TOWARD HIGHER CHROMATIC ANALOGS OF ELLIPTIC COHOMOLOGY AND TOPOLOGICAL MODULAR FORMS
CONFERENCE AT JOHNS HOPKINS UNIVERSITY
MARCH 10, 2007

DOUG RAVENEL

WHAT DOES "CHROMATIC" MEAN?

The stable homotopy category $\mathcal{S}$ localized at a prime p can be studied via a series of increasingly complicated Bousfield localization functors L_n for $n \geq 0$, which detect "v_n-periodic" phenomena. L_n is localization with respect to $v_n^{-1}BP_*$, or equivalently with respect to $K(0) \vee K(1) \vee \ldots K(n)$. For more details, see [**?**].

$$L_0\mathcal{S} \longleftarrow L_1\mathcal{S} \longleftarrow L_2\mathcal{S} \longleftarrow \cdots \longleftarrow \mathcal{S}_{(p)}$$

- L_0 is rationalization. Rational stable homotopy theory is very well understood. It detects only the 0-stem in the stable homotopy groups of spheres.
- L_1 is localization with respect to real or complex K-theory. It detects the image of J and the α family in the stable homotopy groups of spheres. The Lichtenbaum-Quillen conjecture is a statement about L_1 of algebraic K-theory.
- L_2 is localization with respect to elliptic cohomology [**?**] or the theory of topological modular forms of Hopkins *et al.* It detects the β family in the stable homotopy groups of spheres. Davis' nonimmersion theorem for real projective spaces was proved using related methods.
- For $n > 2$ there is no comparable geometric definition of L_n, which can only be constructed by less illuminating algebraic methods related to BP-theory. It detects higher Greek letter families in the stable homotopy groups of spheres. The nth Morava K-theory is closely related to it.

THE m-SERIES OF A FORMAL GROUP LAW

Definition 1. *Let F be 1-dimensional formal group law. For a positive integer m, the m-series is defined inductively by*

$$[m]_F(x) = F(x, [m-1]_F(x))$$

where $[1]_F(x) = x$.

EXAMPLES OF m-SERIES

- For the additive formal group law ($F(x, y) = x + y$), $[m](x) = mx$.
- For the multiplicative formal group law ($F(x, y) = x + y + xy$),

$$\begin{aligned} [m](x) &= (1+x)^m - 1 \\ &= mx + \binom{m}{2}x^2 + \cdots + x^m. \end{aligned}$$

- For $F(x,y)=\frac{x+y}{1+xy}$, we have

$$\begin{aligned}[m](x) &= \textstyle\sum_i \binom{m}{2i+1} x^{2i+1} \Big/ \sum_i \binom{m}{2i} x^{2i} \\ &= \frac{mx+\binom{m}{3}x^3+\dots}{1+\binom{m}{2}x^2+\dots}\end{aligned}$$

The height of a formal group law

Over a field k of characteristic p, the p-series is either 0 or has the form

$$[p]_F(x) = ax^{p^n} + \dots$$

for some nonzero $a \in k$.

Definition 2. *The* **height** *of F is the integer n. If $[p]_F(x) = 0$ (which happens when $F(x,y) = x + y$), the height is defined to be ∞.*

Examples of heights

- The multiplicative formal group law (which is associated with K-theory) has height 1.
- The formal group law associated with an elliptic curve is known to have height at most 2.
- v_n-periodic phenomena (the nth layer in the chromatic tower) are related to formal group laws of height n.

Question

How can we attach formal group laws of height > 2 to geometric objects (such as algebraic curves) and use them get insight into cohomology theories that go deeper into the chromatic tower?

Program

- Let C be a curve of genus g over some ring R.
- Its Jacobian $J(C)$ is an abelian variety of dimension g. $J(C)$ has a formal completion $\widehat{J}(C)$ which is a g-dimensional formal group law.
- If $\widehat{J}(C)$ has a 1-dimensional summand, then Quillen's theorem [**?**] gives us a homomorphism

$$MU_* \xrightarrow{\theta} R$$

If θ is Landweber exact [**?**], then we get an MU-module spectrum E with $\pi_*(E) = R$.

Caveat

Note that a 1-dimensional summand of the formal completion $\widehat{J}(C)$ is *not* the same thing as 1-dimensional factor of the Jacobian $J(C)$. The latter would be an elliptic curve, whose formal completion can have height at most 2. There is a theorem that says if an abelian variety A has a 1-dimensional formal summand of height n for $n > 2$, then the dimension of A (and the genus of the curve, if A is a Jacobian) is at least n.

Artin-Schreier curves

Theorem 3. *(2002) Let $C(p,f)$ be the curve over $\mathbf{F}_p$ defined by the affine equation*

$$y^e = x^p - x \qquad \text{where } e = p^f - 1.$$

(Assume that $f > 1$ when $p = 2$.) Then its Jacobian has a 1-dimensional formal summand of height $(p-1)f$.

The resulting genus is *not* Landweber exact, so this does not lead to a cohomology theory.

PROPERTIES OF THE CURVE $C(p,f)$

- Its genus is $(p-1)(p^f-2)/2$. (Thus it is zero in the excluded case $(p,f) = (2,1)$.)
- It has an action of the group

$$G = \mathbf{F}_p \rtimes \mu_m, \quad \text{where } m = (p-1)(p^f-1)$$

  and μ_m is the group of mth roots of unity, given by

$$(x,y) \mapsto (\zeta^{p^f-1}x + a, \zeta y)$$

  for $a \in \mathbf{F}_p$ and $\zeta \in \mu_m$. This group is isomorphic to a maximal finite subgroup of the hth Morava stabilizer group, and it acts appropriately on the 1-dimensional formal summand of $\widehat{J}(C(p,f))$.

EXAMPLES OF THESE CURVES

- $C(2,2)$ and $C(3,1)$ are elliptic curves whose formal group laws have height 2.
- $C(2,3)$ has genus 3 and its Jacobian has a 1-dimensional formal summand of height 3.
- $C(2,4)$ and $C(3,2)$ each have genus 7 and their Jacobians each have a 1-dimensional formal summand of height 4.

REMARKS

- Theorem **??** was known to and cited by Manin in 1963 [**?**]. Most of what is needed for the proof can be found in Katz's 1979 Bombay Colloquium paper [**?**] and in Koblitz' Hanoi notes [**?**]. (The latter book has been googled and is available online.)
- The original proof rests on the determination of the zeta function of the curve by Hasse-Davenport in 1934 [**?**], and on some properties of Gauss sums proved by Stickelberger in 1890 [**?**]. The method leads to complete determination of $\widehat{J}(C(p,f))$.
- We have reproved Theorem **??** using Honda's theory of commutative formal group laws developed in the early '70s in [**?**] and [**?**]. This proof does not rely on knowledge of the zeta function and can be modified to prove Theorem **??** below.

DEFORMING THE ARTIN-SCHREIER CURVE

We want a lifting of $C(p,f)$ to characteristic 0 that admits a coordinate change similar to the one for the Weierstrass curve used in the construction of tmf. The equation will have the form

$$y^e = x^p + \dots$$

with (nonaffine) coordinate change fixing y and sending

$$x \mapsto x + \sum_{i=1}^{f} t_i y^{(p^f - p^i)/p}.$$

The t_i above are related to the generators of the same name in $BP_*(BP)$.

In order to state this precisely we need some notation. Let

$$I = (i_1, i_2, \dots i_f)$$

be an f-tuple of nonnegative integers and define

$$\begin{array}{rclcrcl} |I| & = & \sum_k i_k & & ||I|| & = & \sum_k (p^k - 1) i_k \\ t^I & = & \prod_k t_k^{i_k} & & I! & = & \prod_k i_k! \end{array}$$

The coefficients in our equation will be formal variables a_I with $|I| \le p$ (where $a_0 = p!$) with topological dimension $2||I||$. We will sometimes write a_I as $a_{||I||}$. For $|I| \le p$, I is uniquely determined by its norm $||I||$. The number of indices I with $0 < |I| \le p$ is $\binom{p+f}{f} - 1$.

Then the equation for our deformed curve is

$$y^e = \sum_{i=0}^{p} \frac{x^{p-i}}{(p-i)!} \sum_{|I|=i} a_I y^{(ei - ||I||)/p} = x^p + a_m x + \dots.$$

The effect of the coordinate change on the coefficients a_I is given by

$$a_I \mapsto \sum_{J+K=I} a_J \frac{t^K}{K!}.$$

An infinitesimal deformation

Theorem 4 (2004). *Let*

$$\begin{array}{rcl} A & = & \mathbf{Z}_p[a_I \colon 0 < |I| \le p] \\ \overline{A} & = & A/(a_m - 1) \\ \overline{A} \supset J & = & (a_i : i \ne m,) \end{array}$$

Then the Jacobian of curve above defined above over the ring $\overline{A}/J^2$ has a 1-dimensional formal summand of height h. The corresponding formal group law has Landweber exact liftings to $\overline{A}$ and $a_m^{-1}A$.

The map $BP_* \to \overline{A}$ is given by

$$v_r = \begin{cases} p a_{m+p^r-1} + a_{p^r-1} & \text{if } 1 \le r \le \min(f, h-1) \\ a_{se+p^i-1} & \text{if } f < r < h \text{ and } p > 2 \\ m - 2a_{2e} & \text{if } r = h \text{ and } p = 2 \\ 1 & \text{if } r = h \text{ and } p > 2 \end{cases}$$

up to unit scalar, where $r = sf + i$ with $1 \le i \le f$.

A fantasy

There is an associated Hopf algebroid

$$\Gamma = A[t_1, \dots, t_f]$$

where each t_i is primitive and the right unit given by the coordinate change formula above.

Conjecture 5. *For each (p, f) as above there is a spectrum generalizing* tmf *whose homotopy can be computed by an Adams-Novikov type spectral sequence with*

$$E_2 = \mathrm{Ext}_\Gamma(A, A).$$

Why A is too big

This conjecture is not likely to be true for $f > 1$ because the ring A is too large. Ideally its Krull dimension should be pf, the sum of the height of the formal group law and the number of coordinate change parameters. The Krull dimension of A is $\binom{p+f}{f} - 1$. For $f = 2$ this is $p(p+3)/2$ instead of the desired $2p$.

A smaller ring R

Replace the equation above with

$$y^e = \prod_{j=1}^{p} \left(x + \sum_{i=1}^{f} r_{i,j} y^{(p^f - p^i)/p} \right)$$

with $|r_{i,j}| = 2(p^i - 1)$.

Thus we get a curve defined over the ring

$$R = \mathbf{Z}_p[r_{i,j} : 1 \le i \le f,\, 1 \le j \le p],$$

which has the desired Krull dimension.

However this ring R leads to an uninteresting Ext group. The coordinate change above induces

$$r_{i,j} \mapsto r_{i,j} + t_i$$

and

$$\mathrm{Ext}^s_\Gamma(R) = \begin{cases} \mathbf{Z}_p[r_{i,j} - r_{p,i} \colon 1 \le j \le p-1] \\ \qquad\qquad\qquad\quad \text{for } s = 0 \\ 0 \qquad\qquad\qquad \text{for } s > 0. \end{cases}$$

A slightly smaller ring B

The equation for the curve is actually defined over the subring

$$B \quad = \quad R^{\Sigma_p} = \mathbf{Z}_p[r_{i,j} : 1 \le i \le f,\, 1 \le j \le p]^{\Sigma_p}$$

where Σ_p acts on R via the second subscript.

This ring is a quotient of A, but its structure appears to be unknown for $f > 1$ except for $p = 2$. B is clearly a module (presumably free of rank $p!^{f-1}$) over the subring

$$C = R^{\Sigma_p^f}$$

where the f copies of Σ_p act independently on the f sets of p generators of R.

The ring C

The structure of C is well known, namely

$$C = \mathbf{Z}_p[\sigma_{i,k} : 1 \le i \le f,\, 1 \le k \le p]$$

where $\sigma_{i,k}$ is the kth elementary symmetric function in the variables $r_{i,1}, \ldots, r_{i,p}$. $\sigma_{i,k}$ is the image of $a_{k(p^i-1)}/(p-k)!$.

Relation to tmf at $p = 2$

For $(p, f) = (2, 2)$ our equation reads

$$y^3 = x^2 + (a_1 y + a_3)x + a_2 y^2 + a_4 y + a_6,$$

so our a_is are the Weierstrass a_is up to sign. In the ring B there is a relation

$$(2a_4 - a_1a_3)^2 = (4a_2 - a_1^2)(4a_6 - a_3^2),$$

which makes it a free module on $\{1, a_4\}$ over the ring

$$C = \mathbf{Z}_2[a_1, a_2, a_3, a_6].$$

Our coordinate change is

$$y \mapsto y \qquad \text{and} \qquad x \mapsto x + t_1y + t_2,$$

while in the construction of tmf it is

$$y \mapsto y + r \qquad \text{and} \qquad x \mapsto x + sy + t.$$

The former can be obtained from the latter by

$$(r, s, t) \mapsto (0, t_1, t_2).$$

It seems likely that our conjecture (with A replaced by B) would lead to the spectrum

$$tmf \wedge \left(S^0 \cup_\nu e^4\right).$$

References

[HD34] H. Hasse and H. Davenport. Die Nullstellensatz der Kongruenz zeta-funktionen in gewissn zyklischen Fällen. *J. Reine Angew. Math.*, 172:151–182, 1934.

[Hon70] Taira Honda. On the theory of commutative formal groups. *J. Math. Soc. Japan*, 22:213–246, 1970.

[Hon73] Taira Honda. On the formal structure of the Jacobian variety of the Fermat curve over a p-adic integer ring. In *Symposia Mathematica, Vol. XI (Convegno di Geometria, INDAM, Rome, 1972)*, pages 271–284. Academic Press, London, 1973.

[Kat81] Nicholas M. Katz. Crystalline cohomology, Dieudonné modules, and Jacobi sums. In *Automorphic forms, representation theory and arithmetic (Bombay, 1979)*, volume 10 of *Tata Inst. Fund. Res. Studies in Math.*, pages 165–246. Tata Inst. Fundamental Res., Bombay, 1981.

[Kob80] Neal Koblitz. *p-adic analysis: a short course on recent work*, volume 46 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1980.

[LRS95] P. S. Landweber, D. C. Ravenel, and R. E. Stong. Periodic cohomology theories defined by elliptic curves. In Mila Cenkl and Haynes Miller, editors, *The Čech Centennial*, volume 181 of *Contemporary Mathematics*, pages 317–338, Providence, Rhode Island, 1995. American Mathematical Society.

[Lan76] P. S. Landweber. Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$. *American Journal of Mathematics*, 98:591–610, 1976.

[Man63] Ju. I. Manin. Theory of commutative formal groups over fields of finite characteristic. *Uspehi Mat. Nauk*, 18(6 (114)):3–90, 1963.

[Qui69] D. G. Quillen. On the formal group laws of oriented and unoriented cobordism theory. *Bulletin of the American Mathematical Society*, 75:1293–1298, 1969.

[Rav92] D. C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, 1992.

[Sti90] L. Stickelberger. Über eine Verallgemeinerung der Kreistheilung. *Mathematische Annalen*, 37:321–367, 1890.