A NEW ANGLE ON THE STABLE HOMOTOPY GROUPS OF SPHERES

JOINT WORK WITH MIKE HILL AND MIKE HOPKINS

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1. Brief introduction to a lengthy subject

Determining the stable homotopy groups of spheres has been a vexing and fascinating problem in algebraic topology for the past 70 years. Recall

- $\pi_{n+k}(S^n)$ is defined to be the set of homotopy classes of maps from S^{n+k} (the unit sphere in $\mathbf{R}^{n+k+1}$) to S^n. The set has a natural abelian group structure.
- $\pi_{n+k}(S^n)$ is known to be independent of n for $n > k+1$. We denote this group by π_k^S or $\pi_k(S^0)$.
- Here are its values for small k.

k	0	1	2	3	4	5	6	7	8	9
π_k^S	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/24$	0	0	$\mathbf{Z}/2$	$\mathbf{Z}/240$	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^3$

- π_k^S is known to be finite for $k > 0$.
- Elements of arbitrarily large order are known to occur for large k.
- The p-component of π_k^S is known for
  - $k \leq 60$ for $p = 2$
  - $k \leq 100$ for $p = 3$
  - $k \leq 1000$ for $p = 5$

Many more details can be found in [Rav86].

2. Chromatic theory

Elaborate algebraic machinery has been developed for studying this problem. It involves homological algebra and ever increasing amounts of algebraic geometry and algebraic number theory.

The values of k mentioned above have not changed in the past 20 years. Research has focused instead on understanding the overall structure of the groups and of the stable homotopy category.

Forty years ago there was very little one could have said or even guessed about this overall structure. Now we have the *chromatic approach* to stable homotopy theory. Roughly speaking it says that, after localizing at a prime p, the problem can be broken up into various "layers," one for each nonnegative integer n, which can be analyzed separately. *Each of them can be completely determined with a finite amount of work.*

These layers can be thought of in at least two different ways:

- Thinking of π_*^S as a complicated function, the first n layers can be assembled into an "nth order approximation" to π_*^S, similar to the first n terms in a power series.
- Thinking of π_*^S as a complicated radio signal, the chromatic layers can be thought of as messages being broadcast at various frequencies. They can be decoded separately. Each layer is said to be *monochromatic*, meaning that its information is all on the same frequency.

Many more details can be found in [Rav92].

3. The Morava stabilizer group

The nth layer in the chromatic filtration is the *Bousfield localization with respect to the nth Morava K-theory*, denoted by $L_{K(n)}S^0$. It has been known since the late '70s that its structure is controlled by the continuous cohomology of a certain profinite group $\mathbb{S}_n$ called the *nth Morava stabilizer group.* It is the automorphism group of a certain 1-dimensional formal group law and can be described explicitly in terms of a certain division algebra over the p-adic numbers.

Here are some of its properties.

- $\mathbb{S}_0$ is the trivial group.
- For $n > 0$, $\mathbb{S}_n$ is an extension of a pro-p-group by $\mathbf{F}_{p^n}^\times$, the group of units in the field $\mathbf{F}_{p^n}$, which is cyclic of order $p^n - 1$.
- $\mathbb{S}_1$ is $\mathbf{Z}_p^\times$, the group of units in the p-adic integers.
- For $n > 1$, $\mathbb{S}_n$ and its pro-p-subgroup are nonabelian.
- $\mathbb{S}_n$ is p-torsion free unless $p - 1$ divides n.
- $\mathbb{S}_n$ has an element of order p^k iff $(p-1)p^{k-1}$ divides n.
- $\mathbb{S}_n$ has virtual cohomological dimension n^2.
- When $p - 1$ divides n, the cohomology ring of $\mathbb{S}_n$ has Krull dimension one. Equivalently, every elementary abelian subgroup of $\mathbb{S}_n$ has oprder p.
- The finite subgroups of $\mathbb{S}_n$ have been determined by Hewett[Hew95].

4. The Hopkins-Miller theorem

The relation between the Morava stabilizer group $\mathbb{S}_n$ and the nth chromatic layer S_n became much more precise with the advent of the Hopkins-Miller theorem in the early '90s. It concerns the action of $\mathbb{S}_n$ on a certain spectrum called E_n, usually referred to as Morava E-theory. Its homotopy groups are explicitly known and easy to describe. Prior to their work we knew of an $\mathbb{S}_n$-action on it *defined only up to homotopy.*

Theorem 1. *[Hopkins-Miller 1992, unpublished]. The action of $\mathbb{S}_n$ on E_n is such that for any closed subgroup $G \subset \mathbb{S}_n$, there is a homotopy fixed point set which we will denote by $EO_n(G)$ with the following properties:*

(i) *For $G = \mathbb{S}_n$, it is $L_{K(n)}S^0$.*

(ii) *It is contravariantly natural in G, i.e., given subgroups*
$$G_1 \subset G_2 \subset \mathbb{S}_n$$
there is a restriction map $EO_n(G_2) \to EO_n(G_1)$. If G_1 has finite index in G_2, then there is a transfer map going the other way.

(iii) *There is a fixed point spectral sequence (also natural in G) of the form*
$$H^*(G; \pi_*(E_n)) \implies \pi_*(EO_n(G))$$
which coincides with the Adams-Novikov spectral sequence for $\pi_*(EO_n(G))$.

The problem with using this result has been the difficulty of explicitly describing the action of $\mathbb{S}_n$ on $\pi_*(E_n)$.

A classical example: the case $(p, n) = (2, 1)$

The following was known long before the Hopkins-Miller theorem was proved, and is the motivation for the "O" in $EO_n(G)$.

- E_1 is the 2-adic completion of complex K-theory.
- $\mathbb{S}_1 \simeq \mathbf{Z}_2^\times$ (the 2-adic units), which is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}/2$, with $\mathbf{Z}/2 = \{\pm 1\}$.
- The action of the generator of $\mathbf{Z}/2$ is by complex conjugation.
- The fixed point set $EO_1(\mathbf{Z}/2)$ is the 2-adic completion of real K-theory, KO. The "O"here stands for "orthogonal group."
- The relation between KO and $L_{K(1)}S^0$ is well understood. See [Rav84].

5. New results

Near Theorem 1. *[HHR 2007] Let $G \subset \mathbb{S}_n$ be a finite subgroup.*

(i) *The action of G on $\pi_*(E_n)$ has a certain explicit description which enables us to compute its cohomology.*

(ii) *When the p-Sylow subgroup of G is C_p, then there are certain differentials in the Hopkins-Miller spectral sequence related to the geometry of the classifying space BC_p.*

(iii) *In this case the Hopkins-Miller spectral sequence is rigid enough to preclude any other differentials, and we can describe $\pi_*(EO_n(G))$.*

Remarks

- For $n = (p-1)f$, the order the maximal subgroup with an element of order p is a metacyclic group of order $p(p-1)(p^f-1)$.
- For $f = 1$, p odd and G as above, the spectrum $EO_{p-1}(G)$ has been studied before by Hopkins-Miller and Gorbunov-Mahowald [GM00], who denoted the spectrum simply by EO_{p-1}. The differentials in that case are closely related to ones discovered long ago by Toda; see [Tod67] and [Tod68]. The spectrum was used recntly by Nave [Nav] to prove the nonexistence of the Smith-Toda complex $V((p+1)/2)$ (see [Tod71]) for $p \geq 7$.
- For $(p,n) = (2,2)$ there are two finite subgroups of interest. One is an extension of the quaternion group by C_3. It fixed point spectrum is the $K(2)$-localization of tmf, which was originally introduced in [HM]. The other case is the abelian extension of C_2 by C_3, which yields the $K(2)$-localization of tmf(3), spectrum related to elliptic curves equipped with a point of order 3.
- For $p = 2$, let G be the maximal subgroup containing an element of order 2. It is cyclic of order $2(2^n-1)$. Then $EO_n(G)$ has been studied previously by Hu-Kriz [HK01] and Kitchloo-Wilson [KW07], who call a variant of it the "real Johnson-Wilson spectrum" $ER(n)$. They use $ER(2)$ (which is tmf(3) in [KW] to prove some nonimmersion results for real projective spaces.
- To my knowledge, no other fixed point spectra of finite groups have been studied before. If n has the form $(p-1)p^{k-1}s$ for s prime to p, then there are k maximal finite subgroups, each having p-Sylow subgroup C_{p^i} for $1 \leq i \leq k$. Their fixed point spectra form a pullback diagram which we hope to study.

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