Lecture 2

A solution to the Arf-Kervaire invariant problem

Instituto Superior Técnico
Lisbon
May 6, 2009

Mike Hill
University of Virginia
Mike Hopkins
Harvard University 2.1
Doug Ravenel
University of Rochester

1 The spectrum M

The spectrum M

Our goal is to prove

Main Theorem. *The Arf-Kervaire elements* $\theta_j \in \pi_{2^{j+1}-2}(S^0)$ *do not exist for* $j \geq 7$.

Our strategy is to find a map $S^0 \to M$ to a nonconnective spectrum M with the following properties.

(i) It has an Adams-Novikov spectral sequence in which the image of each θ_j is nontrivial.
(ii) It is 256-periodic, meaning $\Sigma^{256} M \cong M$.
(iii) $\pi_{-2}(M) = 0$. 2.2

The spectrum M (continued)

We will construct an equivariant C_8-spectrum $\tilde{M}$ and show that its homotopy fixed point set $\tilde{M}^{hC_*}$ (to be defined below) and its actual fixed point set $\tilde{M}^{C_8}$ are equivalent.

- The homotopy of $\tilde{M}^{hC_*}$ can be computed using a spectral sequence similar to that of Hopkins-Miller. Twenty year old algebraic methods can be used to show that it detects the θ_js.
- In order to establish (ii) and (iii), we will use equivariant methods to construct a new spectral sequence (the *slice spectral sequence*) converging to the homotopy of the actual fixed point set $\tilde{M}^{C_8}$. 2.3

2 MU

The complex cobordism spectrum

MU is the Thom spectrum for the universal complex vector bundle, which is defined over the classifying space of the stable unitary group, BU.

- MU has an action of the group C_2 via complex conjugation. The fixed point set is MO, the Thom spectrum for the universal real vector bundle.
- $H_*(MU;\mathbf{Z}) = \mathbf{Z}[b_i : i > 0]$ where $|b_i| = 2i$.
- $H_*(MO;\mathbf{Z}/2) = \mathbf{Z}/2[a_i : i > 0]$ where $|a_i| = i$.
- $\pi_*(MU) = \mathbf{Z}[x_i : i > 0]$ where $|x_i| = 2i$. This is the complex cobordism ring.
- $\pi_*(MO) = \mathbf{Z}/2[y_i : i > 0, i \neq 2^k - 1]$ where $|y_i| = i$. This is the unoriented cobordism ring. 2.4

3 Formal group laws

Formal group laws

The following algebraic structure plays a central role in complex cobordism theory.

A *(1-dimensional commutative) formal group law over a ring R* is a power series

$$F(x,y) = \sum_{i,j\geq 0} a_{i,j}x^i y^j \in R[[x,y]]$$

satisfying

(i) (Commutativity) $F(y,x) = F(x,y)$. This implies $a_{j,i} = a_{i,j}$.

(ii) (Identity element) $F(x,0) = F(0,x) = x$. This implies $a_{1,0} = a_{0,1} = 1$ and $a_{i,0} = a_{0,i} = 0$ for $i \neq 1$.

(iii) (Associativity) $F(x,F(y,z)) = F(F(x,y),z)$. This implies more complicated relations among the $a_{i,j}$. 2.5

Examples of formal group laws

- $x+y$, the additive formal group law.
- $x+y+xy$, the multiplicative formal group law. Note here that $1+F(x,y) = (1+x)(1+y)$.
- $(x+y)/(1-xy)$, the addition formula for the tangent function.
- $$\frac{x\sqrt{1-y^4}+y\sqrt{1-x^4}}{1+x^2y^2},$$

  This formal group law is defined over $\mathbf{Z}[1/2]$. It is the addition formula for the elliptic integral

  $$\int_0^x \frac{dt}{\sqrt{1-t^4}}.$$

  It is originally due to Euler, see *De integratione aequationis differentialis* $(m\mathrm{d}x)/\sqrt{1-x^4} = (n\mathrm{d}y)/\sqrt{1-x^4}$, 1753. 2.6

The Lazard ring and the universal formal group law

Let

$$L = \mathbf{Z}[a_{i,j}]/(\text{relations})$$

where the relations are those implied by the definition of a formal group law. We give this ring a grading by $|a_{i,j}| = 2(i+j-1)$.

There is formal group law G over L given by the formula in the definition. It is universal in the following sense.

Given any formal group law F over any ring R, there is a unique ring homomorphism $\lambda : L \to R$ such that

$$F(x,y) = \lambda(G(x,y)),$$

where $\lambda(G(x,y))$ is the formal group law over R obtained from G by applying λ to each of the $a_{i,j}$. 2.7

Quillen's theorem

Lazard showed that L and $\pi_*(MU)$ are isomorphic as graded rings. Quillen showed that this is not an accident. The isomorphism is defined by a formal group law over $\pi_*(MU)$ defined as follows.

There is a cohomology theory associated with MU under which

$$\begin{aligned} MU^*(\mathbf{C}P^\infty) &= \pi_*(MU)[[x]] \\ \text{and} \quad MU^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) &= \pi_*(MU)[[x\otimes 1, 1\otimes x]]. \end{aligned}$$

The map $\mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$ (corresponding to tensor product of complex line bundles) induces a homomorphism

$$MU^*(\mathbf{C}P^\infty) \to MU^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty)$$

that sends x to a power series in $x\otimes 1$ and $1\otimes x$ which is a formal group law over $\pi_*(MU)$. 2.8

Quillen's theorem (continued)

Quillen's Theorem (1969). *The homomorphism $\theta : L \to \pi_*(MU)$ induced by the formal group law over $\pi_*(MU)$ defined above is an isomorphism.*

This means that the internal structure of MU, and the associated homology and cohomology theories, is intimately related to the structure of formal group laws. 2.9

4 Some relatives of MU

Some relatives of MU

Here is an example of this connection.

After localizing at a prime p, MU splits into a wedge of suspensions of smaller spectra (Brown-Peterson) BP with

$$\pi_*(BP) = \mathbf{Z}_{(p)}[v_n : n > 0] \qquad \text{where } |v_n| = 2p^n - 2.$$

Brown and Peterson originally constructed it (in 1967) via its Postnikov tower.

Quillen's 1969 paper gave a more elegant construction in terms of p-typical formal group laws. A theorem of Cartier says that any formal group law over a $\mathbf{Z}_{(p)}$-algebra is canonically isomorphic to one with certain special properties.

The Brown-Peterson splitting is the topological analog of Cartier's theorem. 2.10

More relatives of MU

The *Morava spectrum* E_n (for a positive integer n) is an E_∞-ring spectrum such that $\pi_*(E_n)$ obtained from $\pi_*(BP)$ as follows:

(i) Invert v_n and kill the higher generators.
(ii) Complete with respect to the ideal $I_n = (p, v_1, \ldots, v_{n-1})$.
(iii) Tensor over $\mathbf{Z}_p$ (the p-adic integers) with the Witt ring $W(\mathbf{F}_{p^n})$; this is equivalent to adjoining $(p^n - 1)$th roots of unity.

The ring $\pi_*(E_n)$ was studied by Lubin-Tate. They showed that it classifies liftings (to Artinian rings) of a certain formal group law F_n over $\mathbf{F}_{p^n}$, the *Honda formal group law*. 2.11

5 The Hopkins-Miller theorem

The Morava stabilizer group S_n

S_n is the automorphism group of the Honda formal group law F_n. It a crucial ingredient in chromatic stable homotopy theory.

Its action on F_n lifts to an action on $\pi_*(E_n)$, the Lubin-Tate ring. This action is defined by certain formulas but is mysterious in practice.

It is a pro-p-group isomorphic to a group of units in a certain division algebra D_n of rank n^2 over the p-adic numbers $\mathbf{Q}_p$.

D_n contains each degree n field extension of $\mathbf{Q}_p$, including the cyclotomic ones.

We will be interested in some finite subgroups of S_n. 2.12

The Hopkins-Miller theorem

The algebraically defined action of S_n on $\pi_*(E_n)$ leads to action on E_n itself, but it is defined only up to homotopy.

In the early 90s Hopkins and Miller showed that the action can be rigidified enough to construct homotopy fixed points sets E_n^{hG} for closed (e.g. finite) subgroups G.

$E_n^{hS_n}$ is $L_{K(n)}S^0$, the localization of the sphere spectrum with respect to the nth Morava K-theory.

Hopkins-Miller Theorem (1992?). *For each closed subgroup $G \subset S_n$ there is a homotopy fixed point set E_n^{hG} and a spectral sequence*

$$H^*(G; \pi_*(E_n)) \implies \pi_*(E_n^{hG}).$$

It coincides with the Adams-Novikov spectral sequence for E_n^{hG}.

2.13

Finite subgroups of S_n

The finite subgroups of S_n have been completely classified by Hewett, but only three of them concern us here. The prime is always 2.

- $C_2 = \{\pm 1\} \subset S_1$, which is $\mathbf{Z}_2^\times$, the units in the 2-adic integers.
- $C_4 \subset S_2$. The group S_2 is in the division algebra D_2 which contains each quadratic extension of the 2-adic numbers. Hence it contains fourth roots of unity.
- $C_8 \subset S_4$. The division algebra D_4 contains eighth roots of unity for similar reasons.

2.14

6 Our first guess at M

A first attempt to define the magic spectrum M

- The spectrum $E_4^{hC_8}$ can be shown to satisfy the first condition required of M, namely its Adams-Novikov spectral sequence detects all of the θ_js. $E_1^{hC_2}$ and $E_2^{hC_4}$ do not have this property.
- The Hopkins-Miller spectral sequence for $E_1^{hC_2}$ is very simple and we will describe it at the end of the third lecture.
- The one for $E_2^{hC_4}$ is very rich and is similar to the one for tmf (topological modular forms), whose $K(2)$-localization is the homotopy fixed point set for a certain subgroup of order 24.
- The one for $E_4^{hC_8}$ is too complicated for us to use it to prove that $\pi_{-2} = 0$.

2.15

A C_8-equivariant substitute for E_4

A G-equivariant spectrum is more than a spectrum with an action of G. We will give the precise definitions shortly.

After describing a C_8-equivariant substitute for E_4, we will present a new spectral sequence, the *slice spectral sequence*, for computing the homotopy of its fixed point set.

A convenient property of the slice spectral sequence is that π_{-2} vanishes at the E_2-level, making property (iii) immediate.

Property (ii) (periodicity) involves some differentials in the slice spectral sequence.

There is an analogous construction for $E_{2^{k-1}}$ as a C_{2^k}-spectrum for any k. The slice spectral sequence for $k = 1$ was the subject of Dan Dugger's thesis, and we will illustrate at at the end of the third lecture. 2.16

7 Equivariant stable homotopy theory

G-spaces

Before we can describe any of this, we need to introduce *equivariant stable homotopy theory.*

Let G be a finite group. A G-space is a topological space X with a continuous left action by G; a based G-space is a G-space together with a basepoint fixed by G.

We can convert an unbased G-spaces X into based one by taking the topological sum of X and a G-fixed basepoint, denoted by X_+.

The product $X \times Y$ of two G-spaces is a G-space under the diagonal action, as is the smash product of two based G-spaces. 2.17

Maps of G-spaces

The space $F(X,Y)$ of based maps $X \to Y$ is itself a G-space with G-action defined by $(\gamma f)(x) = \gamma f(\gamma^{-1}x)$ for $\gamma \in G$.

Its fixed point set $F(X,Y)^G$ is the space of based G-maps $X \to Y$, i.e., those maps commuting with the action of G.

We use the notation $[X,Y]_G$ to denote the set of homotopy classes of based G-maps $X \to Y$.

A map of G-spaces $f : X \to Y$ is said to be a *weak G-equivalence* if for each subgroup $H \subset G$, the induced map $f : X^H \to Y^H$ is a weak equivalence in the nonequivariant sense. 2.18

G-CW complexes via orbits

There are two ways to generalize the construction of CW-complexes to the equivariant world, one based on orbits and a second based on representations.

For the orbit construction, given any subgroup H of G we may form the homogeneous space G/H and its based counterpart, G/H_+.

These are treated as 0-dimensional cells, and they play a role in equivariant theory analogous to the role of points in nonequivariant theory. 2.19

G-CW complexes via orbits (continued)

We form the n-dimensional cells from these homogeneous spaces. In the unbased context, the cell-sphere pair is

$$(G/H \times D^n, G/H \times S^{n-1})$$

and in the based context

$$(G/H_+ \wedge D^n, G/H_+ \wedge S^{n-1}).$$

A cell is said to be *induced* if it comes from a proper subgroup H.

Starting from these cell-sphere pairs, we form G-CW complexes exactly as nonequivariant CW-complexes are formed from the cell-sphere pairs (D^n, S^{n-1}). In such a complex, an element $\gamma \in G$ acts on a cell either by mapping it homeomorphically to another cell or by fixing it. 2.20

G-CW complexes via representations

Let V be an orthogonal representation of G. Denote its one-point compactification by S^V, with ∞ as the basepoint. We denote the trivial n-dimensional real representation by n, giving the symbol S^n its usual meaning.

We may also form the unit disc and unit sphere

$$D(V) = \{v \in V : ||v|| \leq 1\} \text{ and } S(V) = \{v \in V : ||v|| = 1\};$$

we think of them as unbased G-spaces. There is a homeomorphism $S^V \cong D(V)/S(V)$.

We can use these objects to build G-CW complexes as well. In this case G can act on an individual cell by "rotating" it via the representation V. 2.21

More general G-CW complexes

We can also mix these two constructions by considering cell-sphere pairs such as

$$(G \times_H D(V), G \times_H S(V))$$

and

$$(G_+ \wedge_H D(V), G_+ \wedge_H S(V)),$$

where V is a representation of the subgroup H.

In such a complex, individual cells may be either permuted or rotated by an element of G. 2.22

Toward equivariant spectra

Before defining equivariant spectra, we need to recall the definition of an ordinary spectrum.

A *prespectrum* D is a collection of spaces D_n with maps $\Sigma D_n \to D_{n+1}$. The adjoint of the structure map is a map $D_n \to \Omega D_{n+1}$.

We get a spectrum E from the prespectrum D by defining

$$E_n = \varinjlim_k \Omega^k D_{n+k}$$

This makes E_n homeomorphic to ΩE_{n+1}.

For technical reasons it is convenient to replace the collection $\{E_n\}$ by $\{EV\}$ indexed by finite dimensional subspaces V of a countably infinite dimensional real vector space U called a *universe.* 2.23

Toward equivariant spectra (continued)

The homotopy type of EV depends only on the dimension of V and there are homeomorphisms

$$EV \to \Omega^{|W|-|V|} EW \qquad \text{for } V \subset W \subset U.$$

A map of spectra $f : E \to E'$ is a collection of maps of based G-spaces $f_V : EV \to E'V$ which commute with the respective structure maps. 2.24

G-equivariant spectra

Let G be a finite group. Experience has shown that in order to do equivariant stable homotopy theory, one needs G-spaces EV indexed by finite dimensional orthogonal representations V sitting in a countably infinite dimensional orthogonal representation U.

This universe U is said to be *complete* if it contains infinitely many copies of each irreducible representation of G. A canonical example of a complete universe for finite G is the direct sum of countably many copies of the regular real representation of G. 2.25

G-equivariant spectra (continued)

A *G-equivariant spectrum* (G-spectrum for short) indexed on U consists of a based G-space EV for each finite dimensional subspace $V \subset U$ together with a transitive system of based G-homeomorphisms

$$EV \xrightarrow[\cong]{\tilde{\sigma}_{V,W}} \Omega^{W-V} EW$$

for $V \subset W \subset U$. Here $\Omega^V X = F(S^V, X)$ and $W - V$ is the orthogonal complement of V in W. As in the classical case, the G-homotopy type of EV depends only on the isomorphism class of V. 2.26

G-equivariant spectra (continued)

A map of G-spectra $f : E \to E'$ is a collection of maps of based G-spaces $f_V : EV \to E'V$ which commute with the respective structure maps.

Dropping the requirement that the structure maps be homeomorphisms gives us a *G-prespectrum.*

The structure map $\tilde{\sigma}_{V,W}$ is adjoint to a map

$$\sigma_{V,W} : \Sigma^{W-V} EV \to EW,$$

where $\Sigma^V X$ is defined to be $S^V \wedge X$.

A *suspension G-prespectrum* is a G-prespectrum in which the maps above are G-equivalences for V sufficiently large. 2.27

$RO(G)$-graded homotopy groups

Given a representation V one has a suspension G-spectrum $\Sigma^\infty S^V$, which is often denoted abusively (as in the nonequivariant case) by S^V.

As in the nonequivariant case, to define a prespectrum D it suffices to define G-spaces DV for a cofinal collection of representations V.

We define S^{-V} by saying its Wth space for $V \subset W$ is S^{W-V}. This is the analog of formal desuspension in the nonequivariant case. 2.28

$RO(G)$-graded homotopy groups (continued)

Given a virtual representation $\nu = W - V$, we define $S^\nu = \Sigma^W S^{-V}$. *Hence we have a collection of sphere spectra graded over the orthogonal representation ring $RO(G)$.*

We define

$$\pi_\nu^G(X) = [S^\nu, X]_G$$

the $RO(G)$-graded homotopy groups of the G-spectrum X. 2.29

8 MU as a C_2-spectrum

MU as a C_2-spectrum

Let ρ denote the real regular representation of C_2. It is isomorphic to the complex numbers $\mathbf{C}$ with conjugation.

We define a C_2-prespectrum mu by $mu(k\rho) = MU(k)$, the Thom space of the universal $\mathbf{C}^k$-bundle over $BU(k)$, which is a direct limit of complex Grassmannian manifolds. The action of C_2 is by complex conjugation.

Since any orthogonal representation V of C_2 is contained in $k\rho$ for $k \gg 0$, we can define the C_2-spectrum MU by

$$MUV = \lim_{\substack{\to \\ k}} \Omega^{k\rho - V} MU(k).$$

This spectrum in known as real cobordism theory and has been studied by Landweber, Araki, Hu-Kriz and Kitchloo-Wilson. 2.30

Inducing and coinducing up to a larger group

Let $H \subset G$ be groups and let X be a H-space. There are two ways to get a G-space from it. The corresponding functors are the left and right adjoints to the forgetful functor from G-spaces to H-spaces.

There is the *induced G-space* $G \times_H X$. Its underlying space is the disjoint union of $|G/H|$ copies of X.

An example is the the cell-sphere pair

$$(G/H \times D^n, G/H \times S^{n-1}).$$

2.31

Inducing and coinducing up to a larger group (continued)

There is the *coinduced G-space*

$$\begin{aligned} \mathrm{map}_H(G,X) &= \{f \in \mathrm{map}(G,X) \colon f(\gamma\eta^{-1}) = \eta f(\gamma) \\ &\qquad \forall \eta \in H \text{ and } \gamma \in G\} \end{aligned}$$

The underlying space here is the Cartesian product $X^{|G/H|}$.

There is a based analog of the coinduced G-space in which the underlying space is the smash product $X^{(|G/H|)}$.

It extends to H-spectra. For a H-spectrum X we denote the coinduced G-spectrum by $N_H^G X$, the *norm of* X *along the inclusion* $H \subset G$.

2.32

Norming up from MU

We apply this construction to the case $H = C_2$, $G = C_{2^{n+1}}$ and $X = MU$. The underlying spectrum of $N_H^G MU$ is the 2^n-fold smash power $MU^{(2^n)}$.

Let $\gamma \in G$ be a generator and let z_i be a point in MU. Then the action of G on $MU^{(2^n)}$ is given by

$$\gamma(z_1 \wedge \cdots \wedge z_{2^n}) = \bar{z}_{2^n} \wedge z_1 \wedge \cdots \wedge z_{2^n-1},$$

where $\bar{z}_{2^n}$ is the complex conjugate of z_{2^n}.

2.33

Our spectrum M

In particular this makes $MU^{(4)}$ into a C_8-spectrum. *Our spectrum* $\tilde{M}$ *is obtained from it by equivariantly inverting a certain element in its homotopy. Them* $M = \tilde{M}^{C_8}$*, which we will show to be equivalent to* $\tilde{M}^{hC_8}$.

The spectrum $MU^{(4)}$ has two advantages over our earlier candidate E_4.

(i) It is a C_8-equivariant spectrum, while E_4 was merely an ordinary spectrum with a C_8 "action" for which a homotopy fixed point set could be defined.

(ii) The action of C_8 on $\pi_*(MU^{(4)})$ is transparent, unlike its mysterious action on $\pi_*(E_4)$.

2.34