A solution to the Arf-Kervaire invariant problem

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How we construct $\Omega$
The slice spectral sequence

Vic Snaith and Bill Browder in 1981
Photo by Clarence Wilkerson
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A wildly popular dance craze

Can you do the Arf Invariant?
Is it a jig or a reel?
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Drawing by Carolyn Snaith 1981
London, Ontario
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Mike Hill, myself and Mike Hopkins
Photo taken by Bill Browder
February 11, 2010
Our main result

Our main theorem can be stated in three different but equivalent ways:

1. **Manifold formulation:** It says that a certain geometrically defined invariant $\Phi(M)$ (the Arf-Kervaire invariant, to be defined later) on certain manifolds $M$ is always zero.

2. **Stable homotopy theoretic formulation:** It says that certain long sought hypothetical maps between high dimensional spheres do not exist.

3. **Unstable homotopy theoretic formulation:** It says something about the EHP sequence, which has to do with unstable homotopy groups of spheres.

The problem solved by our theorem is nearly 50 years old. There were several unsuccessful attempts to solve it in the 1970s. They were all aimed at proving the opposite of what we have proved.
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Snaith’s book

**Stable Homotopy Around the Arf-Kervaire Invariant**, published in early 2009,
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“As ideas for progress on a particular mathematics problem atrophy it can disappear. Accordingly I wrote this book to stem the tide of oblivion.”
“For a brief period overnight we were convinced that we had the method to make all the sought after framed manifolds
Snaith’s book (continued)

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“For a brief period overnight we were convinced that we had the method to make all the sought after framed manifolds- a feeling which must have been shared by many topologists working on this problem. All in all, the temporary high of believing that one had the construction was sufficient to maintain in me at least an enthusiastic spectator’s interest in the problem.”
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Our main result (continued)

Here is the stable homotopy theoretic formulation.
Our main result (continued)

Here is the stable homotopy theoretic formulation.

**Main Theorem**

The Arf-Kervaire elements $\theta_j \in \pi_{2j+1-2+n}(S^n)$ for large $n$ do not exist for $j \geq 7$.  

The $\theta_j$ in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial. It follows from Browder's theorem of 1969 that such things can exist only in dimensions that are 2 less than a power of 2.
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Our main result (continued)

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After 1980, the problem faded into the background because it was thought to be too hard.
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After 1980, the problem faded into the background because it was thought to be too hard. Our proof is two giant steps away from anything that was attempted in the 70s. We now know that the world of homotopy theory is very different from what they had envisioned then.
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Mark Mahowald’s sailboat

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Pontryagin’s early work on homotopy groups of spheres

Lev Pontryagin 1908-1988

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- Assume $f$ is smooth. We know that any such map is can be continuously deformed to a smooth one.
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Pontryagin’s early work on homotopy groups of spheres

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- Assume \( f \) is smooth. We know that any such map is can be continuously deformed to a smooth one.
- Pick a regular value \( y \in S^n \). Its inverse image will be a smooth \( k \)-manifold \( M \) in \( S^{n+k} \).
- By studying such manifolds, Pontryagin was able to deduce things about maps between spheres.
Pontryagin’s early work (continued)

Let $D^n$ be the closure of an open ball around a regular value $y \in S^n$. 
Pontryagin’s early work (continued)

Let $D^n$ be the closure of an open ball around a regular value $y \in S^n$. If it is sufficiently small, then $V^{n+k} = f^{-1}(D^n) \subset S^{n+k}$ is an $(n+k)$-manifold homeomorphic to $M \times D^n$ with boundary homeomorphic to $M \times S^{n-1}$. 
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A local coordinate system around around the point $y \in S^n$ pulls back to one around $M$ called a framing.
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There is a way to reverse this procedure. A framed manifold $M^k \subset S^{n+k}$ determines a map $f : S^{n+k} \to S^n$. 
Pontryagin’s early work (continued)

To proceed further, we need to be more precise about what we mean by continuous deformation.
Pontryagin’s early work (continued)

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Two maps \( f_1, f_2 : S^{n+k} \to S^n \) are **homotopic** if there is a continuous map \( h : S^{n+k} \times [0, 1] \to S^n \) (called a homotopy between \( f_1 \) and \( f_2 \)) such that
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h(x, 0) = f_1(x) \quad \text{and} \quad h(x, 1) = f_2(x).
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If \( y \in S^n \) is a regular value of \( h \), then \( h^{-1}(y) \) is a framed \((k + 1)\)-manifold \( N \subset S^{n+k} \times [0, 1] \) whose boundary is the disjoint union of \( M_1 = f_1^{-1}(y) \) and \( M_2 = f_2^{-1}(y) \). This \( N \) is called a framed cobordism between \( M_1 \) and \( M_2 \). When it exists the two closed manifolds are said to be framed cobordant.
Pontryagin’s early work (continued)

Here is an example of a framed cobordism for $n = k = 1$. 

![Framed cobordism](image)
Pontryagin’s early work (continued)

\[ \Omega_k := \{ \text{stably framed } k\text{-manifolds} \} / \text{cobordism} \]

**Theorem:** The above construction gives a bijection

\[ \pi_{n+k}(S^n) \cong \Omega_k \]

where

\[ \pi_{n+k}(S^n) := \{ \text{maps } S^{n+k} \to S^n \} / \text{homotopy} \]
Pontryagin’s early work (continued)

Pontryagin (1930’s)

\[ k=0 \]
Pontryagin’s early work (continued)

Pontryagin (1930’s)

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Pontryagin’s early work (continued)

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\[ \pi_n(S^n) = \mathbb{Z} \]
Pontryagin’s early work (continued)

**Pontryagin (1930’s)**

\[ \pi_n(S^n) = \mathbb{Z} \]

(\text{the degree})

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Pontryagin’s early work (continued)

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Pontryagin (1930’s)

$k=0$

$\pi_n(S^n) = \mathbb{Z}$

(the degree)

$k=1$

$\pi_{n+1}(S^n) = \mathbb{Z}/2$
Pontryagin’s early work (continued)

Pontryagin (1930’s)

$k=2$
Pontryagin’s early work (continued)

Pontryagin (1930’s)

\[ k=2 \quad \text{genus } M = 0 \quad \Rightarrow \quad M \text{ is a boundary} \]

(since \( S^2 \) bounds a disk and \( \pi_2(\text{GL}_n(\mathbb{R}))=0 \))
Pontryagin’s early work (continued)

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k=2 \quad \text{genus } M = 0 \quad \Rightarrow \quad M \text{ is a boundary}

\text{(since } S^2 \text{ bounds a disk and } \pi_2(\text{GL}_n(\mathbb{R})) = 0)\]

Suppose the genus of M is greater than 0.
Pontryagin’s early work (continued)

Pontryagin (1930’s)

k=2
Pontryagin’s early work (continued)
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Pontryagin (1930’s)

$k=2$

choose an embedded arc
Pontryagin’s early work (continued)

Pontryagin (1930’s)

k=2

choose an embedded arc

cut the surface open and glue in disks
Pontryagin’s early work (continued)

Pontryagin (1930’s)

k=2

framed surgery
Pontryagin’s early work (continued)

\textbf{Pontryagin (1930’s)}

\textbf{Obstruction: } \varphi : H_1(M; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2
Pontryagin’s early work (continued)

Pontryagin (1930’s)

Obstruction: \( \varphi : H_1(M; \mathbb{Z}/2) \to \mathbb{Z}/2 \)

Argument: Since the dimension of \( H_1(M; \mathbb{Z}/2) \) is even, there is always a non-zero element in the kernel of \( \varphi \), and so surgery can be performed.
Pontryagin’s early work (continued)

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**Argument:** Since the dimension of \( H_1(M; \mathbb{Z}/2) \) is even, there is always a non-zero element in the kernel of \( \varphi \), and so surgery can be performed.

**Conclusion:** \( \Omega_2 = \pi_{n+2}(S^n) = 0. \)
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Pontryagin’s mistake for $k = 2$

The map $\varphi : H_1(M; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2$ is not a homomorphism!
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The map $\varphi : H_1(M; \mathbb{Z}/2) \to \mathbb{Z}/2$ is not a homomorphism!
The Arf invariant of a quadratic form in characteristic 2

Let $\lambda$ be a nonsingular anti-symmetric bilinear form on a free abelian group $H$ of rank $2n$ with mod 2 reduction $\overline{H}$.
The Arf invariant of a quadratic form in characteristic 2

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$$\lambda(a_i, a_i') = 0 \quad \lambda(b_j, b_j') = 0 \quad \text{and} \quad \lambda(a_i, b_j) = \delta_{i,j}.$$
The Arf invariant of a quadratic form in characteristic 2

Let $\lambda$ be a nonsingular anti-symmetric bilinear form on a free abelian group $H$ of rank $2n$ with mod 2 reduction $\overline{H}$. It is known that $\overline{H}$ has a basis of the form \{a_i, b_j: 1 \leq i \leq n\} with

$$\lambda(a_i, a_{i'}) = 0 \quad \lambda(b_j, b_{j'}) = 0 \quad \text{and} \quad \lambda(a_i, b_j) = \delta_{i,j}.$$ 

In other words, $\overline{H}$ has a basis for which the bilinear form’s matrix has the symplectic form

$$
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & 1 \\
1 & 0 \\
& & \ddots \\
& & & & 0 & 1 \\
& & & & 1 & 0
\end{pmatrix}.
$$
A quadratic refinement of $\lambda$ is a map $q : \overline{H} \to \mathbb{Z}/2$ satisfying
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Its Arf invariant is

$$\text{Arf}(q) = \sum_{i=1}^{n} q(a_i)q(b_i) \in \mathbb{Z}/2.$$
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$$\text{Arf}(q) = \sum_{i=1}^{n} q(a_i)q(b_i) \in \mathbb{Z}/2.$$  

In 1941 Arf proved that this invariant (along with the number $n$) determines the isomorphism type of $q$.  

A solution to the Arf-Kervaire invariant problem

Mike Hill
Mike Hopkins
Doug Ravenel

Background and history

Our main result
Pontryagin's early work
The Arf-Kervaire formulation
Questions raised by our theorem

Our strategy
Ingredients of the proof
The spectrum \( \Omega \)
How we construct \( \Omega \)
The slice spectral sequence

From my stamp collection

[Image of a stamp from the Republic of France, depicting Evariste Galois (1811-1832), a revolutionary and mathematician, with the denomination 2.10 + 0.40.]
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1937: Alan Turing’s theory of digital computing
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Money talks: Arf’s definition republished in 2009

Cahit Arf 1910-1997
The Kervaire invariant of a framed \((4m + 2)\)-manifold

Let \(M\) be a \(2m\)-connected smooth closed framed manifold of dimension \(4m + 2\).
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Let \(M\) be a \(2m\)-connected smooth closed framed manifold of dimension \(4m + 2\). Let \(H = H_{2m+1}(M; \mathbb{Z})\), the homology group in the middle dimension. Each \(x \in H\) is represented by an immersion \(i_x : S^{2m+1} \looparrowright M\) with a stably trivialized normal bundle.

Michel Kervaire 1927-2007

Kervaire defined a quadratic refinement \(q\) on its mod 2 reduction in terms of the trivialization of each sphere's normal bundle. The Kervaire invariant \(\Phi(M)\) is defined to be the Arf invariant of \(q\).

For \(m = 0\), Kervaire's \(q\) coincides with Pontryagin's \(\phi\).
The Kervaire invariant of a framed \((4m + 2)\)-manifold

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For \(m = 0\), Kervaire’s \(q\) coincides with Pontryagin’s \(\varphi\).
The Kervaire invariant of a framed \((4m + 2)\)-manifold (continued)

What can we say about \(\Phi(M)\)?
The Kervaire invariant of a framed \((4m + 2)\)-manifold
(continued)

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- For \(m = 0\) there is a framing on the torus \(S^1 \times S^1 \subset \mathbb{R}^4\) with nontrivial Kervaire invariant.
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Pontryagin (1930’s)
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More of what we can say about \(\Phi(M)\).
The Kervaire invariant of a framed $(4m + 2)$-manifold (continued)

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![Topology circa 1960: Kervaire's example](image)

$X = N/\partial N$

(a triangulable manifold)
The Kervaire invariant of a framed \((4m + 2)\)-manifold (continued)

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- Brown-Peterson (1966) showed that it vanishes for all positive even \(m\).

Ed Brown

Frank Peterson 1930-2000
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- Browder (1969) showed that it can be nontrivial only if \(m = 2^{j-1} - 1\) for some positive integer \(j\). This happens iff the element \(h_j^2\) is a permanent cycle in the Adams spectral sequence.

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- In the decade following Browder’s theorem, many topologists tried without success to construct framed manifolds with nontrivial Kervaire invariant in all dimensions 2 less than a power of 2.
The Kervaire invariant of a framed \((4m+2)\)-manifold

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- Our theorem says \(\theta_j\) does not exist for \(j \geq 7\). The case \(j = 6\) is still open.
Questions raised by our theorem

Adams spectral sequence formulation. We now know that the $h_{2j}$ for $j \geq 7$ are not permanent cycles, so they have to support nontrivial differentials. We have no idea what their targets are.

Unstable homotopy theoretic formulation. In 1967 Mahowald published an elaborate conjecture about the role of the $\theta_j$ (assuming that they all exist) in the unstable homotopy groups of spheres. Since they do not exist, a substitute for his conjecture is needed. We have no idea what it should be.

Our method of proof offers a new tool, the slice spectral sequence, for studying the stable homotopy groups of spheres. We look forward to learning more with it in the future. We will illustrate it at the end of the talk.
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- While space \( X \) has a homotopy group \( \pi_k(X) \) for each positive integer \( k \),

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For the sphere spectrum \( S^0 \), \( \pi_k(S^0) \) is the usual homotopy group \( \pi_{n+k}(S^n) \) for \( n > k + 1 \).
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- We use **complex cobordism theory**. This is a branch of algebraic topology having deep connections with algebraic geometry and number theory. It includes some highly developed computational techniques that began with work by Milnor, Novikov and Quillen in the 60s. A pivotal tool in the subject is the theory of formal group laws.
Ingredients of the proof (continued)

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Ingredients of the proof (continued)

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- We also make use of newer less familiar methods from equivariant stable homotopy theory. This means there is a finite group $G$ (a cyclic 2-group) acting on all spaces in sight, and all maps are required to commute with these actions. When we pass to spectra, we get homotopy groups indexed not just by the integers $\mathbb{Z}$, but by $\text{RO}(G)$, the real representation ring of $G$. Our calculations make use of this richer structure.
The spectrum $\Omega$

We will produce a map $S^0 \rightarrow \Omega$, where $\Omega$ is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.
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(i) **Detection Theorem.** It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each $\theta_j$ is nontrivial.
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We will produce a map $S^0 \to \Omega$, where $\Omega$ is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.

(i) Detection Theorem. It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each $\theta_j$ is nontrivial. This means that if $\theta_j$ exists, we will see its image in $\pi_\ast(\Omega)$. 

(ii) Periodicity Theorem. It is 256-periodic, meaning that $\pi_k(\Omega)$ depends only on the reduction of $k$ modulo 256.

(iii) Gap Theorem. $\pi_k(\Omega) = 0$ for $-4 < k < 0$. This property is our zinger. Its proof involves a new tool we call the slice spectral sequence.
The spectrum $\Omega$

We will produce a map $S^0 \to \Omega$, where $\Omega$ is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.

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The background and history

Our main result
Pontryagin's early work
The Arf-Kervaire formulation
Questions raised by our theorem
Our strategy
Ingredients of the proof
The spectrum $\Omega$
How we construct $\Omega$
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How we construct $\Omega$

Our spectrum $\Omega$ will be the fixed point spectrum for the action of $\mathbb{C}_8$ (the cyclic group of order 8) on an equivariant spectrum $\tilde{\Omega}$.

To construct it we start with the complex cobordism spectrum $\text{MU}$. It can be thought of as the set of complex points of an algebraic variety defined over the real numbers. This means that it has an action of $\mathbb{C}_2$ defined by complex conjugation. The fixed point set of this action is the set of real points, known to topologists as $\text{MO}$, the unoriented cobordism spectrum. In this notation, $U$ and $O$ stand for the unitary and orthogonal groups.
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In particular we get a $C_8$-spectrum

$$MU^{(4)} = \text{Map}_{C_2}(C_8, MU).$$
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This spectrum is not periodic, but it has a close relative $\tilde{\Omega}$ which is.
A solution to the Arf-Kervaire invariant problem

Mike Hill
Mike Hopkins
Doug Ravenel

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The slice spectral sequence

A homotopy fixed point spectral sequence
The corresponding slice spectral sequence