THE METHOD OF INFINITE DESCENT IN STABLE HOMOTOPY THEORY I

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Contents

This paper is the first in a series aimed at clarifying and extending of parts of the last chapter of [Rav86], in which we described a method for computing the Adams-Novikov E_2-term and used it to determine the stable homotopy groups of spheres through dimension 108 for $p = 3$ and 999 for $p = 5$. The latter computation was a substantial improvement over prior knowledge, and neither has been improved upon since. It is generally agreed among homotopy theorists that it is not worthwhile to try to improve our knowledge of stable homotopy groups by a few stems, but that the prospect of increasing the know range by a factor of p would be worth pursuing. This possibility may be within reach now, due to a better understanding of the methods of [Rav86, Chapter 7] and improved computer technology. This paper should be regarded as laying the foundation for a program to compute $\pi_*(S^0)$ through roughly dimension $p^3|v_2|$, i.e., 432 for $p = 3$ and 6,000 for $p = 5$.

The method referred to in the title involves the connective p-local ring spectra $T(m)$ of [Rav86, §6.5], which satisfy

$$BP_*(T(m)) = BP_*[t_1, \ldots, t_m] \subset BP_*(BP).$$

$T(0)$ is the p-local sphere spectrum, and there are maps

$$S^0 = T(0) \to T(1) \to T(2) \to \cdots \to BP.$$

The map $T(m) \to BP$ is an equivalence below dimension $|v_{m+1}| - 1 = 2p^{m+1} - 3$.

To descend from $\pi_*(T(m))$ to $\pi_*(T(m-1))$ we need some spectra interpolating between $T(m-1)$ and $T(m)$. Note that $BP_*(T(m))$ is a free module over $BP_*(T(m-1))$ on the generators $\{t_m^j : j \geq 0\}$. In Lemma 1.15 we show that for each h there is a $T(m-1)$-module spectrum $T(m-1)_h$ with

$$BP_*(T(m-1)_h) = BP_*(T(m-1))\{t_m^j : 0 \leq j \leq h\}.$$

We have inclusions

$$T(m-1) = T(m-1)_0 \to T(m-1)_1 \to T(m-1)_2 \to \cdots T(m)$$

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and the map $T(m-1)_h \to T(m)$ is an equivalence below dimension $(h+1)|t_m|-1 = 2(h+1)(p^m-1)-1$.

For example when $m = i = 0$, the spectrum $T(m)_{p^i-1}$ is S^0 while $T(m)_{p^{i+1}-1}$ is the p-cell complex

$$Y = S^0 \cup_{\alpha_1} e^q \cup_{\alpha_1} e^{2q} \cdots \cup_{\alpha_1} e^{(p-1)q},$$

where $q = 2p-2$.

We will be particularly interested in the cases where the subscript h is one less that a power of the prime p. In Theorem 1.21 we give a spectral sequence for computing $\pi_*(T(m-1)_{p^i-1})$ in terms of $\pi_*(T(m-1)_{p^{i+1}-1})$. Its E_1-term is

$$E(h_{m,i}) \otimes P(b_{m,i}) \otimes \pi_*(T(m-1)_{p^{i+1}-1})$$

where the elements

$$\begin{array}{rrcl} & h_{m,i} & \in & E_1^{1,2p^i(p^m-1)} \\ \text{and} & b_{m,i} & \in & E_1^{2,2p^{i+1}(p^m-1)} \end{array}$$

are permanent cycles.

In the case $m = i = 0$ cited above, the E_1-term of this spectral sequence is

$$E(h_{1,0}) \otimes P(b_{1,0}) \otimes \pi_*(Y).$$

where $h_{1,0}$ and $b_{1,0}$ represent the homotopy elements α_1 and β_1 (α_1^2 for $p=2$) respectively.

Thus to compute $\pi_*(S^0)$ below dimension $p^4(2p-2)$ we could proceed as follows. In this range we have

$$BP \cong T(4) \cong T(3)_{p-1}.$$

We then use the spectral sequence of 1.21 to get down to $T(3)$, which is equivalent in this range to $T(2)_{p^2-1}$, then use it twice to get down to $T(2) \cong T(1)_{p^3-1}$, and so on. This would make for a total of ten applications of 1.21. Fortunately we have some shortcuts that enable us to get by with less.

The Adams-Novikov E_2-term for $T(m)$ is

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m))).$$

From now on we will drop the first variable when writing such Ext groups, since we will never consider any value for it other than BP_*. There is a change-of-rings isomorphism that equates this group with

$$\mathrm{Ext}_{\Gamma(m+1)}(BP_*)$$

where

$$\Gamma(m+1) = BP_*(BP)/(t_1, \ldots, t_m) = BP_*[t_{m+1}, t_{m+2}, \ldots].$$

In §3 we will quote earlier determinations of $\mathrm{Ext}^0_{\Gamma(m+1)}(BP_*)$ (Proposition 3.6) and $\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*)$ (Theorem 3.17) in all dimensions, and construct a 4-term exact sequence

$$0 \to BP_* \to D^0_{m+1} \to D^1_{m+1} \to E^2_{m+1} \to 0$$

of $\Gamma(m+1)$-comodules. The two D^i_{m+1} are weak injective, meaning that all of their higher Ext groups (above Ext^0) vanish (we study such comodules systematically in §2), and below dimension $p^2|v_{m+1}|$

$$\mathrm{Ext}^0_{\Gamma(m+1)}(D^i_{m+1}) \cong \mathrm{Ext}^i_{\Gamma(m+1)}(BP_*).$$

It follows that in that range

$$\mathrm{Ext}^s_{\Gamma(m+1)}(E^2_{m+1}) \cong \mathrm{Ext}^{s+2}_{\Gamma(m+1)}(BP_*) \qquad \text{for all } s \geq 0.$$

The comodule E^2_{m+1} is $(2p^{m+2}-2p-1)$-connected. In Theorem 4.5 we determine its Ext groups (and hence those of BP_*) up to dimension $p^2|v_{m+1}|$. There are no Adams-Novikov differentials or nontrivial group extensions in this range (except in the case $m = 0$ and $p = 2$), so this also determines $\pi_*(T(m))$ in the same range.

Thus Theorem 4.5 gives us the homotopy of $T(1)_{p^3-1}$ in our range directly without any use of 1.21. In a future paper [NR] we will determine the Ext group for $T(1)_{p^2-1}$ in this range and study the spectral sequence of 1.21 for the homotopy of $T(m-1)_{p-1}$ below dimension $p^3|v_m|$. There are still no room for Adams-Novikov differentials, so the homotopy and Ext calculations coincide. For $m = 1$ this computation was the subject of [Rav85] and [Rav86, §7.3].

It is only when we pass from $T(m-1)_{p-1}$ to $T(m-1)_0 = T(m-1)$ that we encounter Adams-Novikov differentials below dimension $p^2|v_{m+1}|$. For $m = 1$ the first of these is the Toda differential

$$d_{2p-1}(\beta_{p/p}) = \alpha_1\beta_1^p$$

of [Tod67] and [Tod68]. We have established a partially analogous differential for $m > 1$ in [Rav].

Theorem 0.1. *The first nontrivial differential in the Adams–Novikov spectral sequence for the spectrum $T(1)$ at an odd prime p is*

$$d_{2p-1}(b_{3,0}) = h_{2,0}b_{2,0}^p$$

where $b_{3,0} \in E_2^{2,2p^4-2p}$.

For $m > 1$ the first nontrivial differential in the Adams–Novikov spectral sequence for the spectrum $T(m)$ at an odd prime p is

$$d_{2p-1}(\widehat{v}_1\widehat{b}_{2,0}) = v_2\widehat{h}_{1,0}\widehat{b}_{1,0}^p$$

where $\widehat{v}_1\widehat{b}_{2,0} \in E_2^{2,2p^3\omega+2p\omega-2p-2}$. In this case there is also a nontrivial group extension in $\pi_*(T(m))$, namely

$$p\widehat{b}_{2,0} = v_2\widehat{b}_{1,0}^p.$$

For $p = 3$ this is illustrated for $m = 1$ and $m = 2$ in Figures 1 and 2 respectively.

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1. Basic algebraic methods

We refer to [Rav86, A1] for the definitions of Hopf algebroids, comodules, and and related objects.

First we define some Hopf algebroids that we will need.

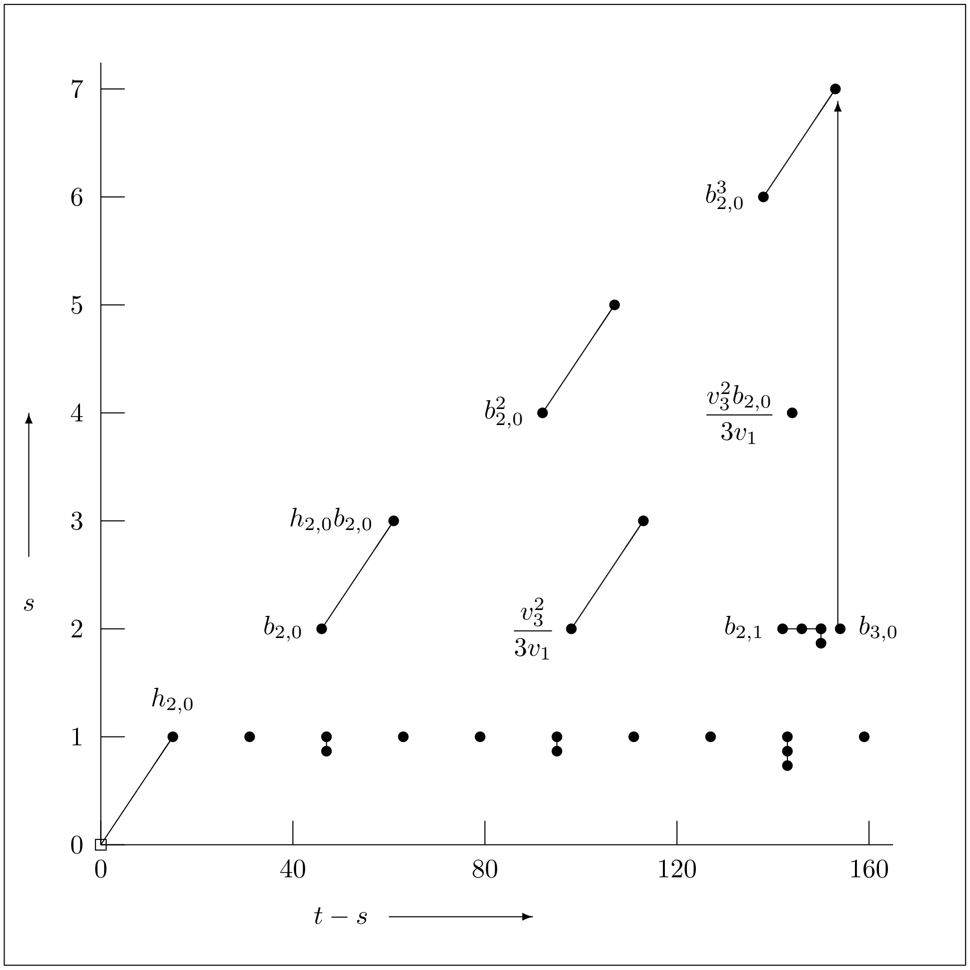

FIGURE 1. The Adams-Novikov E_2-term for $T(1)$ at $p = 3$ in dimensions ≤ 154, showing the first nontrivial differential. Elements on the 0- and 1-lines divisible by v_1 are not shown. Elements on the 2-line and above divisible by v_2 are not shown.

Definition 1.1. *$\Gamma(m+1)$ is the quotient $BP_*(BP)/(t_1, t_2, \ldots, t_m)$,*

$$A(m) = BP_* \Box_{\Gamma(m+1)} BP_* = \mathbf{Z}_{(p)}[v_1, v_2, \ldots, v_m]$$

and

$$G(m+1, k-1) = \Gamma(m+1) \Box_{\Gamma(m+k+1)} BP_* = A(m+k)[t_{m+1}, t_{m+1} \ldots, t_{m+k}]$$

We abbreviate $G(m+1, 0)$ by $G(m+1)$. It will be understood that $G(m+1, \infty) = \Gamma(m+1)$.

In particular, $\Gamma(1) = BP_*(BP)$.

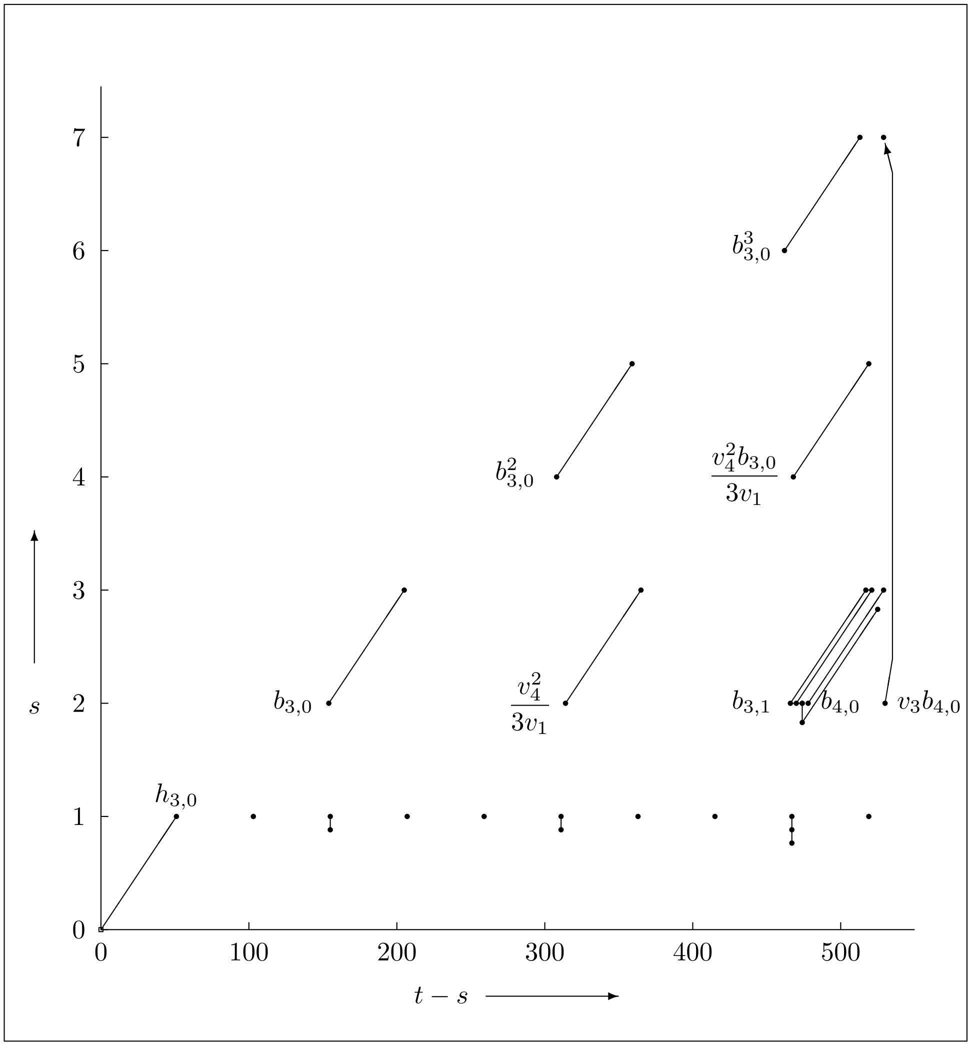

FIGURE 2. The Adams-Novikov E_2-term for $T(2)$ at $p = 3$ in dimensions ≤ 530. Elements on the 0- and 1-lines divisible by v_1 or v_2 are not shown. Elements on the 2-line and above divisible by v_2 or v_3 are not shown except for $v_3 b_{4,0}$ and $v_2 h_{3,0} b_{3,0}^3$, the source and target of the first differential.

Proposition 1.2. *$G(m+1, k-1) \to \Gamma(m+1) \to \Gamma(m+k+1)$ is a Hopf algebroid extension* [Rav86, A1.1.15]*. Given a left $\Gamma(m+1)$-comodule M there is a Cartan-Eilenberg spectral sequence* [Rav86, A1.3.14] *converging to $\mathrm{Ext}_{\Gamma(m+1)}(BP_*, M)$ with*

$$E_2^{s,t} = \mathrm{Ext}^s_{G(m+1,k-1)}(A(m+k), \mathrm{Ext}^t_{\Gamma(m+k+1)}(BP_*, M))$$

and $d_r : E_r^{s,t} \to E_r^{s+r, t-r+1}$.

Corollary 1.3. *Let M be a $\Gamma(m+1)$-comodule concentrated in nonnegative dimensions. Then*

$$\mathrm{Ext}_{\Gamma(m+k+1)}(BP_*, M) = \mathrm{Ext}_{\Gamma(m+1)}(BP_*, G(m+1, k-1) \otimes_{A(m+k)} M).$$

In particular, $\mathrm{Ext}^{s,t}_{\Gamma(m+1)}(BP_*, M)$ *for* $t < 2(p^{m+1}-1)$ *is isomorphic to M for $s = 0$ and vanishes for $s > 0$. Moreover for the spectrum $T(m)$ constructed in* [Rav86, 6.5] *and having* $BP_*(T(m)) = BP_*[t_1, \ldots, t_m]$,

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m))) = \mathrm{Ext}_{\Gamma(m+1)}(BP_*, BP_*).$$

The following characterization of the Cartan-Eilenberg spectral sequence is a special case of [Rav86, A1.3.16].

Lemma 1.4. *The Cartan-Eilenberg spectral sequence of 1.2 is the one associated with the decreasing filtration of the cobar complex $C_{\Gamma(m+1)}(BP_*, M)$ (see below) defined by saying that*

$$\gamma_1 \otimes \cdots \otimes \gamma_s \otimes m \in C^s_{\Gamma(m+1)}(BP_*, M)$$

is in F^i if i of the γ's project trivially to $\Gamma(m+k+1)$.

The *method of infinite descent* for computing $\mathrm{Ext}_{BP_*(BP)}(BP_*, M)$ for a connective comodule M (e.g. the BP-homology of a connective spectrum) is to compute over Ext over $\Gamma(m+1)$ by downward induction on m. To calculate through a fixed range of dimensions k, we choose m so that $k \leq 2(p^{m+1}-1)$ and use 1.3 to start the induction. For the inductive step we could use the Cartan-Eilenberg spectral sequence of 1.2, but it is more efficient to use a different spectral sequence, which we now describe.

The theory of the chromatic spectral sequence of [Rav86, 5.1] generalizes immediately to $\Gamma(m+1)$-comodules. Recall [Rav86, A1.2.11] that for a left Γ-comodule M, the cobar resolution $D^*_\Gamma(M)$ is the cochain complex with $D^s_\Gamma(M) = \Gamma \otimes_A \overline{\Gamma}^{\otimes s} \otimes_A M$, where $\overline{\Gamma} = \ker \epsilon$, and

$$\begin{aligned} & d_s(\gamma_0 \otimes \gamma_1 \otimes \ldots \otimes \gamma_s \otimes m) \\ = \; & \sum_{i=0}^{s} (-1)^i \gamma_0 \otimes \ldots \otimes \gamma_{i-1} \otimes \Delta(\gamma_i) \otimes \gamma_{i+1} \otimes \ldots \otimes m + \\ & (-1)^{s+1} \gamma_0 \otimes \ldots \otimes \gamma_s \otimes \psi(m). \end{aligned}$$

The cobar complex $C^*_\Gamma(M)$ is $A \square_\Gamma D^*_\Gamma(M)$. When $\Gamma = \Gamma(m+1)$, we can define the chromatic cobar complex $CC^*_{\Gamma(m+1)}(BP_*)$ as in [Rav86, 5.1.10]; the additive structure of the latter is given by

$$CC^s_{\Gamma(m+1)}(M) = \bigoplus_{0 \leq n \leq s} C^{s-n}_{\Gamma(m+1)}(M^n \otimes_{BP_*} M).$$

Similarly we can define the chromatic cobar resolution $CD^*_{\Gamma(m+1)}(BP_*)$ additively by

$$CD^s_{\Gamma(m+1)}(M) = \bigoplus_{0 \leq n \leq s} D^{s-n}_{\Gamma(m+1)}(M^n \otimes_{BP_*} M) \tag{1.5}$$

with a suitable coboundary. Here M^n denotes the nth chromatic comodule of [Rav86, 5.1.5] defined inductively by short exact sequences

$$0 \to N^n \to M^n \to N^{n+1} \to 0 \tag{1.6}$$

where $N^0 = BP_*$ and $M^n = v_n^{-1}N^n$. Every element of N^n and M^n is annihilated by a power of the invariant prime ideal $I_n = (p, v_1, \dots, v_{n-1})$. We denote elements in these comodules by fractions of the form

$$\frac{x}{p^{e_0}v_1^{e_1}\dots v_{n-1}^{e_{n-1}}} \tag{1.7}$$

where x is in BP_* (respectively $v_n^{-1}BP_*$) but not in I_n, and all exponents in the denominator are positive. We call such an expression a *chromatic fraction*. It is killed by multiplication by any element in BP_* which leads to a fraction where the denominator has one or more nonpositive exponents.

Definition 1.8. *A comodule M over a Hopf algebroid (A, Γ) is* **weak injective** *(through a range of dimensions) if* $\mathrm{Ext}^s(M) = 0$ *for* $s > 0$.

We will study the properties of such comodules in the next section.

Definition 1.9. *For a left $G(m+1, k-1)$-comodule M let $\widehat{r}_j : M \to \Sigma^{j|t_{m+1}|}M$ be the group homomorphism defined by*

$$M \xrightarrow{\psi_M} G(m+1, k-1) \otimes M \xrightarrow{\rho_j \otimes M} \Sigma^{j|t_{m+1}|}M$$

where $\rho_j : G(m+1, k-1) \to A(m+k)$ is the $A(m+k)$-linear map sending t_{m+1}^j to 1 and all other monomials in the t_{m+i} to 0.

We will refer to this map as a **Quillen operation**.

It follows that

$$\psi(x) = \sum_j t_{m+1}^j \otimes \widehat{r}_j(x) + \dots,$$

where the missing terms involve t_ℓ for $\ell > m+1$.

Lemma 1.10. *The Quillen operation $\widehat{r}_j$ of 1.9 is a comodule map and for $j > 0$ it induces the trivial endomorphism in Ext.*

Proof. The fact that $\widehat{r}_j$ is a comodule map is equivalent to the commutativity of the following diagram, in which we abbreviate $G(m+1, k-1)$ by Γ.

$$\begin{tikzcd} M \arrow[r, "\psi_M"] \arrow[d, "\psi_M"'] & \Gamma \otimes M \arrow[r, "\rho_j \otimes M"] \arrow[d, "\Gamma \otimes \psi_M"'] & \Sigma^{j|t_{m+1}|}M \arrow[d, "\psi_{\Sigma^{j|t_{m+1}|}M}"] \\ \Gamma \otimes M \arrow[r, "\psi_{\Gamma\otimes M}"] & \Gamma \otimes \Gamma \otimes M \arrow[r, "\rho_j \otimes \Gamma \otimes M"] & \Gamma \otimes \Sigma^{j|t_{m+1}|}M \end{tikzcd}$$

The left hand square commutes by the coassociativity of ψ, and the commutativity of the right hand square is obvious.

To show that $\widehat{r}_j$ induces the trivial endomorphism of Ext groups we start with Ext^0. An element in $\mathrm{Ext}^0(M) \subset M$ is by definition in the kernel of $\widehat{r}_j$ for $j > 0$. Now consider the short exact sequence of comodules

$$0 \to M \to \Gamma \otimes M \to \overline{\Gamma} \otimes M \to 0.$$

Since $\Gamma \otimes M$ is weak injective, the long exact sequence of Ext groups reduces to a 4-term sequence

$$0 \to \mathrm{Ext}^0(M) \to M \to \mathrm{Ext}^0(\overline{\Gamma} \otimes M) \to \mathrm{Ext}^1(M) \to 0$$

and isomorphisms

$$\mathrm{Ext}^i(\overline{\Gamma}\otimes M)\cong \mathrm{Ext}^{i+1}(M)$$

for $i>0$. Thus the triviality of $\widehat{r}_j$ in Ext^0 implies its triviality in all higher Ext groups. ☐

Definition 1.11. *Let $T_m^h\subset G(m+1,k-1)$ denote the sub-$A(m+k)$-module generated by $\{t_{m+1}^j\colon 0\le j\le h\}$. A $G(m+1,k-1)$-comodule M is i-**free** if the comodule tensor product $T_m^{p^i-1}\otimes_{A(m+k)} M$ is weak injective.*

We have suppressed the index k from the notation T_m^h because it will usually be clear from the context. In the case $k=\infty$ the Ext group has the topological interpretation given in Lemma 1.15 below. The following lemma is useful in dealing with such comodules.

Lemma 1.12. *For a left $G(m+1)$-comodule M, the group*

$$\mathrm{Ext}^0_{G(m+1)}(A(m+1),T_m^{p^i-1}\otimes_{A(m+k)} M)$$

is isomorphic as an $A(m)$-module to

$$L=\bigcap_{j\ge p^i}\ker\,\widehat{r}_j\subset M.$$

Proof. We will show that the map

$$\begin{array}{rcc} L & \longrightarrow & \mathrm{Ext}^0_{G(m+1)}(A(m+1),T_m^{p^i-1}\otimes_{A(m+1)} M)\\ L\ni y & \mapsto & (1\otimes c)\psi_M(y)\end{array}$$

is an isomorphism.

Let $y\in L$, then we claim that

$$\begin{aligned}(c\otimes 1)\psi_M(y) &= \textstyle\sum_{0\le j<p^i} c(t_{m+1}^j)\otimes \widehat{r}_j(y)\\ &= \textstyle\sum_{0\le j<p^i} (-1)^j t_{m+1}^j\otimes \widehat{r}_j(y)\end{aligned}$$

is primitive. This will follow from the commutativity of the following diagram for an arbitrary comodule M over a Hopf algebroid Γ.

$$\begin{tikzcd}
M \arrow[r, "\psi"] \arrow[d, "\psi"'] & \Gamma\otimes M \arrow[r, "c\otimes M"] & \Gamma\otimes M \arrow[d, "\Delta\otimes\psi"] \\
\Gamma\otimes M \arrow[d, "c\otimes M"'] & & \Gamma\otimes\Gamma\otimes\Gamma\otimes M \arrow[d, "\Gamma\otimes t\otimes M"] \\
\Gamma\otimes M \arrow[drr] & & \Gamma\otimes\Gamma\otimes\Gamma\otimes M \arrow[d, "\mu\otimes\Gamma\otimes M"] \\
& & \Gamma\otimes\Gamma\otimes M
\end{tikzcd} \tag{1.13}$$

In this diagram, t denotes the transposition map, μ denotes the multiplication in Γ, and the unlabelled map sends $\gamma\otimes m$ to $1\otimes\gamma\otimes m$. The composite in the right column is the comodule structure on $\Gamma\otimes M$. Thus the commutativity of the diagram means that $c(m')\otimes m''\in\Gamma\otimes M$ is a comodule primitive, where $m'\otimes m''$ denotes $\psi(m)$.

The see that the diagram commutes we use the fact that a Γ-comodule M is formally dual to a set X acted on by a groupoid G. For $(g_1, g_2, x) \in G \times G \times X$, in the dual diagram we have

$$\begin{tikzcd}
g_2^{-1}(x) & \left(g_2^{-1}g_1^{-1}, g_1(x)\right) \arrow[l] & (g_1g_2, g_1(x)) \arrow[l] \\
\left(g_2^{-1}, x\right) \arrow[u] & & (g_1, g_2, g_1, x) \arrow[u] \\
(g_2, x) \arrow[u] & & (g_1, g_1, g_2, x) \arrow[u] \\
 & & (g_1, g_2, x)\,, \arrow[u] \arrow[ull]
\end{tikzcd}$$

and the commutativity of (1.13) follows.

Conversely, we can show that any primitive $x \in G(m+1) \otimes M$ is written uniquely in this form, i.e., that

$$\mathrm{Ext}^0_{G(m+1)}(G(m+1) \otimes M) \cong M.$$

Let $x = \sum_{j \geq 0} t_{m+1}^j \otimes x_j$ be primitive and assume inductively that $x_j = (-1)^j r_j(x_0)$ for $j < l$. (This is trivially true for $j = 0$).

The Cartan formula implies that

$$\widehat{r}_l(x) = \sum_{t,j \geq 0} \widehat{r}_t(t_{m+1}^j) \otimes \widehat{r}_{l-t}(x_j).$$

Moreover, for $t_{m+1}^j \in T_m^{p^i-1}$ we can deduce the formula

$$\widehat{r}_t(t_{m+1}^j) = \binom{j}{t} t_{m+1}^{j-t}$$

by its comodule structure, and for $y \in M$

$$\widehat{r}_s(\widehat{r}_t(y)) = \binom{s+t}{s} \widehat{r}_{s+t}(y)$$

by the comodule associativity of M.

Because x is primitive, we have $\widehat{r}_l(x) = 0$ for $l > 0$. Thus we have

$$0 = \widehat{r}_l(x) = \sum_{t,j \geq 0} \widehat{r}_t(t_{m+1}^j) \otimes \widehat{r}_{l-t}(x_j) = \sum_{t,j \geq 0} \binom{j}{t} t_{m+1}^{j-t} \otimes \widehat{r}_{l-t}(x_j).$$

Collecting terms where the exponent of t_{m+1} is zero (i.e., $t = j$) gives

$$\begin{aligned}
0 &= \textstyle\sum \widehat{r}_{l-j}(x_j) \\
&= x_l + \textstyle\sum_{j<l} \widehat{r}_{l-j}(x_j) \\
&= x_l + \textstyle\sum_{j<l} (-1)^{l-j} \widehat{r}_{l-j}(\widehat{r}_j(x_0)) \quad \text{by the assumption} \\
&= x_l + \textstyle\sum_{j<l} (-1)^{l-j} \binom{l}{j} \widehat{r}_l(x_0)
\end{aligned}$$

Using the identity $\sum_j (-1)^l \binom{l}{j} = 0$, we obtain $x_l = (-1)^l \widehat{r}_l(x_0)$ as desired.

Since $T_m^{p^i-1} \otimes M \subset G(m+1) \otimes M$, a primitive in it must have the same form, which implies that $\widehat{r}_t(x_0) = 0$ for $t \geq p^i$, i.e., that $x_0 \in L$. □

Lemma 1.14. *Let D be a weak injective comodule over $\Gamma(m+1)$. Then $T_m^{p^i-1} \otimes D$ is also weak injective with*

$$\mathrm{Ext}^0_{\Gamma(m+1)}(T_m^{p^i-1} \otimes D) \cong A(m)\left\{t_{m+1}^j \colon 0 \leq j \leq p^i - 1\right\} \otimes \mathrm{Ext}^0_{\Gamma(m+1)}(D).$$

Given $x_0 \in \mathrm{Ext}^0_{\Gamma(m+1)}(D)$, the element isomorphic to $t_{m+1}^j \otimes x_0$ is

$$\sum_{0 \leq k \leq j} (-1)^k \binom{j}{k} t_{m+1}^k \otimes x_{j-k} \in T_m^{p^i-1} \otimes D$$

where $x_j \in D$ satsifies

$$\psi(x_j) = \sum_{0 \leq k \leq j} \binom{j}{k} t_{m+1}^{j-k} \otimes x_k.$$

Proof. The comodule $T_m^{p^i-1}$ has a skeletal filtration in which F_k is the sub-A-module generated by elements in dimensions $\leq k$. Since $T_m^{p^i-1}$ is free over A, each subquotient F_k/F_{k-1} is a direct sum of a finite number of copies of $\Sigma^k A$.

We can compute $\mathrm{Ext}_{\Gamma(m+1)}(T_m^{p^i-1} \otimes D)$ in terms of $\mathrm{Ext}_{\Gamma(m+1)}(D)$ by a spectral sequence based on the skeletal filtration of $T_m^{p^i-1}$. It must collapse since D has no higher Ext groups, giving the desired isomorphism.

It follows from Lemma 2.2 below that D is also weak injective over $\Gamma(m+2)$ and that $\mathrm{Ext}^0_{\Gamma(m+2)}(D)$ is weak injective over $G(m+1)$. Given $x_0 \in \mathrm{Ext}^0_{\Gamma(m+1)}(D)$ we will construct elements $x_j \in \mathrm{Ext}^0_{\Gamma(m+2)}(D)$ as advertized by induction on j. Suppose we have found x_k for $k < j$. Then the expression

$$y_j = \sum_{0 \leq k < j} \binom{j}{k} t_{m+1}^{j-k} \otimes x_k$$

is a cocycle in the cobar complex $C_{\Gamma(m+1)}(D)$. If it is not a coboundary, then it represents a nontrivial element in Ext^1, contradicting the weak injectivity of D. Hence there must be an element x_j with the desired property.

Then we find that

$$\widehat{r}_k(x_j) = \binom{j}{k} x_{j-k},$$

and a calculation similar to the one in the proof of Lemma 1.12 shows that for $j < p^i$, the element

$$\begin{aligned} \sum_{0 \leq k \leq j} (-1)^k t_{m+1}^k \otimes \widehat{r}_k(x_j) &= \sum_{0 \leq k \leq j} (-1)^k \binom{j}{k} t_{m+1}^k \otimes x_{j-k} \\ &= (-1)^j t_{m+1}^j \otimes x_0 + \dots \end{aligned}$$

is the desired primitive. □

Lemma 1.15. *For each nonnegative m and h there is a spectrum $T(m)_h$ where $BP_*(T(m)_h) \subset BP_*(BP)$ is a free module over*

$$BP_*(T(m)) = BP_*[t_1, \ldots, t_m]$$

on generators $\{t_{m+1}^j : 0 \le j \le h\}$. Its Adams-Novikov E_2-term is

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m)_h)) \cong \mathrm{Ext}_{\Gamma(m+1)}(BP_*, T_m^h).$$

Proof. We will construct these spectra by induction on h. Recall that $T(m) = T(m)_0$ is a p-local summand of the Thom spectrum $X(p^{m+1}-1)$ associated with the map

$$\Omega SU(p^{m+1}-1) \to \Omega SU = BU.$$

This is proved in [Rav86, §6.5]. We have a fibration

$$\Omega SU(p^{m+1}-1) \to \Omega SU(p^{m+1}) \to \Omega S^{2p^{m+1}-1},$$

and $T(m+1)$ is a p-local summand of $X(p^{m+1})$. We know that $\Omega S^{2p^{m+1}-1}$ is homotopy equivalent to the the James construction $J_\infty S^{2p^{m+1}-2}$, which is a CW-complex with one cell in each dimension divisible by $2p^{m+1}-2$. Its $(2p^{m+1}-2)h$-skeleton is denoted by $J_h S^{2p^{m+1}-2}$ and is a certain topological quotient (originally defined by I. M. James) of the h-fold Cartesian product of $S^{2p^{m+1}-2}$. Thus we have a diagram

$$\begin{tikzcd}
\Omega SU(p^{m+1}-1) \arrow[r] & \Omega SU(p^{m+1}) \arrow[r] & \Omega S^{2p^{m+1}-1} \\
\Omega SU(p^{m+1}-1) \arrow[r] \arrow[u, equal] & B_h \arrow[r] \arrow[u] & J_h S^{2p^{m+1}-2}. \arrow[u, "i"']
\end{tikzcd}$$

where the bottom row is the fibration induced from the top row by the inclusion i. We will construct $T(m)_h$ as a p-local summand of the Thom spectrum F_h associated with B_h.

We have a cofiber sequence

$$F_{h-1} \to F_h \to \Sigma^{(2p^{m+1}-2)h} X(p^{m+1}-1).$$

Assuming inductively that $T(m)_{h-1}$ is a p-local summand of F_{h-1}, we get the following diagram in which all spectra have been localized at p.

$$\begin{tikzcd}
F_{h-1} \arrow[r] \arrow[d] & F_h \arrow[r] \arrow[d] & \Sigma^{(2p^{m+1}-2)h} X(p^{m+1}-1) \arrow[d, equal] \\
T(m)_{h-1} \arrow[r] & F'_h \arrow[r] & \Sigma^{(2p^{m+1}-2)h} X(p^{m+1}-1) \\
T(m)_{h-1} \arrow[r] \arrow[u, equal] & T(m)_h \arrow[r] \arrow[u] & \Sigma^{(2p^{m+1}-2)h} T(m) \arrow[u]
\end{tikzcd}$$

The two lower rows are cofiber sequences and each is induced from the one above it by the evident map. □

To pass from $\mathrm{Ext}_{G(m+1,k-1)}(T_m^{p^{i+1}-1}\otimes M)$ to $\mathrm{Ext}_{G(m+1,k-1)}(T_m^{p^i-1}\otimes M)$ we can make use of the tensor product (over $A(m+k)$) of M with the long exact sequence

$$0\to T_m^{p^i-1}\xrightarrow{\ i\ }R^0\xrightarrow{\ d^0\ }R^1\xrightarrow{\ d^1\ }R^2\xrightarrow{\ d^2\ }\cdots, \tag{1.16}$$

where

$$\begin{aligned} R^{2s+e} &= \Sigma^{(ps+e)2p^i(p^{m+1}-1)}T_m^{p^{i+1}-1} \qquad \text{for } e=0,1\\ \text{and}\qquad d^s &= \begin{cases}\widehat{r}_{p^i} & \text{for } s \text{ even}\\ \widehat{r}_{(p-1)p^i} & \text{for } s \text{ odd,}\end{cases}\end{aligned}$$

which leads to a spectral sequence as in [Rav86, A1.3.2].

Theorem 1.17. *For a $G(m+1,k-1)$-comodule M there is a spectral sequence converging to $\mathrm{Ext}_{G(m+1,k-1)}(M\otimes T_m^{p^i-1})$ with*

$$E_1^{*,t}=E(h_{m+1,i})\otimes P(b_{m+1,i})\otimes \mathrm{Ext}^t_{G(m+1,k-1)}(T_m^{p^{i+1}-1}\otimes M)$$

with $h_{m+1,i}\in E_1^{1,0}$, $b_{m+1,i}\in E_1^{2,0}$, and $d_r:E_r^{s,t}\to E_r^{s+r,t-r+1}$. If M is $(i+1)$-free in a range of dimensions, then the spectral sequence collapses from E_2 in the same range.

Moreover d_1 is induced by the action on M of $\widehat{r}_{p^i\Delta_{m+1}}$ for s even and $\widehat{r}_{(p-1)p^i}$ for s odd.

The action of d_1 is as follows. Let

$$x=\sum_{0\le j<p^{i+1}}t_{m+1}^j\otimes m_j\in T_m^{p^{i+1}-1}\otimes M$$

Then d_1 is induced by the endomorphism

$$x\mapsto\begin{cases}-\displaystyle\sum_{0\le k<p^i}\ \sum_{k\le j<p^{i+1}}\binom{j}{k}t_{m+1}^{j-k}\otimes\widehat{r}_{(p^i-k)}(m_j) & \text{for } s \text{ even}\\ -\displaystyle\sum_{0\le k<(p-1)p^i}\ \sum_{k\le j<p^{i+1}}\binom{j}{k}t_{m+1}^{j-k}\otimes\widehat{r}_{((p-1)p^i-k)}(m_j) & \text{for } s \text{ odd.}\end{cases}$$

We will refer to this as the **small descent spectral sequence**.

Proof. Additively this spectral sequence is a special case of the one in [Rav86, A1.3.2] associated with M tensored with the long exact sequence (1.16), and the collapsing for $(i+1)$-free M follows from the fact that the spectral sequence is in that case concentrated on the horizontal axis.

For the identification of d_1, note that by (1.16) it is induced by the endomorphism

$$\begin{aligned} x\ &\mapsto\ \begin{cases}\displaystyle\sum_{0\le j<p^{i+1}}\widehat{r}_{p^i}(t_{m+1}^j)\otimes m_j & \text{for } s \text{ even}\\ \displaystyle\sum_{0\le j<p^{i+1}}\widehat{r}_{(p-1)p^i}(t_{m+1}^j)\otimes m_j & \text{for } s \text{ odd}\end{cases}\\ &=\ \begin{cases}\displaystyle\sum_{p^i\le j<p^{i+1}}\binom{j}{p^i}t_{m+1}^{j-p^i}\otimes m_j & \text{for } s \text{ even}\\ \displaystyle\sum_{(p-1)p^i\le j<p^{i+1}}\binom{j}{(p-1)p^i}t_{m+1}^{j-(p-1)p^i}\otimes m_j & \text{for } s \text{ odd.}\end{cases}\end{aligned}$$

It follows from Lemma 1.10 that $\widehat{r}_{p^i\Delta_{m+1}}$ and $\widehat{r}_{(p-1)p^i\Delta_{m+1}}$ each induce trivial endomorphisms in Ext, so d_1 is also induced by

$$x \quad\mapsto\quad \begin{cases} -\widehat{r}_{p^i}(x) + \displaystyle\sum_{0\le j<p^{i+1}} \widehat{r}_{p^i}(t_{m+1}^j)\otimes m_j & \text{for } s \text{ even} \\ -\widehat{r}_{(p_1)p^i}(x) + \displaystyle\sum_{0\le j<p^{i+1}} \widehat{r}_{(p-1)p^i}(t_{m+1}^j)\otimes m_j & \text{for } s \text{ odd,} \end{cases}$$

which leads to the stated formula.

The multiplicative structure requires some explanation. The elements $h_{m+1,i}$ and $b_{m+1,i}$ correspond under Yoneda's isomorphism [HS70, page 155] to the tensor product of M with the exact sequences

$$0\to T_m^{p^i-1}\to T_m^{2p^i-1}\to \Sigma^{p^i|t_{m+1}|}T_m^{p^i-1}\to 0$$
and
$$0\to T_m^{p^i-1}\to T_m^{p^{i+1}-1}\to \Sigma^{p^i|t_{m+1}|}T_m^{p^{i+1}-1}\to \Sigma^{p^{i+1}|t_{m+1}|}T_m^{p^i-1}\to 0$$

respectively. Products of these elements correspond to the splices of the these. It follows that these two elements are permanent cycles and that the spectral sequence is one of modules over the algebra $E(h_{m+1,i})\otimes P(b_{m+1,i})$. □

In practice we will find higher differentials in this spectral sequence by computing in the cobar complex $C_{G(m+1,k-1)}(M\otimes T_m^{p^i-1})$ or its subcomplex $C_{G(m+1,k-1)}(M)$. As explained in [Rav86, proof of A1.3.2], it can be embedded by a quasi-isomorphism (i.e., a map inducing an isomorphism in cohomology) into the double complex $B=\oplus B^{s,t\ge 0}$ defined by

$$B^{s,t}=C^t_{G(m+1,k-1)}(M\otimes R^s)$$

with coboundary

$$\partial = d+(-1)^s d^s,$$

where d is the coboundary operator in the cobar complex. Our spectral sequence is obtained from the filtration of B by horizontal degree, i.e., the one defined by

$$F^rB=\bigoplus_{s\ge r,t\ge 0} B^{s,t}. \tag{1.18}$$

Theorem 1.17 also has a topological counterpart in the case $M=BP_*$. Before stating it we need to define topological analogs of the operations $\widehat{r}_{p^i}$ and $\widehat{r}_{(p-1)p^i}$. One can show that there are cofiber sequences

$$T(m)_{p^i-1}\to T(m)_{p^{i+1}-1}\to \Sigma^{2p^i(p^{m+1}-1)}T(m)_{p^i(p-1)-1} \tag{1.19}$$

and

$$T(m)_{p^i(p-1)-1}\to T(m)_{p^{i+1}-1}\to \Sigma^{2p^i(p-1)(p^{m+1}-1)}T(m)_{p^i-1}. \tag{1.20}$$

We define

$$T(m)_{p^{i+1}-1}\xrightarrow{\rho_{p^i}}\Sigma^{2p^i(p^{m+1}-1)}T(m)_{p^{i+1}-1}$$

and

$$T(m)_{p^{i+1}-1}\xrightarrow{\rho_{p^i(p-1)}}\Sigma^{2p^i(p-1)(p^{m+1}-1)}T(m)_{p^{i+1}-1}$$

to be the composites

$$T(m)_{p^{i+1}-1} \to \Sigma^{2p^i(p^{m+1}-1)} T(m)_{p^i(p-1)-1} \to \Sigma^{2p^i(p^{m+1}-1)} T(m)_{p^{i+1}-1}$$

and

$$T(m)_{p^{i+1}-1} \to \Sigma^{2p^i(p-1)(p^{m+1}-1)} T(m)_{p^i-1} \to \Sigma^{2p^i(p-1)(p^{m+1}-1)} T(m)_{p^{i+1}-1}.$$

Theorem 1.21. *Let $T(m)_h$ be the spectrum of Lemma 1.15. There is a spectral sequence converging to $\pi_*(T(m)_{p^i-1})$ with*

$$E_1^{s,t} = E(h_{m+1,i}) \otimes P(b_{m+1,i}) \otimes \pi_*(T(m)_{p^{i+1}-1})$$

with $h_{m+1,i} \in E_1^{1,2p^i(p^{m+1}-1)}$, $b_{m+1,i} \in E_1^{2,2p^{i+1}(p^{m+1}-1)}$, and $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$. Moreover d_1 is ρ_{p^i} for s even and $\rho_{(p-1)p^i}$ for s odd. The elements $h_{m+1,i}$ and $b_{m+1,i}$ are permanent cycles, and the spectral sequence is one of modules over the ring

$$R = E(h_{m+1,i}) \otimes P(b_{m+1,i})$$

We will refer to this as the **topological small descent spectral sequence**.

Proof. This the spectral sequence based on the Adams diagram

$$\begin{tikzcd}
X \arrow[d] & \Sigma^a X' \arrow[l] \arrow[d] & \Sigma^b X \arrow[l] \arrow[d] & \Sigma^{a+b} X' \arrow[l] \arrow[d] & \cdots \arrow[l] \\
Y & \Sigma^a Y & \Sigma^b Y & \Sigma^{a+b} Y &
\end{tikzcd}$$

where

$$\begin{aligned}
a &= 2p^i(p^{m+1}-1)-1, \\
b &= 2p^{i+1}(p^{m+1}-1)-2, \\
X &= T(m)_{p^i-1}, \\
X' &= T(m)_{p^i(p-1)-1}, \\
\text{and} \quad Y &= T(m)_{p^{i+1}-1}.
\end{aligned}$$

We will show that the elements $h_{m+1,i}$ and $b_{m+1,i}$ can each be realized by maps of the form

$$S^0 \longrightarrow X \xrightarrow{f} \Sigma^{-?} X$$

For $h_{m+1,i}$, f is the boundary map for the cofiber sequence

$$T(m)^{p^i-1} \to T(m)^{2p^i-1} \to \Sigma^{h+1} T(m)^{p^i-1},$$

and for $b_{m+1,i}$ it is the composite (in either order) of the ones for (1.19) and (1.20). $\square$

Example 1.22. *When $m = i = 0$, the spectrum $T(m)_{p^i-1}$ is S^0 while $T(m)_{p^{i+1}-1}$ is the p-cell complex*

$$Y = S^0 \cup_{\alpha_1} e^q \cup_{\alpha_1} e^{2q} \cdots \cup_{\alpha_1} e^{(p-1)q},$$

where $q = 2p-2$. The E_1-term of the spectral sequence of Theorem 1.21 is

$$E(h_{1,0}) \otimes P(b_{1,0}) \otimes \pi_*(Y).$$

where $h_{1,0}$ and $b_{1,0}$ represent the homotopy elements α_1 and β_1 (α_1^2 for $p = 2$) respectively.

We will use this spectral sequence through a range of dimensions in the following way. For each spectrum $T(m)_{p^{i+1}-1}$ the elements of Adams-Novikov filtration 0 and 1 are all permanent cycles, so we ignore them, replacing $\pi_*(T(m)_{p^{i+1}-1})$ by an appropriate subquotient of $\mathrm{Ext}_{\Gamma(m+1)}(E_{m+1}^2 \otimes T_m^{p^{i+1}-1})$. Let $\mathbf{N}$ be a list of generators of this group arranged by dimension. When an element x has order greater than p, we also list its nontrivial multiples by powers of p. Thus

$$\mathbf{N} \otimes E(h_{m+1,i}) \otimes P(b_{m+1,i})$$

contains a list of generators of the E_1-term in our range. Rather than construct similar lists for each E_r term we use the following method.

Procedure 1.23. *We make two lists $\mathbf{I}$ (input) and $\mathbf{O}$ (output). $\mathbf{I}$ is the subset of $\mathbf{N} \otimes E(h_{m+1,i})$ that includes all elements in our range. Then $\mathbf{O}$ is constructed by dimensional induction starting with the empty list as follows. Assuming $\mathbf{O}$ has been constructed through dimensions $k-1$, add to it the k-dimensional elements of $\mathbf{I}$. If any of them supports a nontrivial differential in the spectral sequence, remove both the source and target from $\mathbf{O}$. (It may be necessary to alter the list of $(k-1)$-dimensional elements by a linear transformation so that each nontrivial target is a "basis" element.) Then if $k > |b_{m+1,i}|$, we append the product of $b_{m+1,i}$ with each element of $\mathbf{O}$ in dimension $k - |b_{m+1,i}|$. This completes the inductive step.*

Note that each element in E_1 of filtration greater than 1 is divisible by $b_{m+1,i}$. Since the spectral sequence is one of R-modules, that same is true of each E_r. In 1.23 we compute the differentials originating in filtrations 0 and 1. If $d_r(x) = y$ is one of them, there is no chance that for some minimal $t > 0$

$$d_{r'}(x') = b_{m+1,i}^t y \qquad \text{with } r' < r$$

because such an x' would have to be divisible by $b_{m+1,i}$. This justifies the removal of $b_{m+1,i}^t x$ and $b_{m+1,i}^t y$ for all $t \geq 0$ from consideration.

2. Weak injective comodules

In this section we will study comodules M over a general Hopf algebroid (A, Γ) over $\mathbf{Z}_{(p)}$. We will abbreviate $\mathrm{Ext}_\Gamma(A, M)$ by $\mathrm{Ext}(M)$.

The definition 1.8 of a weak injective should be compared with other notions of injectivity. A comodule I (or more generally an object in an abelian category) is *injective* if any homomorphism to it extends over monomorphisms, i.e., if one can always fill in the following diagram.

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & M \arrow[u] \arrow[r, "i"] & N \arrow[ul, dotted] \end{tikzcd}$$

This definition is rather limiting. For example if A is a free $\mathbf{Z}_{(p)}$-module, then an injective I must be p-divisible since a homomorphism $A \to I$ must extend over $A \otimes \mathbf{Q}$.

There is also a notion of relative injectivity [Rav86, A1.2.7] requiring I to be a summand of $\Gamma \otimes_A I$, which implies that the diagram above can always be completed when i is split over A. This implies weak injectivity as we have defined it here (see [Rav86, A1.2.8 (b)]), but we do not know if the converse is true. In any case the requirements of our definition can be said to hold only through a range of dimensions.

Lemma 2.1. *A connective comodule M over (A, Γ) is weak injective in a range of dimensions iff* $\mathrm{Ext}^1(M) = 0$ *in the same range.*

Proof. The comodule $\Gamma \otimes_A M$ is weak injective with

$$\mathrm{Ext}^0(\Gamma \otimes_A M) = M.$$

From the short exact sequence

$$0 \longrightarrow M \xrightarrow{\psi} \Gamma \otimes_A M \longrightarrow N \longrightarrow 0$$

(where ψ is the comodule structure map for M) we see that for $s > 0$,

$$\mathrm{Ext}^s(N) \cong \mathrm{Ext}^{s+1}(M).$$

Γ has a skeletal filtration $\{F_t\}$ with $F_0 = A$ and each subquotient F_t/F_{t-1} is a direct sum of finitely many copies of the same suspension of A. We conclude by induction on t that $\mathrm{Ext}^1(M \otimes_A F_t/F_0) = 0$. Passing to the limit gives

$$0 = \mathrm{Ext}^1(N) = \mathrm{Ext}^2(M).$$

Arguing by induction on s we conclude that $\mathrm{Ext}^s(N) = 0$ for all $s > 0$ (making N a weak injective), so $\mathrm{Ext}^s(M) = 0$ for all $s > 1$. Hence M is weak injective as claimed. $\square$

Lemma 2.2. *Let*

$$(D, \Phi) \to (A, \Gamma) \to (A, \Sigma)$$

be an extension [Rav86, A1.1.15] *of graded connected Hopf algebroids of finite type, and suppose that M is a weak injective comodule over Γ. Then M is also weak injective over Σ, and* $\mathrm{Ext}^0_\Sigma(A, M)$ *is weak injective over Φ with*

$$\mathrm{Ext}^0_\Phi(D, \mathrm{Ext}^0_\Sigma(A, M)) \cong \mathrm{Ext}^0_\Gamma(A, M).$$

Proof. The change-of-rings-isomorphism [Rav86, A1.3.12] says that

$$\mathrm{Ext}_\Sigma(A, M) \cong \mathrm{Ext}_\Gamma(A, \Gamma \square_\Sigma M). \tag{2.3}$$

We also know by [Rav86, A1.1.17] that

$$\Gamma \cong \Phi \otimes_D \Sigma$$

as Σ-comodules. It follows that

$$\Gamma \square_\Sigma M \cong (\Phi \otimes_D \Sigma) \square_\Sigma M \cong \Phi \otimes_D M.$$

Recall that Φ has a skeletal filtration in which F_k is the sub-D-module generated by elements of dimension $\leq k$ and each F_k/F_{k-1} is a direct sum of finitely many copies of $\Sigma^k D$. It follows that the skeletal filtration on Φ induces a filtration on $\Gamma \square_\Sigma M$ in which each subquotient is a finite direct sum of copies of suspensions of M. From this we can infer that $\Gamma \square_\Sigma M$ is weak injective over Γ, and therefore (by (2.3)), M is weak injective of Σ as claimed.

Now consider the Cartan-Eilenberg spectral sequence for the extension in question, with

$$E_2 = \mathrm{Ext}_\Phi(D, \mathrm{Ext}_\Sigma(A, M)).$$

Since M is weak injective over Σ, this reduces to an isomorphism

$$\mathrm{Ext}_\Gamma(A, M) = \mathrm{Ext}_\Phi(D, \mathrm{Ext}^0_\Sigma(A, M)).$$

Since M is weak injective over Γ, $N = \mathrm{Ext}^0_\Sigma(A, M)$ is weak injective over Φ with $\mathrm{Ext}^0(N)$ as claimed. $\square$

In order to proceed further we need a method of recognizing weak injectives without computing any higher Ext groups. Here is a useful technical tool.

Definition 2.4. *Let H be a graded connected torsion abelian p-group of finite type, and let H_i have order p^{h_i}. Then the* **Poincaré series** *for H is $h(t) = \Sigma h_i t^i$.*

Example 2.5. *Let $I \subset BP_*$ be the maximal ideal so that $BP_*/I = \mathbf{Z}/(p)$. Then the Poincaré series for $\Gamma(m+1)/I$ is*

$$G_m(t) = \prod_{i>0} (1 - t^{|v_{m+i}|})^{-1}.$$

Theorem 2.6. *Let (A, Γ) be a graded connected Hopf algebroid over $\mathbf{Z}_{(p)}$, and let M be a connected torsion Γ-comodule of finite type. Let $I \subset A$ be the maximal ideal (so that $A/I = \mathbf{Z}/(p)$). Then*

$$g(M) \leq g(\mathrm{Ext}^0(M)) g(\Gamma/I),$$

meaning that each coefficient of the power series on the left is dominated by the corresponding one on the right, with equality holding if and only if M is a weak injective (1.8).

Proof. We will construct a decreasing filtration $\{F^i\}$ on M such that the associated bigraded comodule $E_0 M$ is annihilated by I and $\mathrm{Ext}^0(E_0 M) = E_0 \mathrm{Ext}^0(M)$, so that $\mathrm{Ext}^0(E_0 M)$ has the same Poincaré series as $\mathrm{Ext}^0(M)$. Then we will prove the lemma by by showing it is true for $E_0 M$.

For any comodule M as above we will construct a subcomodule $M' \subset M$ containing IM such that $\mathrm{Ext}^0(IM) = \mathrm{Ext}^0(M')$ and the short exact sequence

$$0 \to M' \to M \to M'' \to 0 \tag{2.7}$$

induces a short exact sequence in Ext^0. Then the desired filtration can be defined by $F^{k+1}M = (F^k M)'$.

Define short exact sequences

$$0 \to M'_i \to M \to M''_i \to 0$$

inductively as follows. Let $M'_0 = IM$ and let $K_i = \mathrm{Ext}^0(M''_i)/\mathrm{im}\ \mathrm{Ext}^0(M)$. Since M''_i is annihilated by I we can choose a splitting $K_i \to \mathrm{Ext}^0(M''_i)$. K_i is then a sub A-module and therefore a subcomodule (with trivial Γ-coaction)) of M''_i and we can define $M''_{i+1} = M''_i/K_i$. Then we have short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & M'_i \arrow[r] \arrow[d, equal] & M'_{i+1} \arrow[r] \arrow[d] & K_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'_i \arrow[r] & M \arrow[r] & M''_i \arrow[r] & 0
\end{tikzcd}$$

K_i was chosen so that $\mathrm{Ext}^0(K_i)$ maps monomorphically to $\mathrm{Ext}^1(M'_i)$, so $\mathrm{Ext}^0(M'_i) = \mathrm{Ext}^0(M'_{i+1})$. It follows that $\mathrm{Ext}^0(IM) = \mathrm{Ext}^0(M')$ where $M' = \lim_{\to} M'_i$.

Now we need to show that $\mathrm{Ext}^0(M)$ maps onto $\mathrm{Ext}^0(M'')$ where $M'' = \lim_{\to} M''_i$. Consider the following diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & K_i \arrow[r, equal] \arrow[d] & K_i \arrow[d] \\
\mathrm{Ext}^0(M) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^0(M''_i) \arrow[r] \arrow[d] \arrow[dl, dotted, "\beta"'] \arrow[dr, "\alpha"] & \mathrm{Ext}^1(M'_i) \arrow[d] \\
\mathrm{Ext}^0(M) \arrow[r] & \mathrm{Ext}^0(M''_{i+1}) \arrow[r] & \mathrm{Ext}^1(M'_{i+1})
\end{tikzcd}$$

The map α is trivial because the two maps to $\mathrm{Ext}^1(M'_i)$ have the same image, so the lifting β exists. When we pass to the limit it induces a splitting map

$$\mathrm{Ext}^0(M'') \to \mathrm{Ext}^0(M),$$

so (2.7) induces a short exact sequence in Ext^0 as claimed.

Defining $F^{k+1}M = (F^k M)'$ gives a decreasing filtration of M subordinate to the I-adic filtration (in the sense that E_0M is annihilated by I) with $\mathrm{Ext}^0(E_0M) = E_0\mathrm{Ext}^0(M)$. Hence it suffices to prove the lemma for E_0M, in other words for comodules N annihilated by I.

Assume this N is $(t-1)$-connected and let N^t be its t-skeleton. The A-module splitting $N \to N^t$ induces a comodule splitting $\Gamma \otimes N \to \Gamma \otimes N^t$. Let $f : N \to \Gamma \otimes N^t$ denote the composite of this map with the comodule structure map on N. Let $\tilde{N}$ and $\overline{N}$ denote the kernel and image of f, so we have a short exact sequence

$$0 \to \tilde{N} \to N \to \overline{N} \to 0$$

with $N^t \subset \overline{N} \subset \Gamma/I \otimes N^t$. It follows that $\tilde{N}$ is more highly connected than N and that $\mathrm{Ext}^0(\overline{N})$ is a quotient of $\mathrm{Ext}^0(N)$. Let $g(M)$ denote the Poincaré series for M. Then

$$g(\overline{N}) \leq g(\mathrm{Ext}^0(\overline{N}))g(\Gamma/I).$$

We can define a complete decreasing filtration on N (different from the one we defined on M earlier) by $F^{i+1}N = \widetilde{F^iN}$. Then we have

$$\begin{aligned} g(N) &= \sum_{i\geq 0} g(F^iN/F^{i+1}N) \\ &= \sum_{i\geq 0} g(\overline{F^iN}) \\ &\leq \sum_{i\geq 0} g(\mathrm{Ext}^0(F^iN))g(\Gamma/I) \qquad \text{since } \overline{F^iN} \subset \mathrm{Ext}^0(F^iN)\otimes\Gamma/I \\ &= g(\mathrm{Ext}^0(N))g(\Gamma/I) \end{aligned}$$

as claimed.

Now suppose we have equality above, i.e., for each i

$$g(\overline{F^iN}) = g(\mathrm{Ext}^0(F^iN))g(\Gamma/I).$$

Since $\overline{F^iN} \subset \mathrm{Ext}^0(F^iN)\otimes\Gamma/I$, this means that $\overline{F^iN} = \mathrm{Ext}^0(F^iN)\otimes\Gamma/I$, which is a weak injective. Then a standard filtration argument says that N is itself a weak injective as claimed. Finally a similar argument says that the weak injectivity of each subquotient of M above implies that of M itself.

Conversely, suppose that M is weak injective. Since the short exact sequence (2.7) induces a short exact sequence in Ext^0, it follows that $\mathrm{Ext}^1(M') = 0$, so M' is weak injective, as is M''. Thus each subquotient in our filtration of M is weak injective. Thus it suffices to prove that if N is a weak injective annihilated by I, then its Poincaré series satisfies the indicated equation.

For such an N we have

$$\mathrm{Ext}^1(A/I, N) = \mathrm{Ext}^1(A, N) = 0.$$

From this we can deduce that $\mathrm{Ext}^1(L, N) = 0$ for any connective L of finite type annihilated by I. In particular the short exact sequence

$$0 \longrightarrow N \xrightarrow{\psi} \Gamma\otimes_A N \longrightarrow \overline{\Gamma}\otimes_A N \longrightarrow 0$$

is split. This means we must have $N \cong \Gamma\otimes_A \mathrm{Ext}^0(N)$. □

It would be nice if for any comodule M one could find a map $M \to W$ to a weak injective inducing an isomorphism in Ext^0, but this is not always possible.

Example 2.8. *Let $(A,\Gamma) = (A(1), G(1))$ and $M = A/(p^2)$. Then a simple calculation shows that*

$$\mathrm{Ext}^0(M) = \mathbf{Z}/(p^2)[v_1^p] \oplus \mathbf{Z}/(p)[v_1^p]\{pv_1, pv_1^2, \ldots, pv_1^{p-1}\}$$

so if such a W exists we would have

$$g(W) = \left(\frac{1}{1-t^{|v_1|}}\right)\left(\frac{1}{1-t^{p|v_1|}} + \frac{1}{1-t^{|v_1|}}\right).$$

Also for $p \geq 3$, $\mathrm{Ext}^1(M)$ is generated by the elements

$$\begin{aligned} &\left\{v_1^{i-1}t_1 + (p(i-1)/2)v_1^{i-2}t_1^2 \colon i \geq 1\right\} \cup \left\{pv_1^{pj-1}t_1 \colon j \geq 1\right\} \\ &\qquad \cup \left\{pv_1^i t_1^{p^k} : i \geq 0,\, k \geq 1\right\} \end{aligned} \tag{2.9}$$

In order to kill the first of these we must adjoin a $|v_1|$-dimensional element x_1 to $A/(p^2)$ with

$$\psi(x_1) = t_1 \otimes 1 + 1 \otimes x_1.$$

This implies that $v_1 - px_1$ is primitive, so in order to avoid enlarging Ext^0 *we must have $px_1 = v_1$ or $x_1 = v_1/p$. Similarly we need to adjoin elements $x_i = v_1^i/p^i$ for all $i \geq 1$. These will kill all of the elements in* Ext^1 *in the first two subsets listed in (2.9). For the others we adjoin elements $v_1^i y_{pj}$ (for $i \geq 0$ and $j > 0$) of order p in dimension $(pj+i)|v_1|$ satisfying*

$$\psi(y_{pj}) = p \sum_{0 \leq k \leq j} \binom{j}{k} t_1^{pj-pk} \otimes y_{pk} \qquad \text{where } y_0 = 1.$$

Adjoining these elements would give us a comodule W with the desired Poincaré series. However the element $p^{p^2-1}x_{p^2} = v_1^{p^2}/p$ is invariant and has order p^3, so the map $M \to W$ does not induce an isomorphism in Ext^0.

3. A 4-term exact sequence

In this section we will consider various $\Gamma(m+1)$-comodules M and will abbreviate $\mathrm{Ext}_{\Gamma(m+1)}(BP_*, M)$ by $\mathrm{Ext}_{\Gamma(m+1)}(M)$ or simply $\mathrm{Ext}(M)$.

Excluding the case $m = 0$ and $p = 2$, we will construct a diagram of 4-term exact sequences of $\Gamma(m+1)$-comodules

(3.1)

$$\begin{tikzcd}
0 \arrow[r] & BP_* \arrow[r] \arrow[d, equal] & D^0_{m+1} \arrow[r] \arrow[d, equal] & D^1_{m+1} \arrow[r] \arrow[d] & E^2_{m+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & BP_* \arrow[r] \arrow[d, equal] & D^0_{m+1} \arrow[r] \arrow[d] & v_1^{-1}E^1_{m+1} \arrow[r] \arrow[d] & E^1_{m+1}/(v_1^\infty) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & BP_* \arrow[r] & M^0 \arrow[r] & M^1 \arrow[r] & N^2 \arrow[r] & 0
\end{tikzcd}$$

where each vertical map is a monomorphism, M^i and N^2 are as in (1.6), the D^i_{m+1} are weak injectives with $\mathrm{Ext}^0(D^0_{m+1}) = \mathrm{Ext}^0(BP_*)$, $\mathrm{Ext}^0(D^1_{m+1}))$ contains $\mathrm{Ext}^1(BP_*)$, and $E^1_{m+1} = D^0_{m+1}/BP_*$. $\mathrm{Ext}^0(BP_*)$ and $\mathrm{Ext}^1(BP_*)$ are given in 3.6 and 3.17 respectively.

We will construct the map from BP_* to the weak injective D^0_{m+1}, inducing an isomorphism in Ext^0 , explicitly in Theorem 3.12. For $m > 0$ we cannot construct a similar map out of $E^1_{m+1} = D^0_{m+1}/BP_*$. Instead we will construct a map to a weak injective D^1_{m+1} which enlarges Ext^0 by as little as possible. We will do this

by producing a comodule $E^2_{m+1} \subset E^1_{m+1}/v_1^\infty$ and using the induced extension
(3.2)

$$\begin{tikzcd} 0 \arrow[r] & E^1_{m+1} \arrow[r] & v_1^{-1}E^1_{m+1} \arrow[r] & E^1_{m+1}/(v_1^\infty) \arrow[r] & 0 \\ 0 \arrow[r] & E^1_{m+1} \arrow[r] \arrow[u, equal] & D^1_{m+1} \arrow[r] \arrow[u] & E^2_{m+1} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

The comodule E^2_{m+1} for $m > 0$ will be described in the next section. For $m = 0$ and p odd a map from E^1_1 to a weak injective D^1_1 inducing an isomorphism in Ext^0 will be constructed in below in Lemma 4.1.

We will use the following notations for $m > 0$. We put hats over the symbols in order to distinguish this notation from the usual one for elements in $\mathrm{Ext}_{BP_*(BP)}(BP_*)$. For $m = 0$ we will use similar notation without the hats.

$$(3.3)\qquad \begin{cases} \widehat{v}_i = v_{m+i}, & \widehat{t}_i = t_{m+i}, \quad \omega = p^m, \\ \widehat{h}_{i,j} = h_{m+i,j}, & \text{and} \quad \widehat{b}_{i,j} = b_{m+i,j}. \end{cases}$$

We will show that in dimensions below $p^2|\widehat{v}_1|$, E^2_{m+1} is the $A(m+1)$-module generated by the set of chromatic fractions

$$(3.4)\qquad \left\{ \frac{\widehat{v}_2^{e_2}}{p^{e_0} v_1^{e_1}} : e_0, e_1 > 0,\ e_2 \geq e_0 + e_1 - 1 \right\},$$

and its Ext group in this range is

$$(3.5)\qquad A(m+1)/I_2 \otimes E(\widehat{h}_{1,0}) \otimes P(\widehat{b}_{1,0}) \otimes \left\{ \frac{\widehat{v}_2^{e_2}}{pv_1} : e_2 \geq 1 \right\},$$

where $\widehat{h}_{1,0} \in \mathrm{Ext}^{1,2(p\omega-1)}$ corresponds to the primitive $\widehat{t}_1 \in \Gamma(m+1)$, and $\widehat{b}_{1,0} \in \mathrm{Ext}^{1,2p(p\omega-1)}$ is its transpotent. In both cases there is no power of v_1 in the numerator when $m = 0$. These statements will be proved below as Theorem 4.5.

An Adams-Novikov differential for $T(m)$ originating in the 2-line would have to land in filtration $2p+1$, which is trivial in the is range of dimensions, so by proving 4.5 we have determined $\pi_*(T(m))$ in this range.

Our first goal here is to compute Ext^0 and Ext^1. The following generalization of the Morava-Landweber theorem [Rav86, 4.3.2] is straightforward.

Proposition 3.6.

$$\mathrm{Ext}^0_{\Gamma(m+1)}(BP_*/I_n) = A(n+m)/I_n.$$

For $n = 0$ each of the generators is a permanent cycle.

Proof. The indicated elements are easily seen to be invariant in $\Gamma(m+1)$. In dimensions less that $|\widehat{v}_1|-1$, $T(m)$ is homotopy equivalent to BP, so the generators v_i for $i \leq m$ are permanent cycles as claimed. $\square$

Theorem 3.7. *Let $W_{m+1,n}$ be the ring*

$$W_{m+1,n} = \Gamma(m+1)/(p, v_1, \dots, v_{n-1}, \eta_R(v_{m+n+1}), \eta_R(v_{m+n+2}), \dots).$$

It is weak injective over $\Gamma(m+1)$ and the map $BP_*/I_n \to W_{m+1,n}$ induces an isomorphism in Ext^0.

Proof. In the comodule algebra $\Gamma(m+1)/I_n$ the ideal

$$(\eta_R(v_{m+n+1}), (\eta_R(v_{m+n+2}), \dots)$$

is invariant and regular. Let

$$L_i = \Gamma(m+1)/(p, v_1, \dots, v_{n-1},\, \eta_R(v_{m+n+1}), \dots, (\eta_R(v_{m+n+i})),$$

so for each $i \geq 0$ we have a short exact sequence

$$0 \to \Sigma^{|v_{m+n+i}|} L_i \to L_i \to L_{i+1} \to 0.$$

Since $L_0 = \Gamma(m+1)/I_n$ is weak injective with

$$\mathrm{Ext}^0(L_0) = BP_*/I_n = A(m+n)[v_{m+n+1}, v_{m+n+2}, \dots]/I_n,$$

we can argue by induction on i that L_i is weak injective with

$$\mathrm{Ext}^0(L_i) = A(m+n)[v_{m+n+i+1}, v_{m+n+i+2}, \dots]/I_n.$$

Letting i go to infinity we get $W_{m+1,n}$ with the desired properties. □

Example 3.8. *If we replace I_n by an invariant regular ideal J_n with n generators, a similar construction will lead to a map from BP_*/J_n to a weak injective W_{m+1,J_n} with Ext^0 being $A(m+1)/J_n$. In general this group is not the same as $\mathrm{Ext}^0(BP_*/J_n)$. The simplest nonprime example of such an ideal is (p^2), which was the basis for the counterexample 2.8.*

Now we will give an alternate description of a map from BP_* to a weak injective D^0_{m+1} inducing an isomorphism in Ext^0. We will leave it to the interested reader to prove that $D^0_{m+1} = W_{m+1,0}$. D^0_{m+1} was incorrectly described in the proof of [Rav86, 7.1.9] as $BP_*[p^{-1}\widehat{v}_i : i > 0]$. It is the sub-$A(m)$-algebra of $p^{-1}BP_*$ generated by certain elements $\widehat{\lambda}_{m+i}$ for $i > 0$ congruent to $\widehat{v}_i/p$ modulo decomposables.

To describe them we need to recall Hazewinkel's formula [Haz77] relating polynomial generators $v_i \in BP_*$ to the coefficients ℓ_i of the formal group law, namely

$$p\ell_i = \sum_{0 \leq j < i} \ell_j v_{i-j}^{p^j}. \tag{3.9}$$

This recursive formula expands to

$$\begin{aligned}
\ell_1 &= \frac{v_1}{p} \\
\ell_2 &= \frac{v_2}{p} + \frac{v_1^{p+1}}{p^2} \\
\ell_3 &= \frac{v_3}{p} + \frac{v_1 v_2^p}{p^2} + \frac{v_2 v_1^{p^2}}{p^2} + \frac{v_1^{1+p+p^2}}{p^3} \\
&\vdots
\end{aligned}$$

We need to define reduced log coefficients $\widehat{\ell}_k$ obtained from the ℓ_{m+k} by subtracting the terms which are monomials in the v_j for $j \leq m$. Thus for $m = 1$ we have

$$\begin{aligned} \widehat{\ell}_1 &= \frac{\widehat{v}_1}{p} \\ \widehat{\ell}_2 &= \frac{\widehat{v}_2}{p} + \frac{v_1 \widehat{v}_1^p}{p^2} + \frac{\widehat{v}_1 v_1^{p^2}}{p^2} \\ &\vdots \end{aligned}$$

The analog of Hazewinkel's formula for these elements is

$$p\widehat{\ell}_i = \begin{cases} 0 & \text{if } i \leq 0 \\ \displaystyle\sum_{0 \leq j < i} \ell_j \widehat{v}_{i-j}^{p^j} + \sum_{0 < j < \min(i, m+1)} \widehat{\ell}_{i-j} v_j^{p^{i-j}\omega} & \text{if } i > 0. \end{cases} \tag{3.10}$$

We use these to define our generators $\widehat{\lambda}_i$ recursively for $i > 0$ by

$$\widehat{\ell}_i = \sum_{0 \leq j < i} \ell_j \widehat{\lambda}_{i-j}^{p^j} \tag{3.11}$$

Theorem 3.12. *The BP_*-module $D_{m+1}^0 \subset p^{-1}BP_*$ described above is a subcomodule over $\Gamma(m+1)$ that is weak injective (1.8) with* $\mathrm{Ext}^0 = A(m)$.

Proof. We show first that it is a subcomodule. The right unit on the ℓ_{m+i} is given by

$$\eta_R(\ell_{m+i}) = \ell_{m+i} + \sum_{0 \leq j < i} \ell_j \widehat{t}_{i-j}^{p^j}.$$

Since ℓ_{m+i} and $\widehat{\ell}_i$ differ by terms that are invariant under η_R it follows that

$$\eta_R(\widehat{\ell}_i) = \widehat{\ell}_i + \sum_{0 \leq j < i} \ell_j \widehat{t}_{i-j}^{p^j}. \tag{3.13}$$

Using (3.11) we have

$$\begin{aligned} \eta_R(\widehat{\ell}_i) &= \sum_{0 \leq j < i} \eta_R(\ell_j) \eta_R(\widehat{\lambda}_{i-j}^{p^j}) \\ &= \sum_{0 \leq j < i} \ell_j \eta_R(\widehat{\lambda}_{i-j}^{p^j}) + \sum_{0 \leq j < i} \sum_{0 \leq k < j-m} \ell_k t_{j-k}^{p^k} \eta_R(\widehat{\lambda}_{i-j}^{p^j}) \end{aligned}$$

Combining this with (3.13) and (3.11) gives

$$\begin{aligned} &\sum_{0 \leq j < i} \ell_j \widehat{\lambda}_{i-j}^{p^j} + \sum_{0 \leq j < i} \ell_j \widehat{t}_{i-j}^{p^j} \\ &\quad = \sum_{0 \leq j < i} \ell_j \eta_R(\widehat{\lambda}_{i-j}^{p^j}) + \sum_{0 \leq j < i} \sum_{0 \leq k < j-m} \ell_k t_{j-k}^{p^k} \eta_R(\widehat{\lambda}_{i-j}^{p^j}) \end{aligned}$$

Summing over all i and reindexing, we can rewrite this as

$$\sum_{i \geq 0,\, j > 0} \ell_i (\widehat{\lambda}_j^{p^i} + \widehat{t}_j^{p^i}) = \sum_{i \geq 0,\, j > 0} \ell_i \eta_R(\widehat{\lambda}_j^{p^i}) + \sum_{i \geq 0,\, j,k > 0} \ell_i \widehat{t}_j^{p^i} \eta_R(\widehat{\lambda}_k^{p^{i+j}\omega}).$$

Applying the formal exponential to both sides gives

$$\begin{aligned} \sum_{j>0}{}^F F(\widehat{\lambda}_j,\widehat{t}_j) &= \sum_{j>0}{}^F \eta_R(\widehat{\lambda}_j) +_F \sum_{j,k>0}{}^F \widehat{t}_j\eta_R(\widehat{\lambda}_k^{p^j\omega}),\\ \text{so}\qquad \sum_{j>0}{}^F \eta_R(\widehat{\lambda}_j) &= \sum_{j>0}{}^F F(\widehat{\lambda}_j,\widehat{t}_j) -_F \sum_{j,k>0}{}^F \widehat{t}_j\eta_R(\widehat{\lambda}_k^{p^j\omega}). \end{aligned}$$

We can use this to solve for $\eta_R(\widehat{\lambda}_j)$ by induction on j, and the answer lies in

$$\Gamma(m+1)\otimes_{BP_*} D^0_{m+1}.$$

A recursive formula is given by

$$\begin{aligned} &\sum_{0\le j<i} \ell_j\eta_R\left(\widehat{\lambda}_{i-j}^{p^j}\right)\\ &= \sum_{0\le j<i}\ell_j\left(\widehat{\lambda}_{i-j}^{p^j}+\widehat{t}_{i-j}^{p^j}-\sum_{0<k<i-j-m}\widehat{t}_k^{p^j}\eta_R\left(\widehat{\lambda}_{i-j-k-m}^{p^{j+k}\omega}\right)\right). \end{aligned}\tag{3.14}$$

This shows that D^0_{m+1} is a comodule over $\Gamma(m+1)$ as claimed. In particular we get

$$\eta_R(\widehat{\lambda}_i)\equiv\widehat{\lambda}_i+\widehat{t}_i \text{ mod decomposables.}\tag{3.15}$$

We will show D^0_{m+1} is a weak injective by filtering it by powers of p, analyzing the Poincaré series of each subquotient, and applying 2.6. Since D^0_{m+1} is free over $\mathbf{Z}_{(p)}$, all the subquotients are isomorphic and it suffices to analyze

$$D^0_{m+1}/(p) = A(m)/(p)[\widehat{\lambda}_1,\widehat{\lambda}_2,\ldots].$$

Note first that $\mathrm{Ext}^0(D^0_{m+1})$ contains $\mathrm{Ext}^0(BP_*) = A(m)$ as a subgroup. It follows that $\mathrm{Ext}^0(D^0_{m+1}/(p))$ contains $A(m)/(p)$ as a subgroup. The Poincaré series of $D^0_{m+1}/(p)$ is the same as that of $\Gamma(m+1)/I\otimes A(m)/(p)$, so the weak injectivity of $D^0_{m+1}/(p)$ and hence D^0_{m+1} itself will follow from showing that its Ext^0 contains no additional elements. The comodule structure of D^0_{m+1} is partially given by (3.15); a similar formula follows for each monomial in the $\widehat{\lambda}_i$, and no linear combination of them can be in Ext^0. There are no additional elements in Ext^0 and the weak injectivity of D^0_{m+1} follows. □

For future reference will need the Poincaré series of $E^1_{m+1} = D^0_{m+1}/BP_*$.

Lemma 3.16. *Let*

$$\begin{aligned} g_m(t) &= \prod_{1\le i\le m}\frac{1}{1-t^{|v_i|}}\\ \text{and}\qquad G_m(t) &= \prod_{i>m}\frac{1}{1-t^{|v_i|}}, \end{aligned}$$

the series for $A(m)/(p)$ and $\Gamma(m+1)/I$ respectively. Then the Poincaré series for $E^1_{m+1} = D^0_{m+1}/BP_*$ is

$$g_m(t)G_m(t)\sum_{i>0}\frac{t^{|\widehat{v}_i|}}{1-t^{|\widehat{v}_i|}}.$$

Proof. We define the I-adic valuation $||\cdot||$ on BP_* by setting $||p|| = ||v_i|| = 1$. It can be extended to $p^{-1}BP_*$, and by restriction to D^0_{m+1}, in an obvious way. From (3.11) we find that $||\widehat{\lambda}_i|| = 0$.

We have bigraded objects E_0BP_* and $E_0D^0_{m+1}$ which can be described by Poincaré series $\tilde{g}(s,t)$ in two variables with s corresponding to the valuation. More precisely, if M is a comodule with a decreasing filtration such that the associated bigraded object E_0M is torsion of finite type, we define

$$\tilde{g}(M) = \sum_{i,j} g_{i,j} s^i t^j$$

where $p^{g_{i,j}}$ is the order of $F^iM_j/F^{i+1}M_j$.

Thus we have

$$\begin{aligned}
\tilde{g}(A(m)) &= \prod_{0\le i\le m} \frac{1}{1-st^{|v_i|}}, \\
\tilde{g}(BP_*) &= \prod_{i\ge 0} \frac{1}{1-st^{|v_i|}} \\
&= \tilde{g}(A(m)) \prod_{i>0} \frac{1}{1-st^{|\widehat{v}_i|}} \\
\text{and} \qquad \tilde{g}(D^0_{m+1}) &= \tilde{g}(A(m)) \prod_{i>0} \frac{1}{1-t^{|\widehat{v}_i|}}.
\end{aligned}$$

It follows that

$$g(E^1_{m+1}) = \lim_{s\to 1} \left(\tilde{g}(D^0_{m+1}) - \tilde{g}(BP_*)\right),$$

which we will compute using some calculus. Thus we have

$$\begin{aligned}
g(E^1_{m+1}) &= \lim_{s\to 1} \frac{1}{1-s} \prod_{0<i\le m} \frac{1}{1-st^{|v_i|}} \left(\prod_{i>0} \frac{1}{1-t^{|\widehat{v}_i|}} - \prod_{i>0} \frac{1}{1-st^{|\widehat{v}_i|}} \right) \\
&= g_m(t) \lim_{s\to 1} \frac{1}{1-s} \left(\prod_{i>0} \frac{1}{1-t^{|\widehat{v}_i|}} - \prod_{i>0} \frac{1}{1-st^{|\widehat{v}_i|}} \right) \\
&= g_m(t) \left. \frac{\partial}{\partial s} \prod_{i>0} \frac{1}{1-st^{|\widehat{v}_i|}} \right|_{s=1} \qquad \text{by definition of } \tfrac{\partial}{\partial s} \\
&= g_m(t) \prod_{i>0} \frac{1}{1-st^{|\widehat{v}_i|}} \left. \sum_{i>0} \frac{t^{|\widehat{v}_i|}}{1-st^{|\widehat{v}_i|}} \right|_{s=1} \\
&= g_m(t) G_m(t) \sum_{i>0} \frac{t^{|\widehat{v}_i|}}{1-t^{|\widehat{v}_i|}}. \qquad \square
\end{aligned}$$

For Ext^1 we have

Theorem 3.17. *Unless $m = 0$ and $p = 2$ (which is handled in* [Rav86, 5.2.6]*), $\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*, BP_*)$ is the $A(m)$-module generated by the set*

$$\left\{ \alpha\begin{pmatrix} \widehat{v}_1^j \\ jp \end{pmatrix} : j > 0 \right\},$$

where α is the connecting homomorphism for the short exact sequence

$$0 \to BP_* \to M^0 \to N^1 \to 0$$

as in (1.6). Its Poincaré series is

$$g_m(t) \sum_{i>0} \frac{t^{p^{i-1}|\widehat{v}_1|}}{1 - t^{p^{i-1}|\widehat{v}_1|}}.$$

Moroever each of these elements is a permanent cycle.

Proof. The Ext calculation follows from [Rav86, 6.5.11] and [Rav86, 7.1.3]. For the Poincaré series, note that the set of $A(m)$-module generators of order p^i is

$$\left\{ \alpha \left(\frac{\widehat{v}_1^{jp^{i-1}}}{p^i} \right) : j > 0 \right\},$$

and its Poincaré series is

$$\frac{t^{p^{i-1}|\widehat{v}_1|}}{1 - t^{p^{i-1}|\widehat{v}_1|}}.$$

To show that each of these elements is a permanent cycle, we will study the Bockstein spectral sequence converging to $\pi_*(T(m))$ with

$$E_1 = \mathbf{Z}/(p)[v_0] \otimes \pi_*(V(0) \wedge T(m)).$$

$V(0) \wedge T(m)$ is a ring spectrum in all cases except $m = 0$ and $p = 2$. We know that $T(m)$ is a ring spectrum for all m and p and that $V(0)$ is one for p odd. The case $p = 2$ and $m > 0$ will be dealt with in Lemma 3.18 below.

Low dimensional calculations reveal that $\widehat{v}_1 \in \mathrm{Ext}^0(BP_*/p)$ is a homotopy element. The elements $\widehat{\alpha}_j = \frac{\widehat{v}_1^j}{p}$ can then be constructed in the usual way using the self-map of $V(0) \wedge T(m)$ inducing multiplictation by $\widehat{v}_1^j$ followed by the pinch map

$$V(0) \wedge T(m) \to \Sigma T(m).$$

In the Bockstein spectral sequence it follows that $\widehat{v}_1^{sp^i}$ survives to E_{i+1}, so $\widehat{\alpha}_{sp^i}$ is divisible (as a homotopy element) by p^i. □

Lemma 3.18. *For $p = 2$ and $m > 0$, $V(0) \wedge T(m)$ is a ring spectrum.*

Proof. For $m = 1$ we make use of the ring spectrum $y(1)$ introduced in [MRS01]. It is the Thom spectrum associated with the composite

$$\Omega S^2 \to BO$$

extending the nontrivial map $S^1 \to BO$. By precomposing with the loops on the Hopf map

$$\Omega SU(2) = \Omega S^3 \to \Omega S^2$$

we get a map $f : T(1) \to y(1)$ inducing a monomorphism in mod 2 homology. We claim that

$$y(1) \cong V(0) \wedge T(1).$$

We have an inclusion $i : V(0) \to y(1)$, and the composite

$$V(0) \wedge T(1) \xrightarrow{i \wedge f} y(1) \wedge y(1) \xrightarrow{\mu} y(1)$$

(where μ is the multiplication on $y(1)$) is the desired homotopy equivalence.

For $m > 1$ we consider the ring spectrum $T(m) \wedge y(1)$ and show that it has $V(0) \wedge T(m)$ as a summand. We know that

$$T(m) \wedge T(1) \cong \vee_{i \geq 0} \Sigma^{2i} T(m)$$

so

$$T(m) \wedge Y(1) \cong T(m) \wedge T(1) \wedge V(0) \cong \vee_{i \geq 0} \Sigma^{2i} T(m) \wedge V(0),$$

so $T(m) \wedge V(0)$ is a ring spectrum. □

Lemma 3.19. *(i) Let $M^1 = v_1^{-1} BP_*/(p^\infty)$ be the chromatic module of (1.6), regarded as a comodule over $\Gamma(m+1)$. Then for $m > 0$ it is weak injective over $\Gamma(m+1)$ with Ext^0 being the $v_1^{-1} A(m)$-module generated by the chromatic fractions (1.7)*

$$\left\{ \frac{1}{p^k} : k > 0 \right\} \cup \left\{ \frac{\widehat{v}_1^i}{ip} : i > 0 \right\}.$$

For $m = 0$ and p odd we have $\mathrm{Ext}^1_{\Gamma(1)}(M^1) = \mathbf{Q}/\mathbf{Z}$ concentrated in degree 0, so M^1 is not weak injective. Its $\mathrm{Ext}^0_{\gamma(1)}$ is the $\mathbf{Z}_{(p)}$-module generated by

$$\left\{ \frac{1}{p^k} : k > 0 \right\} \cup \left\{ \frac{v_1^i}{ip} : i \neq 0 \right\}.$$

(ii) For $m > 0$, the $\Gamma(m+1)$-comodule $v_1^{-1} E^1_{m+1}$ is a weak injective with

$$\mathrm{Ext}^0_{\Gamma(m+1)}(v_1^{-1} E^1_{m+1}) = v_1^{-1} \mathrm{Ext}^1_{\Gamma(m+1)}(BP_*)$$

and there is a short exact sequence

$$0 \to \mathrm{Ext}^1(BP_*)/(v_1^\infty) \to \mathrm{Ext}^0(E^1_{m+1}/(v_1^\infty)) \to \mathrm{Ext}^2(BP_*) \to 0.$$

Proof. All Ext groups here are understood to be over $\Gamma(m+1)$.

(i) We compute $\mathrm{Ext}(M^1)$ by the usual chromatic method using the Bockstein spectral sequence assocated with the short exact sequence

$$0 \to M_1^0 \to M^1 \xrightarrow{p} M^1 \to 0.$$

Unless $m = 0$ and $p = 2$ we have

$$\mathrm{Ext}(M_1^0) = v_1^{-1} A(m+1)/(p) \otimes E(\widehat{h}_{1,0}).$$

For $m > 0$ we have

$$d\left(\frac{\widehat{v}_1^j}{jp^2} \right) = \frac{\widehat{v}_1^{j-1} \widehat{t}_1}{p} \qquad \text{for each } j > 0.$$

It follows that $\mathrm{Ext}(M^1)$ is as claimed.

(ii) For $m > 0$, multiplication by v_1 is a $\Gamma(m+1)$-comodule map. It follows that $v_1^{-1} BP_*$ is a comodule which is flat as a BP_*-module. This means that inverting v_1 preverves exactness and hence weak injectivity. Inverting v_1 in the short exact sequence

$$0 \to BP_* \to M^0 \to N^1 \to 0$$

gives

$$0 \to v_1^{-1} BP_* \to v_1^{-1} M^0 \to M^1 \to 0. \tag{3.20}$$

$v_1^{-1}M^0$ is weak injective since M^0 is, with

$$\mathrm{Ext}^0(v_1^{-1}M^0) = p^{-1}v_1^{-1}A(m).$$

From (i) we see that the long exact sequence of Ext groups associated with (3.20) has the form

$$0 \to \mathrm{Ext}^0(v_1^{-1}BP_*) \to \mathrm{Ext}^0(v_1^{-1}M^0) \to \mathrm{Ext}^0(M^1) \to \mathrm{Ext}^1(v_1^{-1}BP_*) \to 0,$$

and that it leads to

$$\mathrm{Ext}^s_{\Gamma(m+1)}(v_1^{-1}BP_*) = \begin{cases} v_1^{-1}A(m) & \text{for } s = 0 \\ v_1^{-1}\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*) & \text{for } s = 1 \\ 0 & \text{for } s > 1. \end{cases}$$

Now consider the short exact sequence

$$0 \to v_1^{-1}BP_* \to v_1^{-1}D^0_{m+1} \to v_1^{-1}E^1_{m+1} \to 0.$$

Since $v_1^{-1}D^0_{m+1}$ is weak injective, the long exact sequence of Ext groups reduces to pair of isomorphisms, one of which is

$$\mathrm{Ext}^0(v_1^{-1}E^1_{m+1}) = \mathrm{Ext}^1(v_1^{-1}BP_*) = v_1^{-1}\mathrm{Ext}^1(BP_*).$$

For $\mathrm{Ext}^0(E^1_{m+1}/(v_1^\infty))$ consider the short exact sequence

$$0 \to E^1_{m+1} \to v_1^{-1}E^1_{m+1} \to E^1_{m+1}/(v_1^\infty) \to 0$$

The long exact sequence of Ext groups reduces (since $v_1^{-1}E^1_{m+1}$ is weak injective) to the 4-term sequence leading to the desired short exact sequence. □

4. The comodule E^2_{m+1}

In this section we will describe the comodule E^2_{m+1} needed above in (3.2) below dimension $p^2|\widehat{v}_1|$. For $m = 0$ and p odd we can construct D^1_1 in all dimensions directly as follows.

Lemma 4.1. *For p odd there is a map $E^1_1 \to D^1_1$ to a weak injective inducing an isomorphism in* Ext^0.

Proof. As noted in Lemma 3.19, $M^1 = v_1^{-1}E^1_1$ is not a weak injective for $m = 0$.

We will construct D^1_1 as a union of submodules of M^1 as follows. Let $K_0 = E^1_1 \subset M^1$. For each $i \geq 0$ we will construct a diagram

$$\begin{tikzcd}
 & L_{i+1} \arrow[r, equal] & L_{i+1} \\
K_i \arrow[r] & M^1 \arrow[r] \arrow[u] & L_i \arrow[u] \\
K_i \arrow[r] \arrow[u, equal] & K_{i+1} \arrow[r] \arrow[u] & L_i' \arrow[u]
\end{tikzcd}$$

in which each row and column is exact. L_i' will be the sub-BP_*-module of $L_i = M^1/K_i$ generated by the positive dimensional part of $\mathrm{Ext}^0(L_i)$. It is a subcomodule of L_i, K_{i+1} is defined to be the induced extension by K_i, and $L_{i+1} = M^1/K_{i+1}$. Hence K_i, K_{i+1}, and L_i' are connective while L_i and L_{i+1} are not.

We know that in positive dimensions $K_0 = E_1^1$ has the same Ext^0 as M^1. We will show by induction that the same is true for each K_i. In the long exact sequence of Ext groups associated with the right column, the map $\mathrm{Ext}^0(L_i') \to \mathrm{Ext}^0(L_i)$ is an isomorphism in positive dimensions, so the positive dimensional part of $\mathrm{Ext}^0(L_{i+1})$ is contained in $\mathrm{Ext}^1(L_i')$, which has higher connectivity than $\mathrm{Ext}^0(L_i)$.

It follows that the connectivity of L_i' increases with i, and therefore the limit K_∞ has finite type. The connectivity of the positive dimensional part of $\mathrm{Ext}^0(L_i)$ also increases with i, so $\mathrm{Ext}^0(L_\infty)$ is trivial in positive dimensions. From the long exact sequence of Ext groups for the short exact sequence

$$0 \to K_\infty \to M^1 \to L_\infty \to 0$$

we deduce that $\mathrm{Ext}^1(K_\infty) = 0$, so K_∞ is a weak injective by Lemma 2.1. It has the same Ext^0 as E_1^1, so it is our D_1^1. □

Now we are ready to study the hypothetical comodule E_{m+1}^2 of (3.1) for $m > 0$.

Lemma 4.2. *The Poincaré series for E_{m+1}^2 is at least*

$$g_m(t)G_m(t)\sum_{i>0}\frac{t^{p^i|\widehat{v}_1|}(1-t^{|v_i|})}{(1-t^{p^i|\widehat{v}_1|})(1-t^{|\widehat{v}_{i+1}|})}$$

(where $g_m(t)$ and $G_m(t)$ are as in Lemma 3.16) and in dimensions less than $p^2|\widehat{v}_1|$ this simplifies to

$$g_{m+2}(t)\left(\frac{t^{p|\widehat{v}_1|}(1-t^{|v_1|})}{(1-t^{|\widehat{v}_2|})(1-t^{p|\widehat{v}_1|})}\right).$$

A routine computation shows that the ith term in this series can be rewritten as

$$\sum_{1\le j\le i} g\left(BP_*/\left(p, v_1^{p^{i-j}}\right)\right)\frac{t^{p^{i-j+1}|\widehat{v}_j|}}{\left(1-t^{p^{i-j+1}|\widehat{v}_j|}\right)\left(1-t^{p^{i-j}|\widehat{v}_{j+1}|}\right)}.$$

The (i,j)th term of this sum corresponds roughly to

$$BP_*/\left(p, v_1^{p^{i-j}}\right)\left[\frac{\widehat{v}_{j+1}^{p^{i-j}}}{v_1^{p^{i-j}}}, \frac{\widehat{v}_{j+1}^{p^{i-j}}}{p}\right]\left(\frac{\widehat{v}_{j+1}^{p^{i-j}}}{p^{i+1}v_1^{p^{i-j}}} + \dots\right)$$

Proof of Lemma 4.2. The relevant Poincaré series (excluding the case $m=0$ and $p=2$) are

$$\begin{aligned} g(E^1_{m+1}) &= g_m(t)G_m(t)\sum_{i>0}\frac{t^{|\widehat{v}_i|}}{(1-t^{|\widehat{v}_i|})} \qquad \text{by 3.16}\\ &= g_m(t)G_m(t)\left(\frac{t^{|\widehat{v}_1|}}{1-t^{|\widehat{v}_1|}}+\sum_{i>0}\frac{t^{|\widehat{v}_{i+1}|}}{1-t^{|\widehat{v}_{i+1}|}}\right),\\ \text{and}\quad g(\mathrm{Ext}^0(E^1_{m+1})) &= g(\mathrm{Ext}^1(BP_*))\\ &= g_m(t)\sum_{i>0}\frac{t^{p^{i-1}|\widehat{v}_1|}}{1-t^{p^{i-1}|\widehat{v}_1|}} \qquad \text{by 3.17}\\ &= g_m(t)\left(\frac{t^{|\widehat{v}_1|}}{1-t^{|\widehat{v}_1|}}+\sum_{i>0}\frac{t^{p^i|\widehat{v}_1|}}{1-t^{p^i|\widehat{v}_1|}}\right). \end{aligned}$$

If there were a map $E^1_{m+1}\to D^1_{m+1}$ to a weak injective inducing an isomorphism in Ext^0, we would have

$$\begin{aligned} g(D^1_{m+1}) &= G_m(t)g(\mathrm{Ext}^0(E^1_{m+1})) \qquad \text{by 2.6}\\ &= G_m(t)g(\mathrm{Ext}^1(BP_*))\\ &= g_m(t)G_m(t)\left(\frac{t^{|\widehat{v}_1|}}{1-t^{|\widehat{v}_1|}}+\sum_{i>0}\frac{t^{p^i|\widehat{v}_1|}}{1-t^{p^i|\widehat{v}_1|}}\right). \end{aligned}$$

It follows that

$$\begin{aligned} g(E^2_{m+1}) &\geq g_m(t)G_m(t)\left(\frac{t^{|\widehat{v}_1|}}{1-t^{|\widehat{v}_1|}}+\sum_{i>0}\frac{t^{p^i|\widehat{v}_1|}}{1-t^{p^i|\widehat{v}_1|}}\right)-g(E^1_{m+1})\\ &= g_m(t)G_m(t)\sum_{i>0}\left(\frac{t^{p^i|\widehat{v}_1|}}{1-t^{p^i|\widehat{v}_1|}}-\frac{t^{|\widehat{v}_{i+1}|}}{1-t^{|\widehat{v}_{i+1}|}}\right)\\ &= g_m(t)G_m(t)\sum_{i>0}\frac{t^{p^i|\widehat{v}_1|}(1-t^{|v_i|})}{(1-t^{p^i|\widehat{v}_1|})(1-t^{|\widehat{v}_{i+1}|})}. \end{aligned}$$

In our range of dimensions we can replace $g_m(t)G_m(t)$ by $g_{m+2}(t)$, and only the first term of the last sum is relevant. Hence we have

$$g(E^2_{m+1}) \equiv g_{m+2}(t)\left(\frac{t^{p|\widehat{v}_1|}(1-t^{|v_1|})}{(1-t^{|\widehat{v}_2|})(1-t^{p|\widehat{v}_1|})}\right) \qquad \mathrm{mod}\ (t^{p^2|\widehat{v}_1|}). \quad \square$$

Corollary 4.3. *For a subcomodule $E\subset E^1_{m+1}/(v_1^\infty)$, let D denote the induced (as in (3.2)) extension by E^1_{m+1} and let K denote the kernel of the connecting homomorphism*

$$\delta:\mathrm{Ext}^0(E)\to\mathrm{Ext}^1(E^1_{m+1})=\mathrm{Ext}^2(BP_*).$$

Then D is weak injective if and only if the Poincaré series $g(E)$ is $g(K)G_m(t)$ plus the series specified in Lemma 4.2. In particular it is weak injective if δ is a monomorphism and $g(E)$ is the specifed series.

Proof. The specified series is $G_m(t)g(\mathrm{Ext}^0(E^1_{m+1})) - g(E^1_{m+1})$, and

$$g(\mathrm{Ext}^0(D)) = g(\mathrm{Ext}^0(E^1_{m+1})) + g(K).$$

Hence our hypothesis implies

$$\begin{aligned} g(D) &= g(E^1_{m+1}) + g(E) \\ &= g(E^1_{m+1}) + G_m(t)g(\mathrm{Ext}^0(E^1_{m+1})) - g(E^1_{m+1}) + g(K)G_m(t) \\ &= G_m(t)(g(\mathrm{Ext}^0(E^1_{m+1}) + g(K)) \\ &= G_m(t)g(\mathrm{Ext}^0(D)), \end{aligned}$$

which is equivalent to the weak injectivity of D by Theorem 2.6. □

Now we need to identify some elements in $E^1_{m+1}/(v_1^\infty)$.

Lemma 4.4. *The comodule $E^1_{m+1}/(v_1^\infty)$ contains the elements*

(i)
$$\frac{\widehat{v}_1^{1+e_0}}{p^{1+e_0}v_1^{1+e_1}} \qquad \text{for all } e_0, e_1 \geq 0$$

(ii)
$$\frac{\widehat{v}_2^{1+e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}} \qquad \text{for all } e_0, e_1 \geq 0.$$

These generators will be discussed further in Theorem 4.5 below.

Proof. (i) The element in question is the image of $v_1^{-1-e_1}\widehat{\lambda}_1^{1+e_0}$.

(ii) In D^0_{m+1} we have

$$\begin{aligned} \widehat{\lambda}_2 &= \widehat{\ell}_2 - \ell_1\widehat{\lambda}_1^p \\ &= \frac{\widehat{v}_2}{p} + \frac{v_1\widehat{v}_1^p}{p^2} + \frac{v_1^{p\omega}\widehat{v}_1}{p^2} - \frac{v_1\widehat{\lambda}_1^p}{p} \\ &= \frac{\widehat{v}_2}{p} + \frac{v_1}{p}\left(p^{p-1}\widehat{\lambda}_1^p + v_1^{p\omega-1}\widehat{\lambda}_1 - \widehat{\lambda}_1^p\right) \\ \text{so} \quad \frac{\widehat{v}_2}{p} &= \widehat{\lambda}_2 + \frac{v_1}{p}\mu \end{aligned}$$

where $\mu = \widehat{\lambda}_1^p(1-p^{p-1}) - v_1^{p\omega-1}\widehat{\lambda}_1 \in D^0_{m+1}$.

Hence in $p^{-1}v_1^{-1}BP_*$ we have

$$\begin{aligned} \frac{\widehat{v}_2^{1+e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}} &= \frac{p^{e_1}}{v_1^{1+e_1}}\left(\frac{\widehat{v}_2}{p}\right)^{1+e_0+e_1} \\ &= \frac{p^{e_1}}{v_1^{1+e_1}}\left(\widehat{\lambda}_2 + \frac{v_1}{p}\mu\right)^{1+e_0+e_1} \\ &= \frac{p^{e_1}}{v_1^{1+e_1}}\sum_{k\geq 0}\binom{1+e_0+e_1}{k}\widehat{\lambda}_2^{1+e_0+e_1-k}\frac{v_1^k}{p^k}\mu^k \\ &= \sum_{k\geq 0}\binom{1+e_0+e_1}{k}\frac{p^{e_1-k}}{v_1^{1+e_1-k}}\widehat{\lambda}_2^{1+e_0+e_1-k}\mu^k. \end{aligned}$$

The image of this element in $p^{-1}BP_*/(v_1^\infty)$ is

$$\sum_{0\le k\le e_1}\binom{1+e_0+e_1}{k}\frac{p^{e_1-k}}{v_1^{1+e_1-k}}\widehat{\lambda}_2^{1+e_0+e_1-k}\mu^k.$$

The coefficient of each term is an integer, so the expression lies in $D^0_{m+1}/(v_1^\infty)$, and its image in $E^1_{m+1}/(v_1^\infty)$ is the desired element.

□

We will now construct a comodule $E^2_{m+1}\subset E^1_{m+1}/(v_1^\infty)$ satisfying the conditions of Corollary 4.3 with δ monomorphic below dimension $p^2|\widehat{v}_1|$.

Theorem 4.5. *Let $E^2_{m+1}\subset E^1_{m+1}/(v_1^\infty)$ be the $A(m+2)$-module generated by the set*

$$\left\{\frac{\widehat{v}_2^{1+e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}}: e_0,e_1\ge 0\right\}.$$

Below dimension $p^2|\widehat{v}_1|$ it has the Poincaré series specified in Lemma 4.2, it is a comodule, and its Ext group is

$$A(m+1)/I_2\otimes E(\widehat{h}_{1,0})\otimes P(\widehat{b}_{1,0})\otimes\left\{\frac{\widehat{v}_2^{e_2}}{pv_1}: e_2\ge 1\right\}.$$

In particular Ext^0 *maps monomorphically to* $\mathrm{Ext}^2(BP_*)$ *in that range.*

Proof. Recall that the Poincaré series specified in Lemma 4.2 in this range is

$$g_{m+2}(t)\left(\frac{t^{p|\widehat{v}_1|}(1-t^{|v_1|})}{(1-t^{|\widehat{v}_2|})(1-t^{p|\widehat{v}_1|})}\right)=g(BP_*/I_2)\frac{t^{p|\widehat{v}_1|}}{(1-t^{|\widehat{v}_2|})(1-t^{p|\widehat{v}_1|})}.$$

Each generator of E^2_{m+1} can be written as

$$x_{e_0,e_1}=\frac{\widehat{v}_2^{1+e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}}=\frac{\widehat{v}_2}{pv_1}\left(\frac{\widehat{v}_2}{p}\right)^{e_0}\left(\frac{\widehat{v}_2}{v_1}\right)^{e_1}$$

with $e_0,e_1\ge 0$. Since $|\frac{\widehat{v}_2}{pv_1}|=p|\widehat{v}_1|$, the Poincaré series for this set of generators is

$$\frac{t^{p|\widehat{v}_1|}}{(1-t^{|\widehat{v}_2|})(1-t^{p|\widehat{v}_1|})}.$$

We can filter E^2_{m+1} by defining F_i to be the submodule generated by the x_{e_0,e_1} with $e_0+e_1\le i$. Then each subquotient is a direct sum of suspensions of BP_*/I_2, so the Poincaré series is as claimed.

To see that E^2_{m+1} is a comodule, we will use the I-adic valuation as defined in the proof of Lemma 3.16. In our our range the set of elements with valuation at least -1 is the $A(m)$-submodule M generated by

$$\left\{\frac{\widehat{v}_1^i\widehat{v}_2^j}{p^{1+e_0}v_1^{1+e_1}}: e_0,e_1\ge 0,\, i+j\ge 1+e_0+e_1\right\},$$

while E^2_{m+1} is generated by a similar set with $j\ge 1+e_0+e_1$. Thus it suffices to show that the decreasing filtration on M defined by letting F^kM be the submodule generated by all such generators with $j-e_0-e_1\ge k$ is a comodule filtration. For this observe that modulo $\Gamma(m+1)\otimes F^{1+j-e_0-e_1}M$ we have

$$\frac{\eta_R(\widehat{v}_1^i\widehat{v}_2^j)}{p^{1+e_0}v_1^{1+e_1}}\quad\equiv\quad\frac{\widehat{v}_1^i(\widehat{v}_2+v_1\widehat{t}_1^p+p\widehat{t}_2)^j}{p^{1+e_0}v_1^{1+e_1}}\in\Gamma(m+1)\otimes F^{j-e_0-e_1}M,$$

so $E^2_{m+1} = F^1M$ is a subcomodule.

We use the same filtration for the Ext computation. Assuming that $j \geq 1 + e_0 + e_1 > 1$ we have

$$\begin{aligned}
\frac{\eta_R(\widehat{v}_1^i\widehat{v}_2^j) - \widehat{v}_1^i\widehat{v}_2^j}{p^{1+e_0}v_1^{1+e_1}} &\equiv \frac{\widehat{v}_1^i(\widehat{v}_2 + v_1\widehat{t}_1^p + p\widehat{t}_2)^j - \widehat{v}_1^i\widehat{v}_2^j}{p^{1+e_0}v_1^{1+e_1}} \\
&\equiv \binom{j}{e_0+e_1}\frac{\widehat{v}_1^i\widehat{v}_2^{j-e_0-e_1}(v_1\widehat{t}_1^p + p\widehat{t}_2)^{e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}} + \dots \\
&\equiv (e_0, e_1, j-e_0-e_1)\frac{\widehat{v}_1^i\widehat{v}_2^{j-e_0-e_1}\widehat{t}_1^{pe_1}\widehat{t}_2^{e_0}}{pv_1} + \dots
\end{aligned}$$

where the missing terms involve higher powers of $\widehat{v}_2$. The multinomial coefficient $(e_0, e_1, j-e_0-e_1)$ is always nonzero since $j < p$. This means no linear combination of such elements is invariant, and the only invariant generators are the ones with $e_0 = e_1 = 0$, so Ext^0 is as claimed.

We will use this to show that E^2_{m+1} is 1-free (as defined in 1.11), i.e., that $E^2_{m+1} \otimes_{BP_*} T_m^{p-1}$ is weak injective in this range. For $0 \leq k \leq p-1$ we have

$$\frac{\psi(\widehat{v}_1^i\widehat{v}_2^j\widehat{t}_1^k) - \widehat{v}_1^i\widehat{v}_2^j\widehat{t}_1^k}{p^{1+e_0}v_1^{1+e_1}} = (e_0, e_1, j-e_0-e_1)\widehat{t}_1^{pe_1+k}\widehat{t}_2^{e_0} \otimes \frac{\widehat{v}_1^i\widehat{v}_2^{j-e_0-e_1}}{pv_1} + \dots.$$

This means that

$$\mathrm{Ext}^0(E^2_{m+1} \otimes_{BP_*} T_m^{p-1}) = \mathrm{Ext}^0(E^2_{m+1}).$$

It follows that

$$\begin{aligned}
g(\mathrm{Ext}^0) &= g_{m+1}(t)(1-t^{|v_1|})\frac{t^{p|\widehat{v}_1|}}{1-t^{|\widehat{v}_2|}} \\
\text{so} \qquad g(E^2_{m+1}) &= g(\mathrm{Ext}^0)\frac{1}{(1-t^{p|\widehat{v}_1|})(1-t^{|\widehat{v}_2|})}, \\
\text{and} \qquad g(E^2_{m+1} \otimes_{BP_*} T_m^{p-1}) &= g(\mathrm{Ext}^0)\frac{1}{(1-t^{|\widehat{v}_1|})(1-t^{|\widehat{v}_2|})} \\
&= g(\mathrm{Ext}^0)G_m(t)
\end{aligned}$$

This makes $E^2_{m+1} \otimes_{BP_*} T_m^{p-1}$ weak injective in this range by Theorem 2.6.

We can use the small descent spectral sequence of Theorem 1.17 to pass from $\mathrm{Ext}(E^2_{m+1} \otimes_{BP_*} T_m^{p-1})$ to $\mathrm{Ext}(E^2_{m+1})$. It collapses from E_1 since the two comodules have the same Ext^0. This means that $\mathrm{Ext}(E^2_{m+1})$ is as claimed.

The statement about $\mathrm{Ext}^2(BP_*)$ is proved in Lemma 4.6 below. □

Lemma 4.6. *The group* $\mathrm{Ext}^0(E^2_{m+1})$ *specified in Theorem 4.5 maps monomorphically to* $\mathrm{Ext}^2(BP_*)$.

Proof. The chromatic method tells us that $\mathrm{Ext}^2(BP_*)$ is a certain subquotient of $\mathrm{Ext}^0(M^2)$, namely the kernel of the map to $\mathrm{Ext}^0(M^3)$ modulo the image of the map from $\mathrm{Ext}^0(M^1)$. We know that the latter is the $A(m)$-module generated by the elements $\frac{\widehat{v}_1^i}{pi}$, and the elements in question, the $A(m+1)$ multiples of $\frac{\widehat{v}_2^i}{pv_1}$ are not in the image. The latter map trivially to $\mathrm{Ext}^0(M^3)$ because they involve no negative powers of v_2. □

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