A new angle on the stable homotopy groups of spheres

Joint work with Mike Hill and Mike Hopkins

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- Here are its values for small k.

k	0	1	2	3	4	5	6	7
π_k^S	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/24$	0	0	$\mathbf{Z}/2$	$\mathbf{Z}/240$

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- Elements of arbitrarily large order are known to occur for large k.

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Many more details can be found in [Rav86].

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The values of k mentioned above have not changed in the past 20 years. Research has focused instead on understanding the overall structure of the groups and of the stable homotopy category.

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Each of them can be completely determined with a finite amount of work.

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Many more details can be found in [Rav92].

3. The Morava stabilizer group

The nth layer in the chromatic filtration is the *Bousfield localization with respect to the nth Morava K-theory*, denoted by $L_{K(n)}S^0$.

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It is the automorphism group of a certain 1-dimensional formal group law and can be described explicitly in terms of a certain division algebra over the p-adic numbers.

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- $\mathbb{S}_1$ is $\mathbf{Z}_p^\times$, the group of units in the p-adic integers.
- For $n > 1$, $\mathbb{S}_n$ and its pro-p-subgroup are nonabelian.

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- The finite subgroups of $\mathbb{S}_n$ have been determined by Hewett[Hew95].

4. The Hopkins-Miller theorem

The relation between the Morava stabilizer group $\mathbb{S}_n$ and the nth chromatic layer S_n became much more precise with the advent of the Hopkins-Miller theorem in the early '90s. It concerns the action of $\mathbb{S}_n$ on a certain spectrum called E_n, usually referred to as Morava E-theory.

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Its homotopy groups are explicitly known and easy to describe.

Prior to their work we knew of an $\mathbb{S}_n$-action on it *defined only up to homotopy.*

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Theorem 1 *[Hopkins-Miller 1992, unpublished]. The action of* $\mathbb{S}_n$ *on* E_n *is such that for any closed subgroup* $G \subset \mathbb{S}_n$*, there is a homotopy fixed point set which we will denote by* $EO_n(G)$ *with the following properties:*

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(i) *For $G = \mathbb{S}_n$, it is $L_{K(n)}S^0$.*

(ii) *It is contravariantly natural in G, i.e., given subgroups*

$$G_1 \subset G_2 \subset \mathbb{S}_n$$

there is a restriction map $EO_n(G_2) \to EO_n(G_1)$. If G_1 has finite index in G_2, then there is a transfer map going the other way.

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(iii) *There is a fixed point spectral sequence (also natural in G) of the form*

$$H^*(G; \pi_*(E_n)) \implies \pi_*(EO_n(G))$$

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The problem here is the difficulty of explicitly describing the action of $\mathbb{S}_n$ on $\pi_*(E_n)$.

Example: $(p, n) = (2, 1)$

The following was known long before the Hopkins-Miller theorem was proved, and is the motivation for the “O” in $EO_n(G)$.

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- $\mathbb{S}_1 \simeq \mathbf{Z}_2^\times$ (the 2-adic units), which is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}/2$, with $\mathbf{Z}/2 = \{\pm 1\}$.
- The action of the generator of $\mathbf{Z}/2$ is by complex conjugation.

A classical example: the case $(p, n) = (2, 1)$

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- The fixed point set $EO_1(\mathbf{Z}/2)$ is the 2-adic completion of real K-theory, KO.
- The relation between KO and $L_{K(1)}S^0$ is well understood. See [Rav84].

5. New results

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(iii) *In this case the Hopkins-Miller spectral sequence is rigid enough to preclude any other differentials, so it is possible to describe $\pi_*(EO_n(G))$.*

Remarks

- For $n = (p-1)f$, the order the maximal subgroup with an element of order p is a metacyclic group of order $p(p-1)(p^f-1)$.

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- The differentials in that case are closely related to ones discovered long ago by Toda; see [Tod67] and [Tod68].
- The spectrum was used recently by Nave [Nav] to prove the nonexistence of the Smith-Toda complex $V((p+1)/2)$ (see [Tod71]) for $p \geq 7$.

More remarks

- For $(p, n) = (2, 2)$ there are two finite subgroups of interest. One is an extension of the quaternion group by C_3. It fixed point spectrum is the $K(2)$-localization of tmf, which was originally introduced by Hopkins-Mahowald in [HM].

More remarks

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- The other case is the abelian extension of C_2 by C_3, which yields the $K(2)$-localization of tmf(3), spectrum related to elliptic curves equipped with a point of order 3.

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Kitchloo and Wilson use $ER(2)$ (which is closely related to tmf(3)) in [KWa] to prove some new nonimmersion results for real projective spaces.

Last remark

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If n has the form $(p-1)p^{k-1}s$ for s prime to p, then there are k maximal finite subgroups, having p-Sylow subgroup C_{p^i} for $1 \leq i \leq k$.

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Their fixed point spectra form a pullback diagram which we hope to study.

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