TOWARD HIGHER CHROMATIC ANALOGS OF ELLIPTIC COHOMOLOGY

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A homomorphism

$$\varphi : MU_* \to R$$

is called *an R-valued genus* and is equivalent by Quillen's theorem that φ to a 1-dimensional formal group law over R. It is also known that the functor

$$X \mapsto MU_*(X) \otimes_\varphi R$$

is a homology theory if φ satisfies certain conditions spelled out in Landweber's Exact Functor Theorem.

Now suppose E is an elliptic curve defined over R. It is a 1-dimensional algebraic group, and choosing a local paramater at the identity leads to a formal group law $\widehat{E}$, the formal completion of E. Thus we can apply the machinery above and get an R-valued genus.

For example, the *Jacobi quartic*, defined by the equation

$$y^2 = 1 - 2\delta x^2 + \epsilon x^4,$$

is an elliptic curve over the ring

$$R = \mathbf{Z}[1/2, \delta, \epsilon].$$

The resulting formal group law is the power series expansion of

$$F(x, y) = \frac{x\sqrt{1 - 2\delta y^2 + \epsilon y^4} + y\sqrt{1 - 2\delta x^2 + \epsilon x^4}}{1 - \epsilon x^2 y^2};$$

this calculation is originally due to Euler. The resulting genus is known to satisfy Landweber's conditions, and this leads to one definition of elliptic cohomology.

Date: March 19, 2003.

THE HOPKINS-MAHOWALD AFFINE GROUP ACTION. The Weierstrass equation for a general elliptic curve is

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Under the affine coordinate change

$$y \mapsto y + r \qquad \text{and} \qquad x \mapsto x + sy + t$$

we get

$$\begin{aligned} a_6 &\mapsto a_6 + a_4\, r + a_3\, t + a_2\, r^2 \\ &\qquad + a_1\, r\, t + t^2 - r^3 \\ a_4 &\mapsto a_4 + a_3\, s + 2\, a_2\, r \\ &\qquad + a_1(r\, s + t) + 2\, s\, t - 3\, r^2 \\ a_3 &\mapsto a_3 + a_1\, r + 2\, t \\ a_2 &\mapsto a_2 + a_1\, s - 3\, r + s^2 \\ a_1 &\mapsto a_1 + 2\, s. \end{aligned}$$

This can be used to define an action of the affine group on the ring

$$A = \mathbf{Z}[a_1, a_2, a_3, a_4, a_6].$$

Its cohomology is the E_2-term of a spectral sequence converging to $\pi_*(\mathrm{tmf})$.

Theorem 1. *Let $C(p, f)$ be the Artin-Schreier curve over $\mathbf{F}_p$ defined by the affine equation*

$$y^e = x^p - x \qquad \textit{where } e = p^f - 1.$$

(Assume that $f > 1$ when $p = 2$.) Then its Jacobian has a 1-dimensional formal summand of height $(p-1)f$.

Conjecture 2. *Let $\tilde{C}(p, f)$ be the curve over over $\mathbf{Z}_p[u_1, \ldots, u_{(p-1)f-1}]$ defined by*

$$y^e = x^p - x + \sum_{i=0}^{(p-1)f-2} u_{i+1} x^{p-1-[i/f]} y^{p^f-1-p^{i-[i/f]f}}.$$

Then its Jacobian has a formal 1-dimensional summand isomorphic to the Lubin-Tate lifting of the formal group law of height $(p-1)f$.

Properties of $C(p, f)$:

- Its genus is $(p-1)(d-1)/2$.
- It has an action by the group
$$G = \mathbf{F}_p \rtimes \mu_{(p-1)e}$$
given by
$$(x, y) \mapsto (\zeta^d x + a, \zeta y)$$
for $a \in \mathbf{F}_p$ and $\zeta \in \mu_{(p-1)e}$. This group is a maximal finite subgroup of the $(p-1)f$th Morava stablizer group, and it acts appropriately on the 1-dimensional formal summand.
- The case $f = 1$ was studied by Gorbunov-Mahowald.

Examples:

- $C(2, 2)$ and $C(3, 1)$ are elliptic curves whose formal group laws have height 2.
- $C(2, 3)$ has genus 3 and a 1-dimensional formal summand of height 3.
- $C(2, 4)$ and $C(3, 2)$ each has genus 7 and a 1-dimensional formal summand of height 4.

Theorem 3 (Honda). *Let A be a $\mathbf{Z}_p$-algebra with an automorphism σ such that a^σ is congruent to a^p mod p. Then the strict isomorphism classes of n-dimensional formal group laws over A correspond bijectively to the equivalence classes of matrices $H \in M_n(\mathbf{Z}_p)_\sigma\langle\langle T\rangle\rangle$ congruent to pI_n modulo degree 1. H and f are related by the formula*

$$f(x) = (H^{-1} * p)(x).$$

Examples:

- For $n = 1$ and $A = \mathbf{Z}_p$, let H be the 1×1 matrix with entry $u = p - T^h$ for a positive integer h. Then

$$f(x) = \sum_{i \geq 0} \frac{x^{p^{hi}}}{p^i}$$

  and F is the formal group law for the Morava K-theory $K(h)_*$.
- Let $A = \mathbf{Z}_p[[u_1, u_2, \ldots u_{h-1}]]$ for a positive integer h, and let $u_i^\sigma = u_i^p$. Let H be the 1×1 matrix with entry

$$u = p - T^h - \sum_{0<i<h} u_i T^i.$$

  Then $f(x)$ is the logarithm for the Lubin-Tate lifting of the formal group law above.

Theorem 4 (Honda). *For a curve C of genus g, let*

$$\{\omega_1, \ldots, \omega_g\}$$

be a basis for the space of holomorphic 1-forms of C written as power series in a local parameter y, and let

$$\psi_i = \int_0^y \omega_i.$$

If H is a Honda matrix for the vector $(\psi_1, \ldots, \psi_g)$, then it is also one for $\widehat{J}(C)$.

Theorem 5 (Tate). *The determinant of the Honda matrix for a curve of genus g is a polynomial of the form*

$$T^{2g} + \cdots + p^g.$$

A basis for the holomorphic 1-forms for $C(p, f)$ is

$$\{\omega_{i,j} \colon ei + pj < (e-1)(p-1) - 1\},$$

where

$$\omega_{i,j} = \frac{x^i y^j \mathrm{d}x}{y^{e-1}}.$$

We denote its integral of its expansion in terms of y by ψ_{ei+j+1}, which has power series expansion of the form

$$y^{ei+j+1} \sum_{n \geq 0} c_{ei+j+1,n} y^{mn}.$$

Examples:

- For $C(2,3)$ (where $g = 3$ and $m = 7$) the integrals have the form

$$\begin{aligned} \psi_1 &\in y\mathbf{Q}[[y^7]] \\ \psi_2 &\in y^2\mathbf{Q}[[y^7]] \\ \psi_3 &\in y^3\mathbf{Q}[[y^7]] \end{aligned}$$

  The orbits in $\mathbf{Z}/(7)$ under mutiplication by 2 include

$$\{1,\ 2,\ 4\} \qquad \text{and} \qquad \{3,\ 6,\ 5\}\,.$$

- For $C(3,2)$ (where $g = 7$ and $m = 16$) the integrals have the form

$$\begin{aligned} \psi_1 &\in y\mathbf{Q}[[y^{16}]] \\ \psi_2 &\in y^2\mathbf{Q}[[y^{16}]] \\ \psi_3 &\in y^3\mathbf{Q}[[y^{16}]] \\ \psi_4 &\in y^4\mathbf{Q}[[y^{16}]] \\ \psi_5 &\in y^5\mathbf{Q}[[y^{16}]] \\ \psi_9 &\in y^9\mathbf{Q}[[y^{16}]] \\ \psi_{10} &\in y^{10}\mathbf{Q}[[y^{16}]] \end{aligned}$$

  The orbits in $\mathbf{Z}/(16)$ under the multiplication by 3 include

$$\{1,3,9,11\}\,,\ \{15,13,7,5\}\,,\ \{2,6\}\,,\ \{14,10\}\,,\ \text{ and } \{4,12\}\,.$$

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