TOWARD HIGHER CHROMATIC ANALOGS OF ELLIPTIC COHOMOLOGY

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1. What is elliptic cohomology?

Definition 1. *For a ring R, an R-***valued genus** *on a class of closed manifolds is a function φ that assigns to each manifold M an element $\varphi(M) \in R$ such that*

(i) $\varphi(M_1 \coprod M_2) = \varphi(M_1) + \varphi(M_2)$
(ii) $\varphi(M_1 \times M_2) = \varphi(M_1)\varphi(M_2)$
(iii) $\varphi(M) = 0$ *if M is a boundary.*

Equivalently, φ is a homomorphism from the appropriate cobordism ring Ω to R.

Examples of genera:

- The Hirzebruch signature is a $\mathbf{Z}$-valued genus for smooth oriented manifolds.
- The $\widehat{A}$-genus is a $\mathbf{Z}$-valued genus for Spin manifolds.
- The Euler characteristic and Todd genus are $\mathbf{Z}$-valued genera for complex manifolds.

Definition 2. *A 1-dimensional* **formal group law** *over R is a power series $F(x, y) \in R[[x, y]]$ satisfying*

(i) $F(x, 0) = F(0, x) = x.$
(ii) $F(y, x) = F(x, y)$
(iii) $F(x, F(y, z)) = F(F(x, y), z).$

Remarks:

- A commututative 1-dimensional analytic Lie group also leads to such a power series, but here there is no convergence requirement.
- An n-dimensional formal group law (consisting of n power series in $2n$ variables) can be defined in a similar way.

Examples for formal group laws:

(i) $F(x, y) = x + y$, the additive formal group law.
(ii) $F(x, y) = x + y + xy$, the multiplicative formal group law. Here $1 + F(x, y) = (1 + x)(1 + y)$, which makes the associativity condition transparent.
(iii) $F(x, y) = \frac{x+y}{1+xy}$, the formal group law associated with the hyperbolic tangent function via the addition formula.

$$\tanh(x + y) = F(\tanh(x), \tanh(y)).$$

A theorem of Quillen says that in the complex case (where $\Omega = MU_*$, the complex cobordism ring), φ is a equivalent to a 1-dimensional formal group law over R. For example, the formal group law (iii) above corresponds to the signature of a complex manifold.

It is also known that the functor

$$X \mapsto MU^*(X) \otimes_\varphi R$$

is a cohomology theory if φ satisfies certain conditions spelled out in Landweber's Exact Functor Theorem.

Now suppose E is an elliptic curve defined over a ring R. It is a 1-dimensional algebraic group, and choosing a local paramater at the identity leads to a formal group law $\widehat{E}$, the formal completion of E. Thus we can apply the machinery above and get an R-valued genus.

For example, the *Jacobi quartic*, defined by the equation

$$v^2 = 1 - 2\delta u^2 + \epsilon u^4,$$

is an elliptic curve over the ring

$$R = \mathbf{Z}[1/2, \delta, \epsilon].$$

The resulting formal group law is the power series expansion of

$$F(x,y) = \frac{x\sqrt{1-2\delta y^2+\epsilon y^4} + y\sqrt{1-2\delta x^2+\epsilon x^4}}{1-\epsilon x^2y^2};$$

this calculation is originally due to Euler. The resulting genus is known to satisfy Landweber's conditions, and this leads to one definition of elliptic cohomology.

A more general elliptic curve is defined by the Weierstarss equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Under the affine coordinate change

$$y \mapsto y + r \qquad \text{and} \qquad x \mapsto x + sy + t$$

we get

$$\begin{array}{rcl} a_6 & \mapsto & a_6 + a_4\,r + a_3\,t + a_2\,r^2 \\ & & \quad +a_1\,r\,t + t^2 - r^3 \\ a_4 & \mapsto & a_4 + a_3\,s + 2\,a_2\,r \\ & & \quad +a_1(r\,s + t) + 2\,s\,t - 3\,r^2 \\ a_3 & \mapsto & a_3 + a_1\,r + 2\,t \\ a_2 & \mapsto & a_2 + a_1\,s - 3\,r + s^2 \\ a_1 & \mapsto & a_1 + 2\,s. \end{array}$$

This can be used to define a Hopf algrebroid (A, Γ) with

$$\begin{array}{rcl} A & = & \mathbf{Z}[a_1, a_2, a_3, a_4, a_6] \\ \Gamma & = & A[r, s, t] \end{array}$$

and right unit $\eta_R : A \to \Gamma$ given by the formulas above. Its Ext group is the E_2-term of a spectral sequence converging to $\pi_*(\mathrm{tmf})$.

2. What does "chromatic" mean?

The stable homotopy category localized a prime p can be studied via a series of increasingly complicated Bousfield localization functors L_n for $n \geq 0$, which detect "v_n-periodic" phenomena. The follwoing diagram of functors and natural transformations is known as the chromatic tower.

$$L_0 \longleftarrow L_1 \longleftarrow L_2 \longleftarrow L_3 \longleftarrow \cdots$$

- L_0 is rationalization. Rational stable homotopy theory is very well understood. It detects only the 0-stem in the stable homotopy groups of spheres.
- L_1 is localization with respect to K-theory. It detects the image of J and the α family in the stable homotopy groups of spheres. The Lichtenbaum-Quillen conjecture is a statement about L_1 of algebraic K-theory.
- For an odd prime p, L_2 is equivalent to localization with respect to elliptic cohomology as defined above. It detects the β family in the stable homotopy groups of spheres. Davis' nonimmersion theorem for real projective spaces was proved using related methods at the prime 2. The theory of topological modular forms of Hopkins *et al* is a refinement of elliptic cohomolgy.
- In general, L_n can be constructed by algebraic methods related to BP-theory. For $n > 2$ there is no known comparable geometric definition of L_n. It detects higher Greek letter families in the stable homotopy groups of spheres. The nth Morava K-theory is closely related to it.

A key to understanding the algebraic underpinnings of the chromatic point of view is the following.

Definition 3. *Let F be 1-dimensional formal group law over a field k of characteristic p. For a positive integer m, the m series is defined inductively by*

$$[m]_F(x) = F(x, [m-1]_F(x)),$$

where $[1]_F(x) = x$. The p-series is either 0 or has the form

$$[p]_F(x) = ax^{p^n} + \cdots$$

for some nonzero $a \in k$. The **height** *of F is the integer n. It is defined to be ∞ when $[p]_F(x) = 0$, which happens when $F(x, y) = x + y$.*

Examples of m-series:

(i) For the additive formal group law, $[m](x) = mx$.
(ii) For the multiplicative formal group law, $[m](x) = (1 + x)^m - 1$.
(iii) For $F(x, y) = \frac{x+y}{1+xy}$, we have $[m](\tanh(x)) = \tanh(mx)$.

Examples of heights:

- The multiplicative formal group law (which is associated with K-theory) has height 1 at every prime.
- The formal group law associated with the Hirzebruch signature has height 1 at every odd prime, and infinite height at the prime 2.
- The formal group law associated with an elliptic curve is known to have height at most 2. If the height is 1, the curve is said to be *ordinary*; otherwise it is said to be *supersingular*.
- v_n-periodic phenomena (the nth layer in the chromatic tower) are related to formal group laws of height n.

3. What is a higher chromatic analog of elliptic cohomology?

Question: How can we attach 1-dimensional formal group laws of height > 2 to geometric objects (such as algebraic curves) and use them get insight into cohomology theories that go deeper into the chromatic tower?

Program:

- Let C be a curve of genus g over some ring R.
- Its Jacobian $J(C)$ is an abelian variety of dimension g.
- $J(C)$ has a formal completion $\widehat{J}(C)$ which is a g-dimensional formal group law.
- If $\widehat{J}(C)$ has a 1-dimensional summand of height n, then by Quillen's theorem it gives us a genus associated with the curve C.

Caveat: Note that a 1-dimensional summand of the formal completion $\widehat{J}(C)$ is *not* the same thing as 1-dimensional factor of the Jacobian $J(C)$. The latter would be an elliptic curve, whose formal competion can have height at most 2. There is a theorem that says if an abelian variety A has a 1-dimensional formal summand of height n for $n > 2$, then the dimension of A (and the genus of the curve, if A is a Jacobian) is at least n.

Theorem 4. *For prime p and positive integer f, let $C(p,f)$ be the Artin-Schreier curve over $\mathbf{F}_p$ defined by the affine equation*

$$y^e = x^p - x, \qquad \text{where } e = p^f - 1.$$

(Assume that $f > 1$ when $p = 2$.) Then its Jacobian has a 1-dimensional formal summand of height $(p-1)f$.

Conjecture 5. *Let $\tilde{C}(p,f)$ be the curve over over $\mathbf{Z}_p[u_1, \ldots, u_{(p-1)f-1}]$ defined by*

$$y^d = x^p - x + \sum_{i=0}^{(p-1)f-2} u_{i+1} x^{p-1-[i/f]} y^{p^{f-1} - p^{i-[i/f]f}}.$$

Then its Jacobian has a formal 1-dimensional subgroup isomorphic to the Lubin-Tate lifting of the formal group law of height $(p-1)f$.

Properties of $C(p,f)$:

- Its genus is $(p-1)(e-1)/2$. (Thus it is zero in the excluded case $(p,f) = (2,1)$.)
- It has an action by the group
$$G = \mathbf{F}_p \rtimes \mu_{(p-1)e}$$
given by
$$(x,y) \mapsto (\zeta^e x + a, \zeta y)$$
for $a \in \mathbf{F}_p$ and $\zeta \in \mu_{(p-1)e}$. This group is a maximal finite subgroup of the $(p-1)f$th Morava stabilizer group, and it acts appropriately on the 1-dimensional formal summand.
- For $f = 1$ (and $p > 2$) the lifting conjecture was proved by Gorbunov-Mahowald.

Examples of these curves:

- $C(2,2)$ and $C(3,1)$ are elliptic curves whose formal group laws have height 2.
- $C(2,3)$ has genus 3 and a 1-dimensional formal summand of height 3.
- $C(2,4)$ and $C(3,2)$ each has genus 7 and a 1-dimensional formal summand of height 4.

Remarks:

- This result was known to and cited by Manin in 1963. Most of what is needed for the proof can be found in Katz's 1979 Bombay Colloquium paper and in Koblitz' Hanoi notes.
- The original proof rests on the determination of the zeta function of the curve by Davenport-Hasse in 1934, and on some properties of Gauss sums proved by Stickelberger in 1890. The method leads to complete determination of $\widehat{J}(C(p,f))$.
- We have reproved the theorem using Honda's theory of commutative formal group laws developed in the early '70s. This proof does not rely on knowledge of the zeta function and is thus a more promising approach to the lifting problem.

4. Sketch of the Honda theoretic proof.

Notation:

- Let A be a torsion free local ring with maximal ideal $\mathfrak{m}$ and residue field of characteristic p with an automorphism $a \mapsto a^\sigma$ which reduces the Frobenius (or pth power) automorphism modulo $\mathfrak{m}$.
- Let $A_\sigma\langle\langle T\rangle\rangle$ be the ring of noncommutative power series in T over A subject to the rule $Ta = a^\sigma T$.
- Let $M_d(A)$ denote the ring of ring of $d \times d$-matrices over A, and define $M_d(A)_\sigma\langle\langle T\rangle\rangle$ in a similar way.
- Suppose we have a d-dimensional formal group law F over the ring A. F is characterized by its logarithm f, which is a vector of d power series in d variables over the field $A \otimes \mathbf{Q}$. Given such an f, let f^{σ^i} be the vector of power series obtained from f by applying σ^i to each coefficient. Given a matrix $H = \sum_i C_i T^i$ in $M_d(A)_\sigma\langle\langle T\rangle\rangle$, define
$$(H * f)(x_1, \ldots, x_d) = \sum_i C_i f^{\sigma^i}(x_1^{p^i}, \ldots, x_d^{p^i}).$$

Definition 6. *We say that H is a* **Honda matrix** *for F (or for the vector f) and that F* is of type H, *if $H \equiv pI_d$ modulo T (I_d is the $d \times d$ identity matrix) and $(H * f)(x) \equiv 0$ modulo (p). Two such matrices are said to be equivalent if they differ by unit multiplication.*

Examples of Honda matrices:

- For $d = 1$ and $A = \mathbf{Z}_p$, let H be the 1×1 matrix with entry $h = p - T^n$ for a positive integer n. Then
$$f(x) = \sum_{i \geq 0} \frac{x^{p^{ni}}}{p^i}$$
and F is the formal group law for the Morava K-theory $K(n)_*$.
- Let $A = \mathbf{Z}_p[[u_1, u_2, \ldots u_{n-1}]]$ for a positive integer m, and let $u_i^\sigma = u_i^p$. Let H be the 1×1 matrix with entry
$$h = p - T^n - \sum_{0<i<n} u_i T^i.$$
Then $f(x)$ is the logarithm for the Lubin-Tate lifting of the formal group law above.

Theorem 7 (Honda, 1970)**.** *For A as above and $\mathfrak{m}$-adically complete, the strict isomorphism classes of d-dimensional formal group laws over A correspond bijectively to the equivalence classes of matrices*
$$H \in M_d(\mathbf{Z}_p)_\sigma\langle\langle T\rangle\rangle$$
congruent to pI_d modulo degree 1. H and f are related by the formula
$$f(x) = (H^{-1} * p)(x).$$

Theorem 8 (Honda, 1973)**.** *Let C be a curve of genus g over A with smooth reduction modulo $\mathfrak{m}$, and let*

$$\{\omega_1, \dots, \omega_g\}$$

be a basis for the space of holomorphic 1-forms of C written as power series in a local parameter y, and let

$$\psi_i = \int_0^y \omega_i.$$

Then if H is a Honda matrix for the vector $(\psi_1, \dots, \psi_g)$, it is also one for $\widehat{J}(C)$, the formal completion of the Jacobian $J(C)$.

Note that ψ above is a vector of power series in one variable over $A \otimes \mathbf{Q}$, while the logarithm of $\widehat{J}(C)$ is a vector of power seires in g variables. The theorem asserts that they have the same Honda matrix.

Theorem 9 (Tate, 1966)**.** *The determinant of the Honda matrix for the curve C of genus g above is a polynomial of the form*

$$T^{2g} + \cdots + p^g.$$

Recall that our curve $C(p, f)$ is defined by the affine equation

$$y^e = x^p - x \qquad \text{where } e = p^f - 1.$$

A basis for the holomorphic 1-forms for $C(p, f)$ is

$$\{\omega_{i,j} \colon ei + pj < (e-1)(p-1) - 1\},$$

where

$$\omega_{i,j} = \frac{x^i y^j \, \mathrm{d}x}{y^{e-1}}.$$

We denote the integral of its expansion in terms of y by ψ_{ei+j+1}, which has a power series expansion of the form

$$y^{ei+j+1} \sum_{k \geq 0} c_{ei+j+1,n} y^{mk} \qquad \text{where } m = (p-1)e.$$

Examples of Honda matrices of curves:

- For $C(2,3)$ (where $g=3$ and $m=7$), the integrals have the form

$$\begin{aligned} \psi_1 &\in y\mathbf{Q}[[y^7]] \\ \psi_2 &\in y^2\mathbf{Q}[[y^7]] \\ \psi_3 &\in y^3\mathbf{Q}[[y^7]] \end{aligned}$$

  The orbits in $\mathbf{Z}/(7)$ under multiplication by 2 include

$$\{1,\,2,\,4\} \qquad \text{and} \qquad \{3,\,6,\,5\}\,.$$

  The set of indices $\{1,2,3\}$ has two elements in the first orbit and one in the second. This implies that the Honda matrix has the form

$$H = \begin{bmatrix} h_{1,1}(T^3) & h_{1,2}(T^3)T^2 & 0 \\ h_{2,1}(T^3)T & h_{2,2}(T^3) & 0 \\ 0 & 0 & h_{3,3}(T^3) \end{bmatrix}$$

  where

$$h_{i,j}(T^3) = \sum_{k\geq 0} h_{i,j,k}T^{3k}$$

  with $h_{i,i,0}=2$. Thus we have

$$\begin{aligned} \det H &= (h_{1,1}(T^3)h_{2,2}(T^3) - h_{2,1}(T^3)Th_{1,2}(T^3)T^2)h_{3,3}(T^3) \\ &\equiv h_{2,1,0}h^{\sigma}_{1,2,0}h_{3,3,1}T^6 \qquad \bmod (2,T^7) \\ &= T^6+\cdots+8 \qquad \text{by Theorem 9.} \end{aligned}$$

  This means that $h_{3,3,1}$ is a unit, which gives us a 1-dimensional summand of F of height 3 as desired.

- For $C(3,2)$ (where $g=7$ and $m=16$), the set of indices is

$$\{1,\,2,\,3,\,4,\,5,\,9,\,10\}\,,$$

  and one of the orbits in $\mathbf{Z}/(16)$ under multiplication by 3 is

$$\{15,\,13,\,7,\,5\}\,.$$

  The fact that our index set contains a single element in this orbit leads (by an argument similar to the above) to a 1-dimensional summand of height 4.

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