{i=1}^{f} t_i y^{p^{f-1}-p^{i-1}} \\ y &\mapsto y \end{aligned}$$

The $t_i$ above are related to the generators of the same name in $BP_*(BP)$.

In order to state this precisely we need some notation. Let

$$I = (i_1, \ldots, i_f)$$

be a set of nonnegative integers and define

$$\begin{aligned} |I| &= \sum_k i_k \\ ||I|| &= \sum_k (p^k - 1) i_k \\ t^I &= \prod_k t_k^{i_k} \\ I! &= \prod_k i_k! \end{aligned}$$

The coefficients in our equation will be formal variables $a_I$ (where $a_0 = p!$) with topological dimension $2||I||$. Then the equation for our curve is

$$y^e \; = \; \sum_{i=0}^{p} \frac{x^{p-i}}{(p-i)!} \sum_{|I|=i} a_I y^{(ei-||I||)/p}$$

(recall that $e = p^f - 1$) and the effect of the coordinate change on the coefficients $a_I$ is given by

$$a_I \mapsto \sum_{J+K=I} a_J \frac{t^K}{K!}.$$

For $f = 1$ the equation simplifies to the Gorbunov-Mahowald equation

$$y^{p-1} = x^p + \sum_{i=1}^{p} \frac{a_i x^{p-i}}{(p-i)!}$$

with coordinate change

$$a_i \mapsto a_i + \sum_{0<j<i} \frac{a_j t_1^{i-j}}{(i-j)!} + \frac{p! t_1^i}{i!}.$$

**Theorem 7** (2004)**.** *The Jacobian of curve defined above over the ring*

$$A = \mathbf{Z}_p[a_I \colon 0 < |I| \leq p]$$

*has a 1-dimensional formal summand of height $(p-1)f$.*

There is an associated Hopf algebroid

$$\Gamma = A[t_1, \ldots, t_f]$$

where each $t_i$ is primitive and the right unit given by the coordinate change formula above. Note that

$$\eta_R(a_{p\Delta_i}) = a_{p\Delta_i} + \sum_{0<j<p} a_{j\Delta_i}\frac{t_i^{p-j}}{(p-j)!} + t_i^p.$$

This leads to a change-of-rings isomorphism

$$\mathrm{Ext}_\Gamma(A, A) = \mathrm{Ext}_{\Gamma'}(A', A')$$

where

$$\begin{aligned} A' &= A/(a_{p\Delta_1}, \ldots, a_{p\Delta_f}) \\ \text{and} \qquad \Gamma' &= A'[t_1, \ldots, t_f]/(\eta_R(a_{p\Delta_i}) - a_{p\Delta_i}). \end{aligned}$$

Note that $\Gamma'$ is a free $A'$-module of rank $p^f$.

**Conjecture 8.** *For each $(p, f)$ as above there is a spectrum generalizing* tmf *whose homotopy can be computed by an Adams-Novikov type spectral sequence with*

$$E_2 = \mathrm{Ext}_\Gamma(A, A).$$

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