THE FIRST ADAMS-NOVIKOV DIFFERENTIAL FOR THE SPECTRUM $T(m)$

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ABSTRACT. There are p-local spectra $T(m)$ with $BP_*(T(m)) = BP_*[t_1, \ldots, t_m]$. In this paper we determine the first nontrivial differential in the Adams–Novikov spectral sequence for each of them for p odd. For $m = 0$ (the sphere spectrum) this is the Toda differential, whose source has filtration 2 and whose target is the first nontrivial element in filtration $2p + 1$. The same goes for $m = 1$, and for $m > 1$ the target is v_2 times the first such element. The proof uses the Thomified Eilenberg-Moore spectral sequence. We also establish a vanishing line and the behavior near it in the Adams–Novikov spectral sequence for $T(m)$.

This paper concerns the Adams–Novikov spectral sequence for the spectra $T(m)$ introduced in [Rav86, §6.5]. We begin by recalling their basic properties. For each prime p and natural number m there is a p-local spectrum $T(m)$ such that

$$BP_*(T(m)) = BP_*[t_1, \ldots, t_m] \subset BP_*(BP)$$

as a comodule algebra over $BP_*(BP)$. It is a summand of the p-localization of the Thom spectrum of the stable bundle induced by

$$\Omega SU(k) \to \Omega SU = BU$$

for any k satisfying $p^m \leq k < p^{m+1}$. These Thom spectra figure in the proof of the Nilpotence Theorem of [DHS88]. The $T(m)$ themselves figure in the method of infinite descent, the technique for calculating the stable homotopy groups of spheres described in [Rav86, Chapter 7], [Rav04, Chapter 7] and [Rav02].

In particular $T(0)$ is the p-localized sphere spectrum. $T(1)$ is the p-localization of the Thom spectrum of the bundle induced by the map

$$\Omega S^{2p-1} \to BU$$

obtained using the loop space structure of BU to extend the map $S^{2p-2} \to BU$ representing the generator of the homotopy group $\pi_{2p-2}(BU)$.

Let (A, Γ) denote the Hopf algebroid $(BP_*, BP_*(BP))$; see [Rav86, A1] for more information. A change-of-rings isomorphism identifies the Adams-Novikov E_2-term for for $T(m)$ with

$$\mathrm{Ext}_{\Gamma(m+1)}(A,\, A) \tag{1}$$

where

$$\Gamma(m+1) = \Gamma/(t_1, \ldots, t_m) = BP_*[t_{m+1}, t_{m+2}, \ldots]$$

This Hopf algebroid is cocommutative below the dimension of t_{2m+2}, so its Ext group (and the homotopy of $T(m)$) in this range is relatively easy to deal with. We will denote this Ext group by $\mathrm{Ext}_{\Gamma(m+1)}$ for short.

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The following was proved in [Rav86, 6.5.9 and 6.5.12].

Theorem A. Description of $\mathrm{Ext}^0_{\Gamma(m+1)}$**.** *For each $m \geq 0$ and each prime p,*

$$\mathrm{Ext}^0_{\Gamma(m+1)} = \mathbf{Z}_{(p)}[v_1, \ldots, v_m],$$

and we denote this ring by $A(m)$. Each of these generators is a permanent cycle, and there are no higher Ext groups below dimension $|v_{m+1}|-1$. Hence $\pi_*(T(m)) \cong A(m)$ in this range.

Our next result concerns $\mathrm{Ext}^1_{\Gamma(m+1)}$. Before stating it we need some chromatic notation. Consider the short exact sequences of Γ-comodules (and hence of $\Gamma(m+1)$-comodules)

$$0 \longrightarrow N^0 \longrightarrow M^0 \longrightarrow N^1 \longrightarrow 0 \tag{2}$$

and

$$0 \longrightarrow N^1 \longrightarrow M^1 \longrightarrow N^2 \longrightarrow 0 \tag{3}$$

where

$$\begin{aligned} N^0 &= BP_*, \\ M^0 &= p^{-1}BP_* = \mathbf{Q} \otimes BP_*, \\ N^1 &= BP_*/(p^\infty) = \mathbf{Q}/\mathbf{Z}_{(p)} \otimes BP_* \\ M^1 &= v_1^{-1}BP_*/(p^\infty), \\ \text{and} \qquad N^2 &= BP_*/(p^\infty, v_1^\infty). \end{aligned}$$

Let

$$\begin{aligned} \delta_0 : \mathrm{Ext}^s_{\Gamma(m+1)}(N^1) &\to \mathrm{Ext}^{s+1}_{\Gamma(m+1)} \\ \text{and} \qquad \delta_1 : \mathrm{Ext}^s_{\Gamma(m+1)}(N^2) &\to \mathrm{Ext}^{s+1}_{\Gamma(m+1)}(N^1) \end{aligned}$$

denote the associated connecting homomorphisms.

We will write elements in N^1 and M^1 as fractions

$$\frac{x}{pc}$$

where c is a p-local integer and $x \in BP_*$ ($x \in v_1^{-1}BP_*$ for M^1) is not divisible by p, i.e., the fraction has been reduced to lowest terms. If pc is divisible by p^e and not by p^{e+1}, then it is understdood that p^e kills this element.

Similarly elements in N^2 are written as fractions

$$\frac{x}{pcv_1^e}$$

with $c \in \mathbf{Z}_{(p)}$, $e > 0$ and $x \in BP_*$ is nontrivial mod I_2. This element is killed by (pc, v_1^e).

The long exact sequence of Ext groups associated with (2) has a surjective connecting homomorphism

$$\mathrm{Ext}^0_{\Gamma(m+1)}(N^1) \to \mathrm{Ext}^1_{\Gamma(m+1)}.$$

The algebraic statement in the following was proved in [Rav86, 6.5.11] while the topological part is proved in [Rav02].

From now on we will denote v_{m+i} by $\widehat{v}_i$ for $i > 0$ and p^m by ω.

Theorem B. Description of $\mathrm{Ext}^1_{\Gamma(m+1)}$**.** *In all cases except $m=0$ and $p=2$, $\mathrm{Ext}^1_{\Gamma(m+1)}$ is isomorphic (via δ_0) to the $A(m)$-submodule of N^1 generated by the set*

$$\left\{\frac{\widehat{v}_1^i}{ip} : i > 0\right\}.$$

Each of these elements is a permanent cycle, and there are no higher Ext groups below dimension $p|\widehat{v}_1|-2$. We will denote $\delta_0(\frac{\widehat{v}_1}{p})$ by $\widehat{\alpha}_1$. For $m=0$ we will use similar notation without the hats.

For higher Ext groups in low dimensions, we need the following notation for elements in $\mathrm{Ext}^2_{\Gamma(m+1)}$. In each case the indicated element of N^2 is invariant, i.e., it lies in $\mathrm{Ext}^0_{\Gamma(m+1)}(N^2)$. Let

$$(4)\qquad \begin{cases} \widehat{b}_{1,0} \;=\; \widehat{\beta}_1 & = \;\delta_0\delta_1\left(\dfrac{\widehat{v}_2}{pv_1}\right) \\ \qquad\qquad \widehat{\beta}_i & = \;\delta_0\delta_1\left(\dfrac{\widehat{v}_2^i}{pv_1}\right) \\ \widehat{b}_{1,1} \;=\; \widehat{\beta}_{p/p} & = \;\delta_0\delta_1\left(\dfrac{\widehat{v}_2^p}{pv_1^p}\right) \\ \qquad\qquad \widehat{\theta} & = \;\delta_0\delta_1\left(\dfrac{\widehat{v}_3}{pv_1} - \dfrac{v_2\widehat{v}_2^p}{pv_1^{1+p}} + \dfrac{v_2^{p\omega}\widehat{v}_1}{p^2v_1}\right) \\ \qquad\qquad \widehat{\theta}' & = \;\delta_0\delta_1\left(\dfrac{\widehat{v}_1\widehat{v}_3}{pv_1} - \dfrac{v_2\widehat{v}_1\widehat{v}_2^p}{pv_1^{1+p}} + \dfrac{v_2^{p\omega}\widehat{v}_1^2}{2p^2v_1}\right) \end{cases}$$

Again for $m=0$ we will use similar notation without the hats. The elements θ and θ' are not defined.

The element $\widehat{\theta}'$ is in the Massey product

$$\langle\widehat{\alpha}_1,\, p\iota,\, \widehat{\theta}\rangle$$

where $\iota\in\pi_0(T(m))$ denotes the fundamental class. Note that even though the preimage of $\widehat{\theta}$ has order p^2 in N^2, $\widehat{\theta}$ itself has order p since the preimage of $p\widehat{\theta}$ is the coboundary (in the chromatic cobar complex [Rav86, 5.1.10]) of

$$\frac{v_1^{-1}v_2^{p\omega}\widehat{v}_1}{p}.$$

Let B denote the $A(m)$-submodule of N^2 generated by $\widehat{b}_{1,1}$ and $\widehat{\theta}$, and let C be the one generated by

$$(5)\qquad \left\{\frac{\widehat{v}_1^j\widehat{v}_2^i}{ipv_1} : 0 < i \le p,\, 0 \le j \le p^2 - pi\right\}.$$

For the 2-line and above, we have the following, essentially proved as [Rav86, Theorem 7.1.13], [Rav04, Theorem 7.2.6] and [Rav02, Theorem 4.5]. Each of them determines $\mathrm{Ext}^{2,t}_{\Gamma(m+1)}$ for $t < p^2|\widehat{v}_1|$. This is extended to $t < (p^2+p)|\widehat{v}_1|$ in [NR, Theorem 7.12], where $\widehat{\theta}$ and $\widehat{\theta}'$ are denoted by $\widehat{\theta}_0$ and $\widehat{v}_1\widehat{\theta}_0$ respectively.

Theorem C. Description of $\mathrm{Ext}^2_{\Gamma(m+1)}$ in low dimensions. *For $m > 0$, $\mathrm{Ext}^{2,t}_{\Gamma(m+1)}$ for $t \le 2p^2 - 2p + p^2|\widehat{v}_1|$ is isomorphic (via $\delta_0\delta_1$) to $B \oplus C$ as defined above in (5). Moreover $\mathrm{Ext}^{2+\varepsilon+2k}_{\Gamma(m+1)}$ in this range (for $\varepsilon = 0, 1$ and $2k + \varepsilon > 0$) is isomorphic to $\widehat{\alpha}_1^{\varepsilon}\widehat{b}_{1,0}^k C$.*

We are now ready to state the main result of this paper.

Theorem 1. The first differential for $T(m)$ for $m > 0$. *The first (lowest dimensional) nontrivial differential in the Adams–Novikov spectral sequence for the spectrum $T(1)$ at an odd prime p is*

$$d_{2p-1}(\widehat{\theta}) = \widehat{\alpha}_1\widehat{b}_{1,0}^p$$

where $\widehat{\alpha}_1$ is an in Theorem B, and $\widehat{b}_{1,0}$ and $\widehat{\theta}$ are as in (4).

For $m > 1$ the first nontrivial differential in the Adams–Novikov spectral sequence for the spectrum $T(m)$ at an odd prime p is

$$d_{2p-1}(\widehat{\theta}') = v_2\widehat{\alpha}_1\widehat{b}_{1,0}^p.$$

Corollary 2. The first nontrivial group extension in $T(m)$ for $m > 1$. *For each $m > 1$ the first nontrivial group extension in the passage from the E_∞-term of the Adams–Novikov spectral sequence to $\pi_*(T(m))$ is*

$$p\widehat{\theta} = v_2\widehat{b}_{1,0}^p.$$

up to unit scalar multiplication.

For $m = 0$, the corresponding statement is the Toda differential,

$$d_{2p-1}(\beta_{p/p}) = \alpha_1\beta_1^p = \alpha_1 b_{1,0}^p$$

originally proved by Toda in [Tod68] and [Tod67] and stated as [Rav86, Theorem 4.4.22].

Our proof is the opposite of Toda's in the following sense. For all m the first differential occurs in the smallest dimension where there is a potential source and target with compatible bidegrees. Toda shows that his potential target, $\alpha_1\beta_1^p$, vanishes in homotopy, the first instance of his "important relation." It follows that the corresponding element in Ext must be killed by a differential, and there is only one possible source. We show that for $m > 0$ our source cannot represent a homotopy element. It follows that it supports a nontrivial differential, and there is only one possible target.

Toda's target is on the vanishing line described below in Theorem 3, i.e., it is the first element in filtration $2p + 1$. The same is true for our target when $m = 1$, but for $m > 1$ the first such element does not lie above an element on the 2-line with the right bidegree. Its product with v_2 is the first one that does and is therefore the first potential target.

For $m = 1$ the dimension of the element $\widehat{\theta}$ (the source of the differential) is 154 for $p = 3$ and 1238 for $p = 5$. For $p = 3$ this Ext group and the first differential are illustrated for $m = 1$ and $m = 2$ in Figures 1 and 2 respectively.

The following vanishing line result is needed to prove Theorem 1 and may be of independent interest.

Theorem 3. The Adams-Novikov vanishing line for $T(m)$. *In the Adams–Novikov spectral sequence for $T(m)$, $E_2^{s,t} = 0$ for*

$$t < (p\omega - 1)\begin{cases} ps & \text{for } s \text{ even} \\ (ps - p + 2) & \text{for } s \text{ odd.} \end{cases}$$

Moreover each element for $s \geq 2$ and

$$t < (p\omega - 1)\begin{cases} (ps + 2) & \text{for } s \text{ even} \\ (ps - p + 4) & \text{for } s \text{ odd.} \end{cases}$$

is the product of a monomial in $\widehat{\alpha}_1$ and $\widehat{b}_{1,0}$ with an element in $A(m)/I_2$. In particular the first nontrivial element in $E_2^{s,*}$ is $\widehat{b}_{1,0}^{[s/2]}\widehat{\alpha}_1^{s-2[s/2]}$.

This will be proved at the end of the paper.

The first element in $\mathrm{Ext}^2_{\Gamma(m+1)}$ not having the form described above is the image of

$$\frac{\widehat{v}_1\widehat{v}_2}{pv_1},$$

which is in the Massey product

$$\langle \widehat{\alpha}_1, p\iota, \widehat{b}_{1,0}\rangle.$$

To prove Theorem 1 we use certain cohomology operations in $T(m)$-theory for $m > 0$, i.e., maps

$$T(m) \xrightarrow{r_1} \Sigma^{|t_1|}T(m) \qquad \text{and} \qquad T(m) \xrightarrow{r_{p\Delta_m}} \Sigma^{p|t_m|}T(m)$$

derived from the splitting of $T(m)\wedge T(m)$. They have properties similar to Steenrod and Quillen operations, but *they commute with each other*. These operations do not exist for $m = 0$, so the method used here cannot be used to establish the Toda differential.

Lemma 4. Two commuting Quillen operations in $T(m)$-theory.
Let $I = (i_1, i_2, \ldots, i_m)$ be a sequence of m nonnegative integers, and let $t^I = t_1^{i_1} \ldots t_m^{i_m}$. There are operations

$$r_I \in T(m)^{|t^I|}(T(m))$$

dual to the elements

$$t^I \in T(m)_{|t^I|}(T(m))$$

where

$$T(m)_*(T(m)) = \pi_*(T(m))[t_1, \ldots, t_m],$$

and t_i maps to the element of the same name in $BP_*(BP)$.

In particular let r_1 and $r_{p\Delta_m}$ be the duals of t_1 and t_m^p respectively. Then

$$r_1 r_{p\Delta_m} = r_{p\Delta_m} r_1.$$

Proof. We can compute $T(m)_*(T(m))$ using the Atiyah–Hirzebruch spectral sequence with

$$E_2 = H_*(T(m); \pi_*(T(m))) = \pi_*(T(m))[t_1, \ldots, t_m].$$

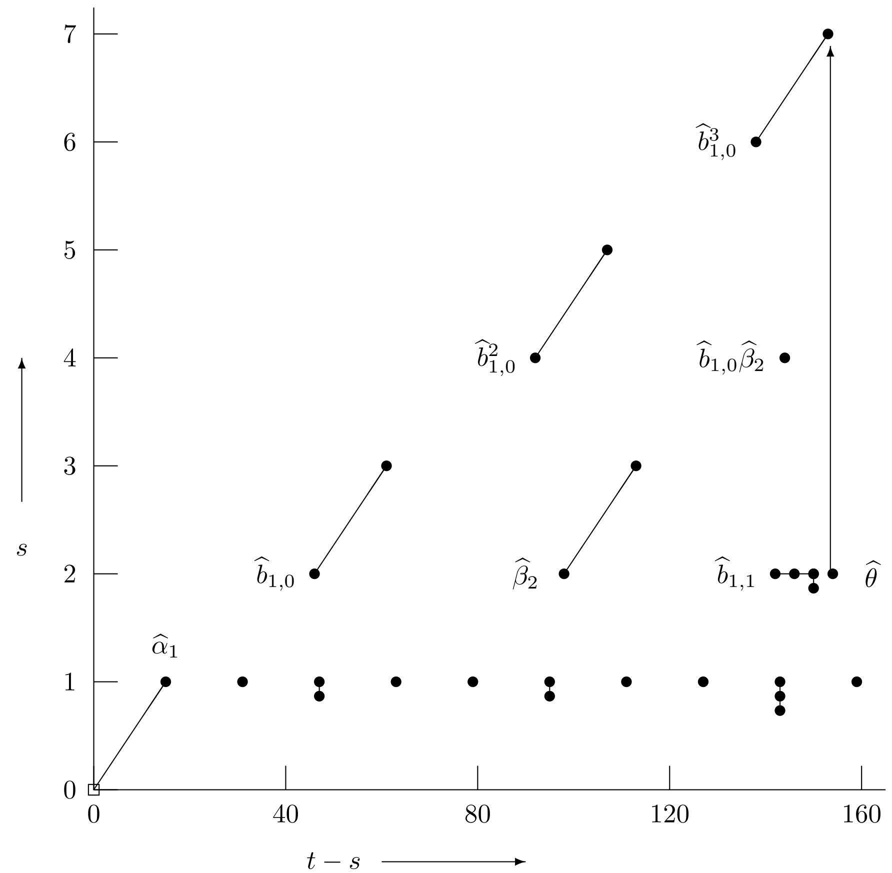

FIGURE 1. The Adams-Novikov E_2-term for $T(1)$ at $p = 3$ in dimensions ≤ 154, showing the first nontrivial differential. Elements on the 0- and 1-lines divisible by v_1 are not shown. Elements on the 2-line and above divisible by v_2 are not shown. Vertical, horizontal and diagonal lines indicate multiplication by p, v_1 and $\widehat{\alpha}_1$ respectively.

We know from Theorem A that $\pi_*(T(m))$ is concentrated in even dimensions below dimension $|\widehat{v}_1| - 1$. It follows that each t_i is a permanent cycle and the spectral sequence collapses, so the Hopf algebroid $T(m)_*(T(m))$ is as claimed. The map

$$T(m)_*(T(m)) \to BP_*(BP)$$

is monomorphic through dimension $|t_m|$, so the coproduct on $t_i \in T(m)_*(T(m))$ is determined by the coproduct on $t_i \in BP_*(BP)$.

The commuting of the two specified operations follows from the fact that the only t^I having either $t_1 \otimes t_m^p$ or $t_m^p \otimes t_1$ in its coproduct expansion is $t_1 t_m^p$, which

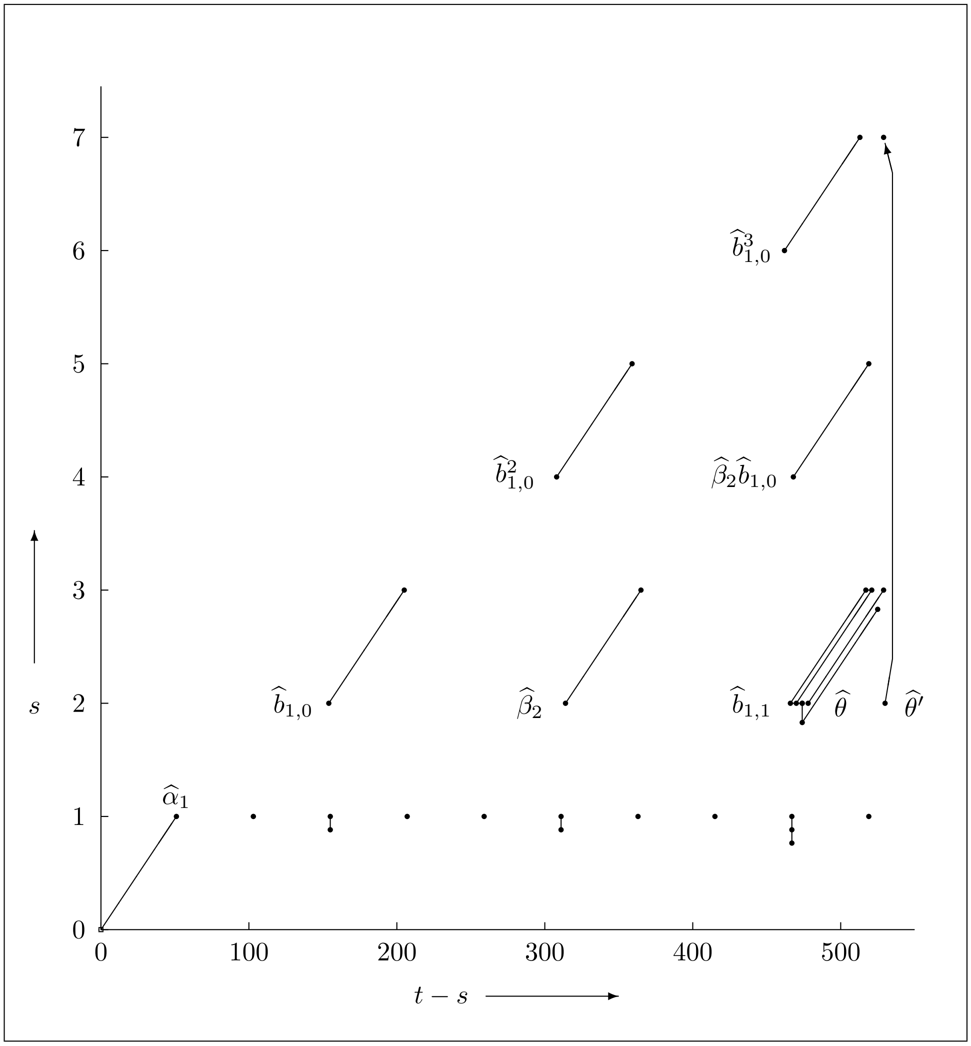

FIGURE 2. The Adams-Novikov E_2-term for $T(2)$ at $p = 3$ in dimensions ≤ 530. Elements on the 0- and 1-lines divisible by v_1 or v_2 are not shown. Elements on the 2-line and above divisible by v_2 or v_3 are not shown except for $v_2\widehat{\alpha}_1\widehat{b}^3_{1,0}$, the target of the first differential. Vertical, horizontal and diagonal lines indicate multiplication by p, v_1 and $\widehat{\alpha}_1$ respectively.

has both terms, i.e.,

$$\Delta(t_1t_m^p) = t_1t_m^p \otimes 1 + 1 \otimes t_1t_m^p + t_1 \otimes t_m^p + t_m^p \otimes t_1 + \dots. \quad \square$$

The action of these operations in Ext has the following interpretation. Consider the Hopf algebroid extension (this term is defined in [Rav86, A1.1.15])

$$(A(m), G(1, m-1)) \to (BP_*, \Gamma) \to (BP_*, \Gamma(m+1)),$$

where $G(1,m-1) = A(m)[t_1,\dots,t_m]$. Then $\mathrm{Ext}_{\Gamma(m+1)}$, which is the Adams-Novikov E_2-term for $T(m)$ by (1), is a $G(1,m-1)$-comodule by [Rav86, A1.3.14 (a)].

Lemma 5. Relation between Quillen operations and comodule structure. *The $G(1,m-1)$-comodule structure on $\mathrm{Ext}(BP_*(T(1)))$ described above is dual to the action of the operations of Lemma 4.*

Proof. The operation r_i of Lemma 4 is induced by the composite map

$$T(m) \xrightarrow{\eta_R} T(m)\wedge T(m) \xrightarrow{r_I} \Sigma^{|t^I|}T(m).$$

Applying the functor $\mathrm{Ext}_\Gamma(BP_*(\cdot))$ and the change-of-rings isomorphism of (1) we get

$$\mathrm{Ext}_{\Gamma(m+1)} \xrightarrow{\eta_R} \mathrm{Ext}_{\Gamma(m+1)}(BP_*[t_1,\dots,t_m]) \xrightarrow{r_I} \Sigma^{|t^I|}\mathrm{Ext}_{\Gamma(m+1)}.$$

The middle term is

$$G(1,m-1)\otimes_{A(m)}\mathrm{Ext}_{\Gamma(m+1)} = A(m)[t_1,\dots,t_m]\otimes_{A(m)}\mathrm{Ext}_{\Gamma(m+1)},$$

and the map induced by η_R is the $G(1,m-1)$-comodule structure map. □

The following lemmas concern the action of specific operations in the relevant dimensions.

Lemma 6. Action of Quillen operations in the Adams–Novikov spectral sequence for $T(m)$. *In the E_2-term of the Adams–Novikov spectral sequence for $T(m)$ we have*

$$\begin{aligned} r_1(\widehat{\theta}') &= -v_m^p\widehat{\theta}, \\ r_{p\Delta_m}(\widehat{\theta}') &= v_1\widehat{\theta} = -v_2\widehat{b}_{1,1}, \\ r_{p\Delta_m}(v_m^p\widehat{\theta}) &= 0, \\ r_{p\Delta_m}(v_2v_m^p\widehat{b}_{1,0}^p) &= 0, \\ \textit{and}\qquad r_1(v_2\widehat{b}_{1,1}) &= 0. \end{aligned}$$

For $m=1$ we also have

$$\begin{aligned} r_1(\widehat{\theta}) &= 0 \\ \text{and}\qquad r_p(\widehat{\theta}) &= \widehat{b}_{1,1}. \end{aligned}$$

Lemma 7. Action of Quillen operations in $\pi_*(T(m))$. *In $\pi_*(T(m))$,*

$$r_{p\Delta_m}(v_m^p\widehat{\theta}) = 0 \qquad \textit{and} \qquad r_1(v_2\widehat{b}_{1,1}) = v_2\widehat{b}_{1,0}^p$$

up to unit scalar multiplication. For $m=1$,

$$r_1(\widehat{b}_{1,1}) = \pm\widehat{b}_{1,0}^p.$$

We will prove these two lemmas below. To prove Theorem 1 we will assume that $\widehat{\theta}'$ represents a homotopy element x and show that

$$r_1r_{p\Delta_m}(x) \neq r_{p\Delta_m}r_1(x), \tag{8}$$

which contradicts that fact that r_1 and $r_{p\Delta_m}$ commute. This means that $\widehat{\theta}'$ cannot survive to represent a homotopy element. The indicated differential

$$d_{2p-1}(\widehat{\theta}') = v_2\widehat{\alpha}_1\widehat{b}_{1,0}^p$$

is the only one it can support because the indicated target is the only element in an appropriate bidegree by Theorem 3. When $m = 1$ we can make a similar calculation on $\widehat{\theta}$ and conclude that

$$d_{2p-1}(\widehat{\theta}) = \widehat{\alpha}_1\widehat{b}_{1,0}^p.$$

As before let x denote a hypothetical homotopy element represented by $\widehat{\theta}'$. Lemma 6 determines $r_1(x)$ modulo elements of higher filtration, so we have

$$r_1(x) = -v_m^p(\widehat{\theta} + cv_2\widehat{b}_{1,0}^p)$$

for some scalar c. Note that $\widehat{b}_{1,0}^p$ is the first element in filtration $2p$, so below its dimension, computing in Ext is equivalent to computing in homotopy. It follows that

$$r_{p\Delta_m}r_1(x) = -r_1(v_m^p\widehat{\theta} + cv_2v_m^p\widehat{b}_{1,0}^p) = 0.$$

On the other hand we have

$$r_1r_{p\Delta_m}(x) = r_1(-v_2\widehat{b}_{1,1}) = v_2\widehat{b}_{1,0}^p$$

up to unit scalar multiplication. This completes the proof of Theorem 1, modulo Lemmas 6 and 7.

For Corollary 2, note that in $\mathrm{Ext}_{\Gamma(m+1)}$

$$\widehat{\theta}' \in \langle\widehat{\alpha}_1,\, p\iota,\, \widehat{\theta}\rangle.$$

Since $\widehat{\theta}'$ is not a homotopy element, the Toda bracket corresponding to the Massey product above must not be defined. Since we know that $\widehat{\alpha}_1$ represents an element of order p, it follows that the homotopy element represented by $\widehat{\theta}$ does not have order p. Hence p times it must have filtration $2p$, and the only possibility there is $v_2\widehat{b}_{1,0}^p$.

Remark: Possible higher differentials. *We conjecture that* $d_{2p^m-1}(\widehat{b}_{2,m-1}) = \widehat{\alpha}_1\widehat{b}_{1,0}^{p^m}$ *for all* $m > 0$ *(we prove this here for* $m = 1$*), where for* $m > 1$

$$\widehat{b}_{2,m-1} = \delta_0\delta_1\left(\frac{\widehat{v}_3^{\omega/p}}{pv_1^{\omega/p}} - \frac{v_2^{\omega/p}\widehat{v}_2^{\omega}}{pv_1^{(p+1)\omega/p}} + \frac{v_2^{\omega^2}\widehat{v}_2^{\omega/p^2}}{pv_1^{(p+1)\omega/p^2}}\right).$$

Computations similar to those to be described below show that if y *is a homotopy element representing* $\widehat{b}_{2,m-1}$*, then* $r_{\Delta_m}r_{p^m}(y) = b_{1,0}^{\omega}$ *but* $r_{p^m}r_{\Delta_m}(y) = 0$*, which leads to a contradiction as above. The element* $\widehat{\alpha}_1\widehat{b}_{1,0}^{p^m}$ *is in the right dimension to be a target; one has to show that this differential is not preempted by an earlier one in the same dimension. If we let* m *go to* ∞ *as in* [Rav00]*, this element disappears. This suggests we should look at a Hopf algebroid graded over* $\mathbf{Q}[\omega]$ *rather than just over* $\mathbf{Z} \oplus \mathbf{Z}\omega$*.*

Proof of Lemma 6. We can compute the action of r_1 and $r_{p\Delta_m}$ on these elements by computing the actions of the corresponding Quillen operations on the numerators of the chromatic fractions modulo the ideals associated with the denominators. For example we need to know $r_I(v_2\widehat{v}_1\widehat{v}_2^p)$ modulo (p, v_1^{p+1}), and so on. We have

$$\begin{aligned} r_{p\Delta_m}(\widehat{\theta}') &= -r_{p\Delta_m}\left(\frac{v_2\widehat{v}_1\widehat{v}_2^p}{pv_1^{1+p}}\right) \\ &= -\frac{v_2\widehat{v}_2^p}{pv_1^p} \\ &= -v_2\widehat{b}_{1,1}, \end{aligned}$$

and

$$\begin{aligned} r_1(\widehat{\theta}') &= \delta_0\delta_1\left(r_1\left(\frac{\widehat{v}_1\widehat{v}_3}{pv_1} - \frac{v_2\widehat{v}_1\widehat{v}_2^p}{pv_1^{p+1}} + \frac{v_2^{p\omega}\widehat{v}_1^2}{2p^2v_1}\right)\right) \\ &= \delta_0\delta_1\left(\frac{r_1(\widehat{v}_1)\widehat{v}_3}{pv_1} + \frac{\widehat{v}_1 r_1(\widehat{v}_3)}{pv_1}\right. \\ &\qquad -\frac{r_1(v_2)\widehat{v}_1\widehat{v}_2^p}{pv_1^{p+1}} - \frac{v_2 r_1(\widehat{v}_1)\widehat{v}_2^p}{pv_1^{p+1}} - \frac{v_2\widehat{v}_1 r_1(\widehat{v}_2^p)}{pv_1^{p+1}} \\ &\qquad \left.+\frac{r_1(v_2^{p\omega})\widehat{v}_1^2}{2p^2v_1} + \frac{v_2^{p\omega} r_1(\widehat{v}_1^2)}{2p^2v_1} + v_2^{p\omega}\widehat{v}_1^2 r_1\left(\frac{1}{2p^2v_1}\right)\right) \\ &= \delta_0\delta_1\left(-\frac{v_m^p\widehat{v}_3}{pv_1} - \frac{\widehat{v}_1\widehat{v}_2^p}{pv_1} + \frac{v_1^p\widehat{v}_1\widehat{v}_2^p}{pv_1^{p+1}} + \frac{v_2 v_m^p\widehat{v}_2^p}{pv_1^{p+1}} - \frac{v_2^{p\omega}v_m^p\widehat{v}_1}{p^2v_1} - \frac{v_2^{p\omega}\widehat{v}_1^2}{2pv_1^2}\right) \\ &= -v_m^p\widehat{\theta}. \end{aligned}$$

We leave the additional computations for $m = 1$ as an exercise. □

Proof of Lemma 7. We need the Thomified Eilenberg-Moore spectral sequence of [MRS01]. Given a fibration of spaces

$$X \to E \to B$$

with a stable vector bundle over E, we get a spectral sequence converging to the homotopy of Y, the Thom spectrum for the induced bundle over X.

If the fibration is

$$\Omega SU(p\omega - 1) \to \Omega SU \to B$$

with the evident vector bundle over $\Omega SU = BU$, we get the usual Adams–Novikov spectral sequence for $X(p\omega - 1)$, the Thom spectrum associated with $\Omega SU(p\omega - 1)$, which has $T(m)$ as a summand.

If we take the Cartesian product of this fibration with

$$\Omega^2 S^{2p\omega-1} \to \text{pt.} \to \Omega S^{2p\omega-1},$$

the E_2-term is a subquotient of the tensor product of the one above with $H_*(\Omega^2 S^{2p\omega-1})$ equipped with the Eilenberg-Moore filtration, and the spectral sequence converges to

$$\pi_*(\Omega^2 S^{2p\omega-1} \wedge X(p\omega - 1)).$$

The map

$$\Omega^2 S^{2p\omega-1} \to \Omega SU(p\omega - 1)$$

sends this homology to

$$E(\widehat{\alpha}_1, h_{m+1,1}, \ldots) \otimes P(\widehat{b}_{1,0}, \widehat{b}_{1,1}, \ldots)$$

in the Adams–Novikov spectral sequence for $T(m)$. Thus we have a map to the Adams–Novikov spectral sequence for $T(m)$ from a spectral sequence converging to $T(m)_*(\Omega^2 S^{2p\omega-1})$. In E_2 this gives us

$$E_2' \to \operatorname{Ext}_{\Gamma(m+1)}$$

where E_2' is a subquotient of

$$E(\widehat{\alpha}_1, h_{m+1,1}, \ldots) \otimes P(\widehat{b}_{1,0}, \widehat{b}_{1,1}, \ldots) \otimes \operatorname{Ext}_{\Gamma(m+1)}. \tag{9}$$

We want to deduce the lemma from the fact that in $H_*(\Omega^2 S^{2p\omega-1})$,

$$P_*^1(\widehat{b}_{1,1}) = \widehat{b}_{1,0}^p \qquad \text{and} \qquad P_*^{p\Delta_m}(\widehat{b}_{1,1}) = 0.$$

For this note that the following diagrams commute.

$$\begin{tikzcd} T(m) \arrow[r, "r_1"] \arrow[d, "\iota"'] & \Sigma^{2p-2}T(m) \arrow[d, "\iota"] \\ H/(p) \arrow[r, "-P^1"] & \Sigma^{2p-2}H/(p) \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} T(m) \arrow[r, "r_{p\Delta_m}"] \arrow[d, "\iota"'] & \Sigma^{p|t_m|}T(m) \arrow[d, "\iota"] \\ H/(p) \arrow[r, "\chi(P^{p\Delta_m})"] & \Sigma^{p|t_m|}H/(p), \end{tikzcd}$$

where $H/(p)$ denotes the mod p Eilenberg–Mac Lane spectrum, the map $\iota : T(m) \to H/(p)$ is the bottom mod p cohomology class, and $-P^1$ and $\chi(P^{p\Delta_m})$ denote the conjugates of the Steenrod operations dual to ξ_1 and ξ_m^p respectively.

It follows that in E_2' we have

$$r_1(\widehat{b}_{1,1} \otimes 1) = -\widehat{b}_{1,0}^p \otimes 1 + \ldots$$

where the missing terms involve elements with lower dimensional first factors. However this operation must respect the Snaith splitting of $\Omega^2 S^{2p\omega-1}$, which means there are no lower dimensional first factors, so

$$r_1(\widehat{b}_{1,1} \otimes 1) = -\widehat{b}_{1,0}^p \otimes 1 \qquad \text{and} \qquad r_1(\widehat{b}_{1,1}) = -\widehat{b}_{1,0}^p \in \pi_*(T(m))$$

precisely. For $m > 1$ it follows that $r_1(v_2\widehat{b}_{1,1}) = -v_2\widehat{b}_{1,0}^p$ as required. A similar argument shows that $r_{p\Delta_m}(\widehat{b}_{1,1} \otimes 1) = 0$, and the result follows. □

Remark: Why this method does not apply to the original Toda differatial. *The method described here cannot be used to establish the Toda differential of* [Tod67] *and* [Tod68], *i.e.,*

$$d(b_{1,1}) = \alpha_1 b_{1,0}^p,$$

because we do not have the operation r_1 acting in stable homotopy. However as in the proof of Lemma 7 an essential fact is the action of the Steenrod algebra in $H_*(\Omega^2 S^{2p-1})$, in particular the fact that $P_*^1(b_{1,1}) = b_{1,0}^p$. As above one can use the Thomified Eilenberg-Moore spectral sequence to show that the element $b_{1,1} \in \operatorname{Ext}_\Gamma^2$ cannot be a permanent cycle.

Proof of Theorem 3. Let

$$f_m(s) = (p\omega - 1)\begin{cases} ps & \text{for } s \text{ even} \\ (ps - p + 2) & \text{for } s \text{ odd,} \end{cases}$$

so the theorem says that $\mathrm{Ext}^{s,t}_{\Gamma(m+1)} = 0$ for $t < f_m(s)$. Let $g_m(s)$ be the corresponding function for the actual vanishing line, i.e., the smallest number such that

$$\mathrm{Ext}^{s,g_m(s)}_{\Gamma(m+1)} \neq 0.$$

We will use the small descent spectral sequence of [Rav02, 1.17]. For each $h \geq 0$ there is a $T(m)$-module spectrum $T(m)_h$ with

$$BP_*(T(m)_h) = BP_*(T(m))\{t_{m+1}^j \colon 0 \leq j \leq h\}.$$

In particular

$$T(m) = T(m)_0 \qquad \text{and} \qquad T(m+1) = \lim_{\to} T(m)_h.$$

The small descent spectral sequence computes $\mathrm{Ext}_\Gamma(BP_*(T(m)_{p^i-1}))$ in terms of $\mathrm{Ext}_\Gamma(BP_*(T(m)_{p^{i+1}-1}))$. Its E_1-term is

$$E_1^{*,s} = E(\widehat{h}_{1,i}) \otimes P(\widehat{b}_{1,i}) \otimes \mathrm{Ext}^s_\Gamma(BP_*(T(m)_{p^{i+1}-1}))$$

where the elements

$$\begin{array}{rccl} & \widehat{h}_{1,i} & \in & E_1^{1,0} \qquad \text{with } |\widehat{h}_{1,i}| = 2p^i(p\omega - 1) \\ \text{and} & \widehat{b}_{1,i} & \in & E_1^{2,0} \qquad \text{with } |\widehat{b}_{1,i}| = 2p^{i+1}(p\omega - 1) \end{array}$$

are permanent cycles.

Letting i go to ∞, we can conclude that

$$\mathrm{Ext}_{\Gamma(m+1)} = \mathrm{Ext}_\Gamma(BP_*(T(m)))$$

is a subquotient of

$$E(\widehat{\alpha}_1, \widehat{h}_{1,1}, \dots) \otimes P(\widehat{b}_{1,0}, \widehat{b}_{1,1}, \dots) \otimes \mathrm{Ext}_{\Gamma(m+2)}.$$

This means that

$$g_m(s) = \min(f_m(s),\, g_{m+1}(s)),$$

i.e., that $\mathrm{Ext}_{\Gamma(m+1)}$ has the desired vanishing line if $\mathrm{Ext}_{\Gamma(m+2)}$ does.

We will now use downward induction on m, i.e., the method of infinite descent, as follows. We know that $T(m')$ is equivalent to BP below dimension $|v_{m'+1}|-1$. It follows that for any fixed t, $\mathrm{Ext}^{s,t}_{\Gamma(m'+1)}$ vanishes for $s > 0$ and m' sufficiently large. This means that for a fixed value of s there is an $m' > m$ such that $g_{m'}(s) \geq f_m(s)$. This implies that

$$\begin{array}{rrcl} & g_{m'-1}(s) & = & \min(f_{m'-1}(s),\, g_{m'}(s)) \geq f_m(s), \\ \text{so} & g_{m'-2}(s) & = & \min(f_{m'-2}(s),\, g_{m'-1}(s)) \geq f_m(s) \qquad \text{if } m' - 2 \geq m, \end{array}$$

and so on, leading to the desired conclusion that $g_m(s) = f_m(s)$.

The second assertion of the theorem concerns elements for which $t < f_m(s) + |\widehat{v}_1|$. We will look at the small descent spectral sequence converging to $\mathrm{Ext}_{\Gamma(m+1)}$ with

$$E_1 = E(\widehat{\alpha}_1) \otimes P(\widehat{b}_{1,0}) \otimes \mathrm{Ext}_\Gamma(BP_*(T(m)_{p-1})).$$

The following differentials occur in the range in question.

$$\begin{array}{rrcl} & d_1(\widehat{v}_1) & = & p\widehat{\alpha}_1, \\ & d_2(\widehat{h}_{1,1}) & = & p\widehat{b}_{1,0} \\ \text{and} & d_2(\widehat{h}_{2,0}) & = & v_1\widehat{b}_{1,0}. \end{array}$$

These imply that $\widehat{b}_{1,0}$ and hence $\widehat{b}_{1,0}^{[s/2]}\widehat{\alpha}_1^{s-2[s/2]}$ for each $s \geq 2$ are annihilated by I_2 and by nothing else in this range, and the result follows. $\square$

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