On the Nilpotence Order of β_1

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For $p > 2$, $\beta_1 \in \pi^S_{2p^2-2p-2}(S^0)$ is the first positive even-dimensional element in the stable homotopy groups of spheres. A classical theorem of Nishida [Nis73] states that all elements of positive dimension in the stable homotopy groups of spheres are nilpotent. In fact, Toda [Tod68] proved $\beta_1^{p^2-p+1} = 0$. For $p = 3$ he showed that $\beta_1^6 = 0$ while $\beta_1^5 \neq 0$. In [Rav86] the second author computed the first thousand stems of the stable homotopy groups of spheres at the prime 5. One of the consequences of this computation is that $\beta_1^{18} = 0$ while $\beta_1^{17} \neq 0$.

Our purpose here is to study the problem for larger primes. Our result is the following.

Theorem *For $p \geq 7$, $\beta_1^{p^2-p-1}$ is nontrivial.*

We do not know whether $\beta_1^{p^2-p}$ is trivial or not.

We will prove our result by examining the E_2-term of the Adams–Novikov spectral sequence (which coincides in our range of dimensions with the classical Adams spectral sequence) for $V(3)$, where all powers of β_1 are nontrivial. We look for elements that have the rights dimensions to support differentials killing powers of β_1. We find that the first possibility lies in the dimension that would kill $\beta_1^{p^2-p}$.

Preliminaries.

Denote by BP the Brown-Peterson spectrum at the prime p and set $q = 2p - 2$. We have

$$BP_* = Z_{(p)}[v_1, v_2, \cdots] \text{ and } BP_*(BP) = BP_*[t_1, t_2, \cdots]$$

where the homological degrees of v_i and t_i are given by $|v_i| = |t_i| = 2(p^i - 1)$. For any $BP_*(BP)$ comodule M, we will write

$$\mathrm{Ext}^*(M) = \mathrm{Ext}^*_{BP_*(BP)}(BP_*, M).$$

One method for calculating this Ext group is to use the the cobar complex. Recall the Hopf algebroid $(BP_*, BP_*(BP))$ come equipped with a right unit

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map $\eta : BP_* \to BP_*(BP)$ and a coproduct $\Delta : BP_*(BP) \to BP_*(BP) \otimes_{BP_*} BP_*(BP)$. Given any $BP_*(BP)$-comodule M with coaction $\psi : M \to M \otimes_{BP_*} BP_*(BP)$, one has $\mathrm{Ext}^*(M) = H^*(\Omega^* M, d)$ where the cobar complex $\Omega^* M$ is the differential graded $Z_{(p)}$-module with

$$\Omega^t M = M \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} \cdots \otimes_{BP_*} BP_*(BP)$$

(t factors of BP_*B) and differential d of degree $+1$ given by

$$\begin{aligned} d(m \otimes x_1 \otimes \cdots \otimes x_t) &= \sum m' \otimes m'' \otimes x_1 \otimes \cdots \otimes x_t \\ &+ \sum_1^t (-1)^i m \otimes x_1 \otimes \cdots \otimes x_i' \otimes x_i'' \otimes \cdots \otimes x_t \\ &-(-1)^t m \otimes x_1 \otimes \cdots \otimes x_t \otimes 1 \end{aligned}$$

where $\Delta x_i = \sum x_i' \otimes x_i''$ and $\psi m = \sum m' \otimes m''$.

Given any finite complex X, the E_2-term of the Adams-Novikov spectral sequence converging to the stable homotopy groups of X at the prime p is given by $\mathrm{Ext}^*(BP_*(X))$. Let $I_n = (p, v_1, ..., v_{n-1})$ be an ideal of BP_*. Using known formulae for Δ and η, it is easy to show that β_1 is represented in $\Omega^* BP_*$ by $b_{1,0}$ where

$$b_{j,k} = -\sum_{i=1}^{p-1} \frac{1}{p} \begin{pmatrix} p \\ i \end{pmatrix} t_j^{ip^k} \otimes t_j^{(p-i)p^k}.$$

Smith-Toda $V(n)$ complexes.

For $n \leq 3$ and $p \geq 7$, there are certain finite complexes called $V(n)$ constructed by Smith-Toda [Smi71][Tod71] with the property that $BP_*(V(n)) = BP_*/I_{n+1}$. Define $V(-1) = S^0$ and let $V(0)$ be the mod p Moore space. $V(n)$ is constructed as the cofiber of certain v_n self-map of $V(n-1)$ such that the canonical map from $V(n-1)$ to $V(n)$ induces the natural projection on BP_* homology. Toda showed [Tod71, Theorem 4.4] that $V(3)$ is a ring spectrum for $p \geq 11$, and at $p = 7$ it is a module spectrum over $V(2)$.

May spectral sequence for $V(3)$.

The key idea for determining the nilpotence order of β_1 in $\pi_*^S(S^0)$ is to consider the images of powers of β_1 in the stable homotopy groups of $V(3)$. First we recall the computation by Toda [Tod71] of the E_2-term of a certain May spectral sequence related to $V(3)$. In the range we are interested in, we have an isomorphism

$$\mathrm{Ext}^*_{BP_*(BP)}(BP_*, BP_*(V(3))) = \mathrm{Ext}^*_{P(3)_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$$

where $P(3)_*$ is the Hopf algebra $P(t_1, t_2, t_3)$ whose coproduct is given by

$$\begin{aligned} \Delta(t_1) &= 1 \otimes t_1 + t_1 \otimes 1 \\ \Delta(t_2) &= 1 \otimes t_2 + t_1 \otimes t_1^p + t_2 \otimes 1 \\ \Delta(t_3) &= 1 \otimes t_3 + t_1 \otimes t_2^p + t_2 \otimes t_1^{p^2} + t_3 \otimes 1 \end{aligned}$$

Consider the dual of Steenrod's reduced powers $P_* = P(t_1, t_2, \cdots)$. We can compute its cohomology using the modified form of the May spectral sequence introduced in [Rav86, 3.2.5]. In it we have

$$E_1^{*,*,*} = E(h_{i,j}) \otimes P(b_{i,j}) \quad \text{with} \quad \delta_r : E_r^{s,t,u} \to E_r^{s+1,t,u-r}$$

where

$$\begin{array}{rcl} h_{i,j} & \in & E_1^{1,2p^j(p^i-1),2i-1} \\ b_{i,j} & \in & E_1^{2,2p^{j+1}(p^i-1),p(2i-1)} \end{array}$$

for $i > 0$ and $j \geq 0$. We indicate May differentials by δ_r to avoid confusion with Adams-Novikov differentials d_r.

In our range we need only consider $h_{i,j}$ with $i+j \leq 3$ and $b_{i,j}$ with $i+j \leq 2$. The first differentials are given (up to sign) by

$$\left.\begin{array}{rcl} \delta_1(h_{i,j}) & = & \sum_{0<k<i} h_{k,j} h_{i-k,j+k} \\ \delta_r(b_{1,j}) & = & 0 \qquad \text{for all } r \\ \delta_r(b_{2,0}) & = & 0 \qquad \text{for } r < 2p-1 \\ \delta_{2p-1}(b_{2,0}) & = & h_{1,2}b_{1,0} - h_{1,1}b_{1,1} \end{array}\right\} \tag{1}$$

The modified May E_2-term is $P(b_{1,0}, b_{1,1}, b_{2,0})$ tensored with the cohomology of the complex

$$(E(h_{i,j} : i+j \leq 3), \delta_1)$$

This cohomology is easy to compute in our range of dimensions. An additive basis for it is shown in Table 1. The 24 generators are listed in order of dimension. They are divided into four groups of six according to the number of factors of the form $h_{k,3-k}$. For future use we indicate the dimension of each generator modulo that of $b_{1,0}$. This computation was described by Toda in [Tod71], and we adopt his notation for these elements.

Proof of the theorem.

Let i_3 be the composite of the following natural maps

$$S^0 \to V(0) \to V(1) \to V(2) \to V(3).$$

Denote by e_3 the least positive integer such that $i_{3*}(\beta_1^{e_3}) = 0$ in $\pi_*^S(V(3))$. By Toda's result, $e_3 \leq p^2-p+1$. Since $b_{1,0}^{e_3}$ is non-zero in the E_2-term of the Adams–Novikov spectral sequence for $V(3)$, it must be killed by some differential in the spectral sequence. Similarly, let μ be the structure map for the $V(2)$-module spectrum $V(3)$:

$$\mu : V(2) \wedge V(3) \to V(3).$$

Then the same is true of the element $h_1 b_{1,0}^{e_3}$ since we have $\mu_*(h_1 \wedge i_3 \circ \beta_1^{e_3}) = 0$ in $\pi_*^S(V(3))$ where $h_1 \in \pi_*^S(V(2))$.

In our range the Adams-Novikov E_2-term is isomorphic to $\mathrm{Ext}_{P_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$, which is a subquotient of

Table 1: The cohomology of $E(h_{i,j} : i+j \leq 3)$

Element	$(s,\ t/q)$	$t-s \bmod \lvert b_{1,0}\rvert$
1	$(0,\ 0)$	0
$h_0 = h_{1,0}$	$(1,\ 1)$	$2p-3$
$h_1 = h_{1,1}$	$(1,\ p)$	1
$g_0 = h_{1,0}h_{2,0}$	$(2,\ p+2)$	$4p-4$
$k_0 = h_{1,1}h_{2,0}$	$(2,\ 2p+1)$	$2p$
h_0k_0	$(3,\ 2p+2)$	$4p-3$
$h_2 = h_{1,2}$	$(1,\ p^2)$	$2p-1$
h_0h_2	$(2,\ p^2+1)$	$4p-4$
$g_1 = h_{1,1}h_{2,1}$	$(2,\ p^2+2p)$	$2p+2$
$l_1 = h_{1,0}h_{2,0}h_{3,0}$	$(3,\ p^2+2p+3)$	$8p-5$
$l_2 = h_{1,1}h_{2,0}h_{2,1}$	$(3,\ p^2+3p+1)$	$4p+1$
h_1l_1	$(4,\ p^2+3p+3)$	$8p-4$
$k_1 = h_{1,2}h_{2,1}$	$(2,\ 2p^2+p)$	$4p$
$l_3 = h_{1,0}h_{1,2}h_{3,0}$	$(3,\ 2p^2+p+2)$	$8p-5$
h_1k_1	$(3,\ 2p^2+2p)$	$4p+1$
h_2l_1	$(4,\ 2p^2+2p+3)$	$10p-6$
$m_1 = h_{1,1}h_{2,0}h_{2,1}h_{3,0}$	$(4,\ 2p^2+4p+2)$	$8p$
h_0m_1	$(5,\ 2p^2+4p+3)$	$10p-3$
$l_4 = h_{1,2}h_{2,1}h_{3,0}$	$(3,\ 3p^2+2p+1)$	$8p-1$
h_0l_4	$(4,\ 3p^2+2p+2)$	$10p-4$
h_1l_4	$(4,\ 3p^2+3p+1)$	$8p$
g_0l_4	$(5,\ 3p^2+3p+3)$	$12p-5$
k_0l_4	$(5,\ 3p^2+4p+2)$	$10p-1$
$h_0k_0l_4$	$(6,\ 3p^2+4p+3)$	$12p-4$

$$H^*(E(h_{i,j})) \otimes P(b_{1,0}, b_{1,1}, b_{2,0}).$$

Recall the ANSS differentials respect the $V(2)$-module structure map μ and β_1 is also a homotopy element in $\pi_*^S(V(2))$. Therefore in order to consider all possible differentials killing a power of $b_{1,0}$ or its product with h_1, up to multiplication by powers of $b_{1,0}$ it is sufficient to consider only those of the form

$$\left.\begin{array}{rcl} d_r(xb_{1,1}^{e_{1,1}}b_{2,0}^{e_{2,0}}) & = & b_{1,0}^{e} \\ d_r(x'b_{1,1}^{e'_{1,1}}b_{2,0}^{e'_{2,0}}) & = & h_1 b_{1,0}^{e'} \end{array}\right\} \qquad (2)$$

where x and x' are generators listed in Table 1. The dimension of the source $xb_{1,1}^{e_{1,1}}b_{2,0}^{e_{2,0}}$ must be congruent to 1 modulo the dimension of $b_{1,0}$, i.e., modulo $2p^2-2p-2$, and that of $x'b_{1,1}^{e'_{1,1}}b_{2,0}^{e'_{2,0}}$ must be congruent to 2. For $b_{1,1}$ and $b_{2,0}$ we have

$$|b_{1,1}| \equiv 2p-2 \quad \text{and} \quad |b_{2,0}| \equiv 2p.$$

Toda's computations (as explained in [Rav86, page 290]) imply that

$$d_{(p-1)q+1}(h_0 b_{1,1}^{p-1}) = b_{1,0}^{p^2-p+1}$$

so $e_3 \leq p^2 - p + 1$ and we need not consider instances of (2) in dimensions higher than this. In particular we need only consider those cases where $e_{1,1} + e_{2,0} \leq p - 1$.

Next observe that

$$\left.\begin{array}{rrcllcll}
\text{if} & |x'| & \equiv & 2kp+2 & \text{then} & |x' b_{1,1} b_{2,0}^{p-2-k}| & \equiv & 2, \\
\text{if} & |x| & \equiv & 2kp+1 & \text{then} & |x b_{1,1} b_{2,0}^{p-2-k}| & \equiv & 1, \\
\text{if} & |x'| & \equiv & 2kp & \text{then} & |x' b_{2,0}^{p-1-k}| & \equiv & 2, \\
\text{if} & |x| & \equiv & 2kp-1 & \text{then} & |x b_{2,0}^{p-1-k}| & \equiv & 1, \\
\text{if} & |x| & \equiv & 2kp-3 & \text{then} & |x b_{1,1}^{p-1} b_{2,0}^{1-k}| & \equiv & 1, \\
\text{if} & |x'| & \equiv & 2kp-4 & \text{then} & |x' b_{1,1}^{p-2} b_{2,0}^{2-k}| & \equiv & 2, \\
\text{if} & |x| & \equiv & 2kp-5 & \text{then} & |x b_{1,1}^{p-2} b_{2,0}^{2-k}| & \equiv & 1 \quad \text{and} \\
\text{if} & |x'| & \equiv & 2kp-6 & \text{then} & |x' b_{1,1}^{p-3} b_{2,0}^{3-k}| & \equiv & 2.
\end{array}\right\} \quad (3)$$

This accounts for all the odd values of $|x|$ and even values of $|x'|$ in Table 1 and all of the acceptable values of the exponents of $b_{1,1}$ and $b_{2,0}$ in (2). Note that in the last four cases of (3), there are upper bounds on acceptable values of k.

With this in mind, inspection of Table 1 shows that the only possible sources of (2) are

$$b_{2,0}^{p-1} \quad \text{with} \quad t-s = 1 + |h_1| + (p^2-1)|b_{1,0}|, \tag{4}$$
$$h_0 b_{1,1}^{p-1} \quad \text{with} \quad t-s = 1 + (p^2-p+1)|b_{1,0}|, \tag{5}$$
$$h_1 b_{1,1} b_{2,0}^{p-2} \quad \text{with} \quad t-s = 1 + p^2|b_{1,0}|, \tag{6}$$
$$g_0 b_{1,1}^{p-2} \quad \text{with} \quad t-s = 1 + |h_1| + (p^2-2p+1)|b_{1,0}|, \tag{7}$$
$$k_0 b_{2,0}^{p-2} \quad \text{with} \quad t-s = 1 + |h_1| + (p^2-p)|b_{1,0}|, \tag{8}$$
$$h_2 b_{2,0}^{p-2} \quad \text{with} \quad t-s = 1 + (p^2-1)|b_{1,0}|, \tag{9}$$
$$h_0 h_2 b_{2,0}^{p-2} \quad \text{with} \quad t-s = 1 + |h_1| + (p^2-1)|b_{1,0}|, \tag{10}$$
$$g_1 b_{1,1} b_{2,0}^{p-3} \quad \text{with} \quad t-s = 1 + |h_1| + (p^2-1)|b_{1,0}|, \tag{11}$$
$$l_2 b_{1,1} b_{2,0}^{p-4} \quad \text{with} \quad t-s = 1 + (p^2-p)|b_{1,0}|, \tag{12}$$
$$k_1 b_{2,0}^{p-3} \quad \text{with} \quad t-s = 1 + |h_1| + (p^2-2)|b_{1,0}|, \tag{13}$$
$$h_1 k_1 b_{1,1} b_{2,0}^{p-4} \quad \text{with} \quad t-s = 1 + (p^2-1)|b_{1,0}|, \tag{14}$$
$$m_1 b_{2,0}^{p-5} \quad \text{with} \quad t-s = 1 + |h_1| + (p^2-2p-1)|b_{1,0}|, \tag{15}$$
$$l_4 b_{2,0}^{p-5} \quad \text{with} \quad t-s = 1 + (p^2-p-2)|b_{1,0}|, \tag{16}$$
$$h_1 l_4 b_{2,0}^{p-5} \quad \text{with} \quad t-s = 1 + |h_1| + (p^2-p-2)|b_{1,0}|, \quad \text{and} \tag{17}$$
$$k_0 l_4 b_{2,0}^{p-6} \quad \text{with} \quad t-s = 1 + (p^2-2p-1)|b_{1,0}|. \tag{18}$$

Of these possible sources, (4), (6), (9), (10), (11), (13) and (14) can be discarded because their dimensions are too high. We can elminate two more by using the May differential δ_{2p-1} of (1). In the modified May spectral sequence we have the following differentials, up to nonzero scalar multiplication.

$$\begin{aligned} \delta_{2p-1}(m_1 b_{2,0}^{p-5}) &= m_1 b_{2,0}^{p-6}(h_{1,2}b_{1,0} - h_{1,1}b_{1,1}) = k_0 l_4 b_{1,0} b_{2,0}^{p-6}, & (19) \\ \delta_{2p-1}(l_4 b_{2,0}^{p-5}) &= l_4 b_{2,0}^{p-6}(h_{1,2}b_{1,0} - h_{1,1}b_{1,1}) = h_1 l_4 b_{1,1} b_{2,0}^{p-6}. & (20) \end{aligned}$$

These imply that the elements of (15) and (16) are not present in the E_2-term of the Adams–Novikov spectral sequence.

Hence we have reduced the list (4)–(18) to the following.

$$\left.\begin{array}{rl} h_0 b_{1,1}^{p-1} & \text{with} \quad t-s = 1 + (p^2-p+1)|b_{1,0}|, \\ g_0 b_{1,1}^{p-2} & \text{with} \quad t-s = 1 + |h_1| + (p^2-2p+1)|b_{1,0}|, \\ k_0 b_{2,0}^{p-2} & \text{with} \quad t-s = 1 + |h_1| + (p^2-p)|b_{1,0}|, \\ l_2 b_{1,1} b_{2,0}^{p-4} & \text{with} \quad t-s = 1 + (p^2-p)|b_{1,0}|, \\ h_1 l_4 b_{2,0}^{p-5} & \text{with} \quad t-s = 1 + |h_1| + (p^2-p-2)|b_{1,0}|, \quad \text{and} \\ k_0 l_4 b_{2,0}^{p-6} & \text{with} \quad t-s = 1 + (p^2-2p-1)|b_{1,0}|. \end{array}\right\} \quad (21)$$

From (21), it follows immediately that e_3 is at least $p^2 - 2p - 1$. Since h_1 is a homotopy element in $\pi_*^S(V(2))$ and the ANSS differentials are module maps over $\pi_*^S(V(2))$, the last differential in (21) can now be eliminated since otherwise there would be a differential killing $h_1 b_{1,0}^{p^2-2p-1}$ which is a contradiction to (21). Now this by itself does not imply that $e_3 \geq p^2 - p$ since it is conceivable that $k_0 l_4 b_{1,0}^j b_{2,0}^{p-6}$ kills $b_{1,0}^{p^2-2p-1+j}$ for some $j > 0$. However this latter possibility can not happen by (19).

It now follows that the source of the first differential killing a power of β_1 is either $l_2 b_{1,1} b_{2,0}^{p-4}$, $l_2 b_{1,1} b_{1,0} b_{2,0}^{p-4}$ or $h_0 b_{1,1}^{p-1}$, *so the nilpotence order is either* $p^2 - p$ *or* $p^2 - p + 1$.

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