

Suspension Spectra are Harmonic

In memory of José Adem

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The notion of a harmonic spectrum in stable homotopy theory was introduced by the second author in [Rav84]. Roughly speaking, a spectrum is harmonic if its homotopy is computable with the chromatic spectral sequence. It was shown in [Rav84, Theorem 4.4] that a large class of connective spectra (including all finite spectra) are harmonic. On the other hand, the Eilenberg-Mac Lane spectrum for a torsion group was shown to be dissonant (the opposite of harmonic), i.e., chromatic methods give no information at all about its homotopy type.

The purpose of this paper is to show that *all suspension spectra are harmonic* (Theorem 8), thereby giving an affirmative answer to a question raised in [Rav84]. This result has some interesting corollaries. One is a new proof of the Whitehead theorem for Morava K-theory, Theorem 2. (This was originally proved, although not explicitly stated by Bousfield in [Bou82].) Another (Theorem 12) is that a suspension spectrum is uniquely determined by any of its connective covers.

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Morava K-theories

For each prime p and each integer $n \geq 0$ there is a generalized homology theory $K(n)_*$ called the n^{th} Morava K-theory. These were invented by Jack

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Morava, but he never published an account of them. The definition is too complicated to give here; instead we will list some of their properties. Most of the following result is proved in [JW75]; a proof of (v) can be found in [Rav84].

Proposition 1 *For each prime p there is a sequence of homology theories $K(n)_*$ for $n \geq 0$ with the following properties. (We follow the standard practice of omitting p from the notation.)*

- (i) $K(0)_*(X) = H_*(X; \mathbf{Q})$ and $\overline{K(0)}_*(X) = 0$ when $\overline{H}_*(X; \mathbf{Z})$ is all torsion.
- (ii) $K(1)_*$ is one of $p - 1$ isomorphic summands of mod p complex K -theory.
- (iii) $K(0)_*(\text{pt.}) = \mathbf{Q}$ and for $n > 0$, $K(n)_*(\text{pt.}) = \mathbf{Z}/(p)[v_n, v_n^{-1}]$ where the dimension of v_n is $2p^n - 2$. This ring is a graded field in the sense that every graded module over it is free. $K(n)_*(X)$ is a module over $K(n)_*(\text{pt.})$.
- (iv) There is a Künneth isomorphism

$$K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*(\text{pt.})} K(n)_*(Y).$$

- (v) Let X be a finite CW-complex. If $\overline{K(n)}_*(X)$ (the reduced Morava K -theory of X) vanishes, then so does $\overline{K(n-1)}_*(X)$.
- (vi) There is an Atiyah-Hirzebruch spectral sequence converging to $K(n)_*(X)$ for any space X with

$$E_2 = H_*(X; \mathbf{Z}/(p)) \otimes K(n)_*(\text{pt.}).$$

- (vii) If X is finite then the Atiyah-Hirzebruch spectral sequence collapses for large n , giving

$$K(n)_*(X) = H_*(X; \mathbf{Z}/(p)) \otimes K(n)_*(\text{pt.}).$$

In particular if a finite complex has trivial Morava K -theory (for all n and p) then it is contractible.

- (viii) The Atiyah-Hirzebruch spectral sequence collapses if $H_*(X; \mathbf{Z})$ is torsion free.

We will give a new proof of the following.

Theorem 2 *Let $f : X \rightarrow Y$ be a map between simply connected spaces with $K(n)_*(f)$ an isomorphism for all n and p . Then f is a weak homotopy equivalence, i.e., it induces an isomorphism of homotopy groups and is an equivalence if X and Y have the homotopy type of CW-complexes.*

An interesting corollary, which is actually equivalent to the theorem, is the following.

Corollary 3 *If X is a simply connected space with*

$$K(n)_*(X) = K(n)_*(\text{pt.})$$

for all n and p , then X is weakly contractible.

In Theorem 2 and Corollary 3 we could replace $K(n)_*$ by the related theories $E(n)_*$. (This assertion can be deduced from [Rav84, 2.1(d)].) These were introduced by Johnson-Wilson in [JW73]; $E(n)$ is a *BP*-module spectrum with

$$\pi_*(E(n)) = \mathbf{Z}_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}].$$

Generalizing the Whitehead theorem

The classical Whitehead theorem [Whi49] is the following.

Theorem 4 *Let $f : X \rightarrow Y$ be a map between simply connected spaces with $H_*(f)$ an isomorphism. Then f is a weak homotopy equivalence.*

One can ask to which homology theories E_* this can be generalized. Clearly the result will *not* hold if there is noncontractible simply connected CW-complex X for which

$$E_*(X) = E_*(\text{pt.}),$$

i.e., an E_* -acyclic space X .

If E is connective (i.e., if $E_{-i}(\text{pt.}) = 0$ for $i \ll 0$) then the class of E_* -acyclic spaces depends only on certain *arithmetic* properties of $E_*(\text{pt.})$, namely for which primes p multiplication by p in $E_*(\text{pt.})$ is either an isomorphism or has a nontrivial kernel.

For nonconnective theories such as classical complex K-theory, the situation is more delicate. For example it is known that

$$K_*(K(\mathbf{Z}/(p), 2)) = K_*(\text{pt.}),$$

so there is no Whitehead theorem for K -theory.

The theory of interest in Theorem 2 is E_* where

$$E = \bigvee_p \bigvee_{n \geq 0} K(n), \tag{5}$$

the wedge of the Morava K-theories for all n and p . It has suitable arithmetic properties, but it is not connective, so Theorem 2 is not obvious.

A stable question related to generalizing the Whitehead theorem is the following. *For which spectra is it true that a map f is a weak equivalence whenever $E_*(f)$ is an isomorphism?* It is true for E_* -local spectra, which are defined as follows.

Definition 6 *A spectrum X is E_* -local if for each E_* -acyclic spectrum Y , there are no essential maps from Y to X . A space X is E_* -local if each map $f : Y_1 \rightarrow Y_2$ with $E_*(f)$ an isomorphism induces an isomorphism*

$$[Y_2, X] \xrightarrow{f^*} [Y_1, X].$$

The main theorem in the subject, which is due to Bousfield ([Bou75] and [Bou79]), says that for any theory E_* , any space or spectrum X is E_* -equivalent to one which is local. For more discussion, see [Rav84].

Definition 7 A spectrum is **harmonic** if it is E_* -local for E as in (5). If $E_*(X) = 0$ then X is **dissonant**. A space is harmonic or dissonant if its suspension spectrum is.

Then Theorem 2 is a consequence of the following result, which answers a question posed in [Rav84].

Theorem 8 Every suspension spectrum is harmonic.

Corollary 9 A simply connected dissonant space is weakly contractible.

Proof that Theorem 8 implies Theorem 2. Suppose that

$$f : X \longrightarrow Y$$

is a map between simply connected spaces with $E_*(f)$ an isomorphism. Then it follows that $E_*(C_f)$ is trivial, where C_f denotes the cofibre of f , and C_f is therefore weakly contractible by 9, so f is a weak equivalence as claimed. ■

Notice that localization (and hence the notions of harmonic and dissonant) in the category of spaces is *not the same* as in the category of spectra. In particular, the statement that every simply connected space is harmonic (in the category of spaces) is a consequence of Theorem 2, while Theorem 8 is not. We will explain.

By definition a simply connected CW-complex X is harmonic (in the category of spaces) if for every E_* -equivalence of simply connected spaces $f : Y_1 \rightarrow Y_2$, the induced map $f^* : [Y_1, X] \leftarrow [Y_2, X]$ is an isomorphism. But by Theorem 2, f is a weak equivalence, so f^* is an isomorphism.

Now suppose that X is a suspension spectrum. It is harmonic if the above holds for spectra Y_1 and Y_2 . However, we do not have a Whitehead theorem for spectra, so it is not clear that f^* is an isomorphism.

Properties of harmonic spectra

The following two results can be found in [Rav84].

Proposition 10 For any homology theory E_* ,

(i) Any homotopy inverse limit of E_* -local spectra is E_* -local.

(ii) If

$$W \longrightarrow X \longrightarrow Y \longrightarrow \Sigma W$$

is a cofibre sequence and any two of W , X and Y are E_* -local, then so is the third.

(iii) If $X \vee Y$ is E_* -local, then so are X and Y .

On the other hand, a homotopy direct limit of local spectra need not be local. We will use three different inverse limit constructions in this paper: an Adams resolution in the proof of Lemma 14, a Postnikov tower in the proof of Theorem 8, and a geometric construction of the Eilenberg-Moore spectral sequence in the proof of Lemma 17.

Proposition 11 *The mod p Eilenberg-Mac Lane spectrum H/p is dissonant. More generally, if $\pi_*(X)$ is bounded above and torsion, then X is dissonant.*

Given this result, Theorem 8 has the following consequence.

Theorem 12 *Let X be a suspension spectrum with $\pi_*(X) \otimes \mathbf{Q} = 0$. Let $X\langle m \rangle$ denote the $(m-1)$ -connected cover of X (as a spectrum), i.e., there is a map $X\langle m \rangle \rightarrow X$ inducing an isomorphism in stable homotopy in dimensions m and above. Then the homotopy type of X is uniquely determined by that of $X\langle m \rangle$.*

Proof. Assume for simplicity that X is p -local. The general case will follow by an arithmetic square argument which we leave to the reader.

The fibre F of the map $X\langle m \rangle \rightarrow X$ has homotopy that is all torsion and is bounded above. Thus F is dissonant by 11. It follows that $X\langle m \rangle$ and X have the same E_* -localization (for E as in (5)), namely X , so X is the E_* -localization of $X\langle m \rangle$. ■

Theorem 13 *If X is a connective spectrum such that $H_*(X_{(p)}; \mathbf{Z})$ is a free $\mathbf{Z}_{(p)}$ -module for each prime p , then it is harmonic.*

If we add a finiteness hypothesis, then this becomes a special case of a more general result ([Rav84, 4.4]) which implies that all finite complexes are harmonic.

We will give the proof of Theorem 13 below. It will depend on the following two lemmas.

Lemma 14 *A connective p -local spectrum X is harmonic if $BP \wedge X$ is harmonic, where BP denotes the Brown-Peterson spectrum.*

This lemma in turn depends on the following.

Lemma 15 *Let W be a (possibly infinite) wedge of spheres. Then $BP \wedge W$ is harmonic.*

Proof of Theorem 13. It suffices to prove that $X_{(p)}$ is harmonic for each prime p . If X is as in Theorem 13, then $BP \wedge X_{(p)}$ is a wedge of suspensions of BP , and hence harmonic by Lemma 15. Then $X_{(p)}$ is harmonic by Lemma 14, and Theorem 13 follows. ■

Proof of Lemma 15. In [Rav84, Theorem 4.2] it is shown by direct calculation that BP is harmonic. We need to know the same for an arbitrary wedge of suspensions of BP , but the proof given there does not obviously generalize to this case. Instead, we argue as follows.

By [Rav84, Theorem 2.1] and [Rav87, Lemma 3] a spectrum is $L_n BP_*$ -local if and only if it is local with respect to $\bigvee_{0 \leq i \leq n} K(i)$. In particular such a spectrum is harmonic. Hence by [Rav84, Prop. 1.24] (which says that an E -module spectrum is E_* -local for a ring spectrum E), the spectrum

$$X_n = W \wedge L_n BP$$

is harmonic for each n . It follows that $\lim_{\leftarrow} X_n$ is also harmonic. We claim that this inverse limit is equivalent to $W \wedge BP$, so the latter is harmonic as desired. We have

$$\pi_*(X_n) = H_*(W) \otimes \pi_*(L_n BP)$$

and $\pi_*(L_n BP)$ was computed in [Rav84]. The maps in the inverse system respect this direct sum decomposition, so we have

$$\begin{aligned} \pi_*(\lim_{\leftarrow} X_n) &= \lim_{\leftarrow} \pi_*(X_n) \\ &= H_*(W) \otimes \lim_{\leftarrow} \pi_*(L_n BP) \\ &= H_*(W) \otimes \pi_*(BP) \\ &= \pi_*(W \wedge BP). \quad \blacksquare \end{aligned}$$

Proof of Lemma 14. We use the canonical BP -based Adams resolution for X . This is a diagram which displays X as a homotopy inverse limit

$$X = \operatorname{holim}_{\leftarrow} X_s$$

with X_0 contractible and cofibre sequences

$$X \wedge \overline{BP}^{(s)} \wedge BP \longrightarrow X_{s+1} \longrightarrow X_s.$$

where \overline{BP} denotes the cofibre of the unit map $S^0 \rightarrow BP$ and $\overline{BP}^{(s)}$ is its s -fold smash product. The smash product $\overline{BP}^{(s)} \wedge BP$ is a wedge of suspensions of BP , so its smash product with X is harmonic by hypothesis. It follows by induction on s that each X_s is harmonic, so X itself is. \blacksquare

The proof of Theorem 8

We need the following definition.

Definition 16 *A space has **depth 0** if its integral homology groups are free abelian or (in the p -local case) if its homology groups are free $\mathbf{Z}_{(p)}$ -modules. It has **depth** $\leq k+1$ if it is the fibre of a map between spaces (with simply connected target) each having **depth** $\leq k$.*

Note that the category of spaces having **depth** $\leq k$ is closed under finite products and weak infinite products.

Now we need two lemmas.

Lemma 17 *If a space has finite depth, then it is harmonic.*

Lemma 18 *Every simply connected Eilenberg-Mac Lane space has finite depth.*

Now consider the Postnikov tower for a simply connected space X . It displays X as a homotopy inverse limit

$$X = \operatorname{holim}_{\leftarrow} X_i$$

(not to be confused with the Adams resolution used earlier) where $X_2 = K(\pi_2(X), 2)$ and there are fibre sequences

$$X_{i+1} \longrightarrow X_i \longrightarrow K(\pi_{i+1}(X), i+2).$$

It follows from Lemma 18 that each X_i has finite depth, so each of them is harmonic by Lemma 17. Hence X is harmonic by Proposition 10.

This completes the proof of Theorem 8 and hence our Whitehead theorem, modulo the proofs of Lemmas 17 and 18.

The proof of Lemma 17

We will of course argue by induction on the depth, using Theorem 13 to start. Let

$$X \longrightarrow E \longrightarrow B$$

be a fibre sequence where E and B each have depth $\leq k$.

The geometric construction for the Eilenberg-Moore spectral sequence (due originally to L. Smith [Smi69, Smi70] and Rector [Rec70]) displays X as the total space of the cosimplicial space

$$[i] \mapsto E \times \overbrace{B \times B \times \cdots B}^{i \text{ factors}}.$$

Applying the functor “suspension spectrum” to everything gives a cosimplicial spectrum. It follows from the strong convergence of the Eilenberg-Moore spectral sequence that the total spectrum of this cosimplicial spectrum is the suspension spectrum of X . This writes the suspension spectrum of X as a homotopy inverse limit

$$X = \operatorname{holim}_{\leftarrow} X_i$$

(not to be confused with the Postnikov tower or the Adams resolution used earlier) with the following properties: the spectrum X_1 is the suspension spectrum of E , and there are cofibre sequences

$$X_{i+1} \longrightarrow X_i \longrightarrow K_i$$

where $\Sigma^{i-1}K_i$ is a stable summand of

$$E \times \overbrace{B \times B \times \cdots B}^{i \text{ factors}}.$$

(Note that this is a stable construction in that there is a space called $\Sigma^{i-1}X_i$ which need not desuspend.)

Now there is no theorem saying *a priori* that the product of harmonic spaces is harmonic, but it is clear from the definition that any finite product of spaces of depth $\leq k$ has depth $\leq k$. Hence each $E \times B^i$ is harmonic by the inductive hypothesis, so each K_i is by 10(iii). It follows that each X_i is harmonic by induction on i (using 10(ii)), so X itself is harmonic by 10(i).

The proof of Lemma 18

It suffices to show that $K(\mathbf{Z}, i + 2)$ has finite depth for each $i \geq 0$, since any abelian group admits a free resolution. We claim that it suffices to prove it for $K(\mathbf{Z}_{(p)}, i + 2)$ for each i and p . There are short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Q} & \longrightarrow & \prod_p \mathbf{Q}/\mathbf{Z}_{(p)} \longrightarrow 0 \\ 0 & \longrightarrow & \mathbf{Z}_{(p)} & \longrightarrow & \mathbf{Q} & \longrightarrow & \mathbf{Q}/\mathbf{Z}_{(p)} \longrightarrow 0 \end{array}$$

which leads to corresponding fibre sequences of Eilenberg-Mac Lane spaces. We know that $K(\mathbf{Q}, i)$ has depth 0 since its homology is torsion free. If $K(\mathbf{Z}_{(p)}, i)$ has finite depth, then so does $K(\mathbf{Q}/\mathbf{Z}_{(p)}, i)$ since $\mathbf{Q}/\mathbf{Z}_{(p)}$ is a p -torsion group. The first short exact sequence would then imply that $K(\mathbf{Z}, i)$ has finite depth.

We will show that $K(\mathbf{Z}_{(p)}, i + 2)$ has depth $\leq i$. We need some results from Steve Wilson's thesis [Wil73] and [Wil75]. There he studies the spaces BP_i in the Ω -spectrum for BP . The main result of [Wil73] is that each of them has torsion free homology and hence depth 0.

Recall that the Brown-Peterson spectrum BP has homotopy groups

$$\pi_*(BP) = \mathbf{Z}_{(p)}[v_1, v_2, \dots]$$

with $|v_i| = 2p^i - 2$. There are BP -module spectra $BP\langle n \rangle$ (known as Johnson-Wilson spectra) with

$$\pi_*(BP\langle n \rangle) = \mathbf{Z}_{(p)}[v_1, v_2, \dots, v_n].$$

In particular, $BP\langle 0 \rangle = H_{(p)}$, the Eilenberg-Mac Lane spectrum for $\mathbf{Z}_{(p)}$.

There are cofibre sequences of spectra

$$\Sigma^{|v_n|} BP\langle n \rangle \xrightarrow{v_n} BP\langle n \rangle \longrightarrow BP\langle n - 1 \rangle$$

which yield fibre sequences of spaces

$$BP\langle n - 1 \rangle_i \longrightarrow BP\langle n \rangle_{i+1+|v_n|} \longrightarrow BP\langle n \rangle_{i+1}. \quad (19)$$

In [Wil75] Wilson showed that for $i \leq 2 + 2p + 2p^2 + \dots + 2p^n$, $BP\langle n \rangle_i$ is a factor of BP_i and therefore has depth 0.

Lemma 18 follows from the $n = 0$ case of the following.

Lemma 20 *Let $f(n) = 2 + 2p + 2p^2 + \dots + 2p^n$. Then for each $n \geq 0$ and $i \geq 0$, the space $BP\langle n \rangle_{i+f(n)}$ has depth $\leq i$.*

Proof. Wilson proved this for $i = 0$ and we will argue by induction on i . From (19) we get a fibre sequence

$$BP\langle n \rangle_{i+f(n)} \longrightarrow BP\langle n + 1 \rangle_{i+f(n)+1+|v_{n+1}|} \longrightarrow BP\langle n + 1 \rangle_{i+f(n)+1}. \quad (21)$$

Since

$$f(n) + |v_{n+1}| = f(n + 1) - 2,$$

we have

$$i + f(n) + 1 + |v_{n+1}| = f(n + 1) + i - 1.$$

This means that the base and total space of (21) each have depth $\leq i - 1$ by the inductive hypothesis, and the result follows. ■

References

- [Bou75] A. K. Bousfield. The localization of spaces with respect to homology. *Topology*, 14:133–150, 1975.
- [Bou79] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18:257–281, 1979.
- [Bou82] A. K. Bousfield. On homology equivalences and homological localizations of spaces. *American Journal of Mathematics*, 104:1025–1042, 1982.
- [JW73] D. C. Johnson and W. S. Wilson. Projective dimension and Brown-Peterson homology. *Topology*, 12:327–353, 1973.
- [JW75] D. C. Johnson and W. S. Wilson. BP-operations and Morava’s extraordinary K-theories. *Mathematische Zeitschrift*, 144:55–75, 1975.
- [Rav84] D. C. Ravenel. Localization with respect to certain periodic homology theories. *American Journal of Mathematics*, 106:351–414, 1984.
- [Rav87] D. C. Ravenel. The geometric realization of the chromatic resolution. In W. Browder, editor, *Algebraic topology and algebraic K-theory*, pages 168–179, 1987.
- [Rec70] D. L. Rector. Steenrod operations in the Eilenberg-Moore spectral sequence. *Commentarii Mathematici Helvetici*, 45:540–552, 1970.
- [RW80] D. C. Ravenel and W. S. Wilson. The Morava K-theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture. *American Journal of Mathematics*, 102:691–748, 1980.
- [Smi69] L. Smith. On the construction of the Eilenberg-Moore spectral sequence. *Bulletin of the American Mathematical Society*, 75:873–878, 1969.
- [Smi70] L. Smith. *Lectures on the Eilenberg-Moore spectral sequence*. Volume 134 of *Lecture Notes in Mathematics*, Springer-Verlag, 1970.
- [Whi49] J. H. C. Whitehead. *Bulletin of the American Mathematical Society*, 55:213–249, 1949.
- [Wil73] W. S. Wilson. The Ω -spectrum for Brown-Peterson cohomology, Part I. *Commentarii Mathematici Helvetici*, 48:45–55, 1973.
- [Wil75] W. S. Wilson. The Ω -spectrum for Brown-Peterson cohomology, Part II. *American Journal of Mathematics*, 97:101–123, 1975.