THE MORAVA K-THEORIES OF EILENBERG-MACLANE SPACES AND THE CONNER-FLOYD CONJECTURE

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Introduction. Of the many generalized homology theories available, very few are computable in practice except for the simplest of spaces. Standard homology and K-theory are the only ones which can be considered somewhat accessible. In recent years, complex cobordism, or equivalently, Brown-Peterson homology, has become a useful tool for algebraic topology. The high state of this development is particularly apparent with regard to BP stable operations, which are understood well enough to have many applications to stable homotopy; see for example [16]. Despite this achievement, it is still virtually impossible to compute the Brown-Peterson homology of any but the nicest of spaces; for example: some simple classifying spaces, spaces with no torsion and spaces with few cells.

As a replacement for Brown-Peterson homology in this respect, we present the closely related generalized homologies known as the Morava K-theories. These are a sequence of homology theories, $K(n)_*(—)$, $n > 0$, for each prime p. The $n = 1$ case is essentially standard mod p complex K-theory. These theories are periodic of period $2(p^n - 1)$ and fit together to give Morava's beautiful structure theorem for complex cobordism; see [11]. Because of their close relationship to complex bordism, information about them will sometimes suffice for bordism, and thus geometric, problems. This is the case with our proof of the Conner-Floyd conjecture.

The Morava K-theories each possess Künneth isomorphisms for all spaces. This feature enhances their computability tremendously. We demonstrate this point by computing the Morava K-theories of the Eilenberg-MacLane spaces. These spaces are difficult to handle even for

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standard homology, and at present, untouchable for BP; see [12]. Our computation always proceeds directly using the Morava K-theories; never needing to refer to standard homology in any way.

From the perspective of the Morava K-theories, the Eilenberg-MacLane spaces for finite abelian groups appear as finite complexes. In fact, the n-th Morava K-theory "sees" only the first n Eilenberg-MacLane spaces. In our computations we continue the program of [25] by exploiting Hopf rings throughout. The Morava K-theory of all the spaces in a multiplicative Ω-spectrum always gives rise to such a graded ring object over the category of $K(n)_*$ coalgebras. For the ring $Z/(p^j)$, the n-th Morava K-theory of the associated Eilenberg-MacLane spaces is just the free Hopf ring on the Morava K-theory of the first Eilenberg-MacLane space!

These computations are all done using a highly non-collapsing bar spectral sequence for $K(n)_*(—)$. In order to do this, we use the Hopf ring structure to define a new pairing under which an element in the bar spectral sequence for the q-th Eilenberg-MacLane space is multiplied by one in the Morava K-theory of the m-th space and their product lies in the spectral sequence for the $(m + q)$-th space. This pairing is compatible with the map induced by the cup product pairing of the q-th and m-th spaces into the $(q + m)$-th space, and it allows us to identify elements in the bar spectral sequence in terms of the Hopf ring structure, most importantly, the elusive transpotence elements in the second filtration. In addition it allows us to compute the nontrivial differentials inductively. The Hopf ring structure is then applied to solve all extension problems. A similar, but simpler computation can use this pairing to obtain the standard homology of the Eilenberg-MacLane spaces by a purely homological process without reference to chains or operations. We hope to pursue this in a future paper. The spectral sequence pairing has since been generalized to cover all pairings of spectra [31].

With this paper we hope to persuade the reader of three things: the usefulness of Hopf rings as a descriptive and computational tool; the power of the new spectral sequence pairing; the computability of the Morava K-theories.

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We can now begin the description of our results.

Recall that Brown-Peterson homology, $\mathrm{BP}_*(-)$, has coefficients

$$\mathrm{BP}_* \simeq Z_{(p)}[v_1, v_2, \ldots]$$

with $|v_n| = 2(p^n - 1)$. There is a natural transformation from Brown-Peterson homology to the n-th Morava K-theory

$$\mathrm{BP}_*(-) \to K(n)_*(-)$$

where $K(n)_* \simeq F_p[v_n, v_n^{-1}]$ with $v_i \to 0$ if $i \neq n$, and $v_n \to v_n$. For p an odd prime $K(n)_*(-)$ is a nicely multiplicative generalized homology theory. Because of problems with the multiplication at $p = 2$ we restrict our attention to the odd primes throughout this paper.

Because $K(n)_*$ is a graded field, $K(n)_*(-)$ always has a Künneth isomorphism; $K(n)_*(X \times Y) \simeq K(n)_*X \otimes_{K(n)_*} K(n)_*Y$. Thus $K(n)_*(X)$ is always a coalgebra and in addition, if X is an H-space, it is a Hopf algebra. Let $\mathbf{K}_q = \mathbf{K}(Z/(p^j), q)$ be the mod (p^j) Eilenberg-MacLane space. Each $K(n)_*\mathbf{K}_q$ is a Hopf algebra and $K(n)_*\mathbf{K}_* = \{K(n)_*\mathbf{K}_q\}_{q \geq 0}$ is a Hopf ring; i.e., there are maps

$$\circ : K(n)_*\mathbf{K}_q \otimes_{K(n)_*} K(n)_*\mathbf{K}_m \to K(n)_*\mathbf{K}_{q+m}$$

which have certain compatibility conditions and these maps and conditions come from the cup product for $Z/(p^j)$ cohomology.

The description of $K(n)_*\mathbf{K}_0$ is simple. It is just the group-ring on $Z/(p^j)$ over $K(n)_*$. The Hopf algebra, $K(n)_*\mathbf{K}_1$, for the first Eilenberg-MacLane space is most important. It is computed using the Gysin sequence.

THEOREM 5.7. *Let p be an odd prime, then $K(n)_*\mathbf{K}_1$ is free over $K(n)_*$ on elements $a_i \in K(n)_{2i}\mathbf{K}_1$, $0 \leq i < p^{nj}$, with coproduct*

$$\psi(a_k) = \sum_{i=0}^{k} a_i \otimes a_{k-i}$$

and algebra structure given by the usual divided power structure with the relations $a^{*p}{}_{(n+i-1)} = v_n^{p^i} a_{(i)}$, $a_{(i)} = a_{p^i}$. □

For $I = (i_1, i_2, \ldots, i_q)$, $0 \leq i_k < nj$, we define

$$a_I = a_{(i_1)} \circ a_{(i_2)} \circ \cdots \circ a_{(i_q)}$$

in $K(n)_* \mathbf{K}_q$. Our description begins with:

LEMMA 11.2

(a) $a_{(i)} \circ a_{(k)} = -a_{(k)} \circ a_{(i)}$,
(b) $a_{(i)} \circ a_{(i)} = 0$,
(c) $a_{(i)} \circ a_{(k)} = 0$ *if* $i < n$ *and* $k < n(j-1)$
$= v_n^m a_{(i-n)} \circ a_{(k+n)}$ *if* $n \leq i$ *and* $k < n(j-1)$
some $m \in Z$. □

It follows immediately that any a_J can be rearranged to be zero or some power of v_n times an a_I with

$$0 \leq \bar{i}_1 < \bar{i}_2 < \cdots < \bar{i}_q < n,$$
$$0 \leq i_1 < i_2 < \cdots < i_q < nj,$$

and $n(j-1) < i_2$, where

$$\bar{i} \equiv i \bmod(n) \qquad \text{with } 0 \leq \bar{i} < n.$$

In fact, these a_I are all non-zero and distinct (11.1(b)) and their p-th powers follow from $K(n)_* \mathbf{K}_1$ and Hopf ring properties. They are (11.1(c)):

$$a_I^{*p} = 0 \qquad \text{if } i_1 < n-1,\ i_q < nj-1$$
$$a_I^{*p} = v_n^m a_{(i_1-n+1, i_2+1, \ldots, i_q+1)} \qquad \text{if } n-1 \leq i_1,\ i_q < nj-1;$$
$$a_I^{*p} = (-1)^{q-1} v_n^m a_{(n[i_1/n], n(j-1)+\bar{i}_1+1, \ldots, i_{q-1}+1)}$$
$$\text{if } i_q = nj-1, \text{ some } m \in Z.$$

Not only are all of the above a_I's non-trivial but they are in 1-1 correspondence with the $j\binom{n}{q}$ various p-th powers (11.1(b)) of the generators in what follows. Let $T_k^R(x) = R[x]/(x^{p^k})$. The $n = 1$ case was done by Anderson and Hodgkin [2].

THEOREM 11.1(a). *Let p be an odd prime and $\mathbf{K}_q = \mathbf{K}(Z/(p^j), q)$, $j > 0$ then as algebras*

$$K(n)_*\mathbf{K}_q \simeq \otimes_I T_{\rho(I)}^{K(n)*}(a_I), \qquad 0 < q < n,$$
$$n(j-1) < i_1 < i_2 < \cdots < i_q < nj$$

where $\rho(I) = mn + \bar{\hat{i}}_c + 1$ with m and $\bar{\hat{i}}_c$ defined as follows. Given I as above, let $\hat{I} = (\hat{i}_1, \ldots, \hat{i}_{n-q})$ with $\{\bar{i}_k\} \cup \{\bar{\hat{i}}_k\} = \{0, 1, \ldots, n-1\}$, $\hat{i}_1 < \cdots < \hat{i}_{n-q}$. Let $j - 1 = m(n-q) + c - 1$, $0 < c \le n - q$.

$$K(n)_*\mathbf{K}_n \simeq \otimes_I K(n)_*[a_I]/(a_I^{*p} - (-1)^{q-1}v_n^{c(I)}a_I).$$
$$c(I) = p^{n(j-1)} - (p-1)(p^{n(j-2)} + p^{n(j-3)} + \cdots + p^{nk})$$
$$I = (nk, n(j-1)+1, \ldots, nj-1),\ 0 \le k < j;$$
$$K(n)_*\mathbf{K}_q \simeq K(n)_*, \qquad q > n. \qquad \square$$

The coalgebra structure follows from this, the Hopf ring properties and 5.7, (11.1(e) and (f)). From here it is easy to prove the elegant description of this result.

COROLLARY 11.3. *The Hopf ring $K(n)_*\mathbf{K}_*$ is the free $K(n)_*[Z/(p^j)]$ Hopf ring on the Hopf algebra $K(n)_*\mathbf{K}_1$.*

The standard reduction $Z/(p^{j+1}) \to Z/(p^j)$ induces a map

$$\beta_*: K(n)_*\mathbf{K}(Z/(p^{j+1}), q) \to K(n)_*\mathbf{K}(Z/(p^j), q)$$

which is completely described by

$$\beta_*(a_I) = v_n{}^m a_{(i_1-n, i_2-n, \ldots, i_q-n)}, \qquad \text{some } m \in Z.$$

This as well as the following is Proposition 11.4. The inclusion $Z/(p^j) \hookrightarrow Z/(p^{j+1})$ induces a corresponding map α_* which is completely described by

$$\alpha_*(a_I) = v_n{}^m a_{(i_1, i_2+n, \ldots, i_q+n)}, \qquad \text{some } m \in Z.$$

Thus β_* is a surjection and α_* is an injection (13.1).

COROLLARY 12.2. *For $q > 0$,*

$$\lim_{\overleftarrow{j}} K(n)_* \mathbf{K}(Z/(p^j), q) \simeq K(n)_* \mathbf{K}(Z, q+1). \qquad \square$$

The Morava K-theory of the Eilenberg-MacLane spaces for other finitely generated abelain groups is furnished by the Künneth isomorphism.

The main geometric corollary of this work is a proof of the Conner-Floyd conjecture.

Let $\mathrm{MSO}_*(X)$ be the oriented bordism of X. This is a module over MSO_*. Conner and Floyd give manifolds $M^{2(p^k-1)}$ which are Milnor basis-elements of $\mathrm{MSO}_*/(p)$.

THEOREM 10.1. (The Conner-Floyd Conjecture [6, p. 146]. *Let p be an odd prime, then for the obvious map*

$$\gamma_n : \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ copies}} \to \underbrace{BZ/(p) \times \cdots \times BZ/(p)}_{n \text{ copies}}$$

representing an element

$$\gamma_n \in MSO_n(\underbrace{BZ/(p) \times \cdots \times BZ/(p)}_{n \text{ copies}}),$$

the annihilator ideal

$$A(\gamma_n) = (p, M^{2(p-1)}, \ldots, M^{2(p^{n-1}-1)}) \subset MSO_*. \qquad \square$$

Conner and Floyd showed that $A(\gamma_n) \supset (p, M^{2(p-1)}, \ldots, M^{2(p^{n-1}-1)})$. The difficulty of computing bordism groups is well illustrated by the number of years it took to prove this limited result.

The Conner-Floyd conjecture is a p-primary problem so it can quickly be reduced to a $\mathrm{BP}_*(—)$ question. Furthermore a comparison with $\mathrm{BP}_*\mathbf{K}_n$ is sufficient for the solution. Let $v_k^{-1}\mathrm{BP}_*(—)$ be the homology theory obtained by inverting v_k. The Morava K-theories of Eilenberg-MacLane spaces and the Morava structure theorem can then be used to make the following computation, part (b) of which suffices for the proof of the Conner-Floyd conjecture.

THEOREM 10.5. *Let $v_0 = p$ be an odd prime.*

(a) $v_k^{-1}\widetilde{BP}_*(\mathbf{K}(Z/(p), n)) = 0, \qquad 0 \le k < n.$

(b) *As a BP_** module,

$$v_n^{-1}\,\widetilde{BP}_*(\mathbf{K}(Z/(p), n)) \simeq \oplus_{i=1}^{p-1} M_0{}^n(i)$$

where $M_0{}^n(i) \equiv M_0{}^n = v_n{}^{-1}N_0{}^m$ and $N_0{}^m$ is defined inductively by $N_0{}^0 \equiv BP_*$ and $0 \to N_0{}^{n-1} \to v_{n-1}{}^{-1}N_0{}^{n-1} \to N_0{}^n \to 0$, with $v_0{}^{-1}v_1{}^{-1} \cdots v_{n-1}{}^{-1} \in M_0{}^n(i)$ in degree $n + 2(i-1)(1 + p^1 + \cdots + p^{n-1})$ and $\rho(\iota_n) = v_0{}^{-1}v_1{}^{-1} \cdots v_{n-1}{}^{-1} \in M_0{}^n(1)$ where $\iota_n \in BP_n(\mathbf{K}(Z/(p), n))$ is the fundamental class and ρ is the reduction $BP_*(-) \to v_n{}^{-1}BP_*(-)$. □

To compute $K(n)_*\mathbf{K}_*$ we use the bar spectral sequence enriched with a Hopf ring related structure as follows. We know that $\mathbf{K}_{i+1}$ is $B\mathbf{K}_i$, the bar construction on $\mathbf{K}_i$. The bar spectral sequence comes from the bar filtration

$$pt = B_0\mathbf{K}_i \subset \cdots \subset B_s\mathbf{K}_i \subset \cdots \subset B\mathbf{K}_i \equiv \mathbf{K}_{i+1}.$$

We show, Corollary 1.9, that the cup product pairing

$$\mathbf{K}_{i+1} \wedge \mathbf{K}_m \to \mathbf{K}_{i+1+m}$$

factors as

$$B_s\mathbf{K}_i \wedge \mathbf{K}_m \to B_s\mathbf{K}_{i+m}.$$

Let $E^r_{**}(K(n)_*\mathbf{K}_i) \Rightarrow K(n)_*\mathbf{K}_{i+1}$ be the bar spectral sequence coming from the above filtration. The factoring just given implies the next theorem.

THEOREM 2.2. *There are maps*

$$\circ : E^r_{s,*}(K(n)_*\mathbf{K}_i) \otimes_{K(n)_*} K(n)_*(\mathbf{K}_m) \to E^r_{s,*}(K(n)_*\mathbf{K}_{i+m})$$

with $d^r(x \circ y) = d^r(x) \circ y$. The E^∞_{**} term of this map is compatible with the map

$$\circ: K(n)_*(\mathbf{K}_{i+1}) \otimes_{K(n)_*} K(n)_*(\mathbf{K}_m) \to K(n)_*(\mathbf{K}_{i+m+1})$$

coming from the cup product pairing $\mathbf{K}_{i+1} \wedge \mathbf{K}_m \to \mathbf{K}_{i+m+1}$. □

We need a more refined description of this pairing for it to be an effective computational tool. We show that the map

$$\begin{array}{ccc}
(B_s\mathbf{K}_i/B_{s-1}K_i) \wedge K_m & \longrightarrow & B_s\mathbf{K}_{i+m}/B_{s-1}\mathbf{K}_{i+m} \\
\simeq \downarrow & & \simeq \downarrow \\
(S^s \wedge \underbrace{\mathbf{K}_i \wedge \cdots \wedge \mathbf{K}_i) \wedge K_m}_{s\text{-copies}} & \longrightarrow & S^s \wedge \underbrace{\mathbf{K}_{i+m} \wedge \cdots \wedge \mathbf{K}_{i+m}}_{s\text{-copies}}
\end{array}$$

is induced in the obvious way by $\mathbf{K}_i \wedge \mathbf{K}_m \to \mathbf{K}_{i+m}$. Thus since

$$E^1_{s,*}(K(n)_*\mathbf{K}_i) \simeq \widetilde{K(n)}_*(S^s) \otimes^s \widetilde{K(n)}_*\mathbf{K}_i$$

the map for $r = 1$ of 2.2 is given by

$$(y_1 \otimes \cdots \otimes y_s) \circ x = \Sigma \pm (y_1 \circ x^{(1)}) \otimes \cdots \otimes (y_s \circ x^{(s)})$$

where $\tilde{\psi}(s) = \Sigma\, x^{(1)} \otimes \cdots \otimes x^{(s)}$ is the iterated reduced coproduct for $x \in K(n)_*\mathbf{K}_m$. This allows us to carry out explicit computations inductively.

The paper is divided into thirteen sections.

1. The multiplicative pairing for Eilenberg-MacLane spectra
2. A bar spectral sequence of Hopf rings
3. Hopf ring review
4. Morava's extraordinary K-theories
5. $K(n)_*\mathbf{K}_1$
6. Tor and differentials
7. The Frobenius and Verschiebung maps
8. The spectral sequence for $\mathbf{K}_1$
9. The mod p Eilenberg-MacLane spaces
10. The Conner-Floyd conjecture
11. The mod p^j Eilenberg-MacLane spaces
12. The integral Eilenberg-MacLane spaces
13. Exact sequences, $K(n)^*(-)$, and the Johnson question

The mod p^j case, $j > 1$, is proven by induction on j and is considerably more complicated than the mod p case. The mod p case demonstrates all of the important techniques, grounds the induction, and is sufficient for the Conner-Floyd conjecture. Hence, to avoid the clutter, we have separated the mod p case out in Section 9 and only sketched the p^j, $j > 1$, cases in Section 11.

1. The multiplicative pairing for Eilenberg-MacLane spectra. In this section we fix the notation

$$\mathbf{K}_* = \{\mathbf{K}_i\}_{i\geq 0} \equiv \{\mathbf{K}(Z/(p^j), i)\}_{i\geq 0} = \mathbf{K}(Z/(p^j), *),$$

the Eilenberg-MacLane spectrum for the ring $Z/(p^j)$. Two characterizations for $\mathbf{K}_i$ up to homotopy are

$$\pi_n \mathbf{K}_i \simeq \begin{cases} Z/(p^j) & \text{if } n = i \\ 0 & \text{if } n \neq i \end{cases} \tag{1.1}$$

and

$$H^i(X; Z/(p^j)) \simeq [X, \mathbf{K}_i] \qquad \text{naturally for all } X. \tag{1.2}$$

Cup product is induced by a map

$$\circ = \circ_{i,m} \colon \mathbf{K}_i \wedge \mathbf{K}_m \to \mathbf{K}_{i+m} \tag{1.3}$$

Several explicit constructions of this map have been given over the last 25 years. We will give a very simple construction which we believe to be new. Although it can be readily extracted from some previous general constructions we will do it from scratch for simplicity. It has a very close link with the bar spectral sequence. This, together with the Hopf ring point of view, allows us to exploit the pairing in a strong computational way.

Milgram [18] has given an especially convenient form of the bar construction for abelian H-spaces such as $\mathbf{K}_i$. In particular, $\mathbf{K}_{i+1}$ can be constructed inductively from $\mathbf{K}_i$ by

$$\mathbf{K}_{i+1} \equiv BK_i \equiv \coprod_{n\geq 0} \sigma^n \times \mathbf{K}_i^n/\sim \tag{1.4}$$

where σ^n is the standard n-simplex, X^n denotes the product of n copies of X, $\sim$ means there are identifications to be made, and $\coprod$ is disjoint union. We will give the identifications explicitly later.

The map 1.3 is constructed inductively on i. Assuming $\circ_{i,m}$ has been defined we will define $\circ_{i+1,m}$ by replacing $\mathbf{K}_{i+1}$ and $\mathbf{K}_{i+m+1}$ with their bar constructions:

$$\left\{\coprod_{n\geq 0} \sigma^n \times \mathbf{K}_i^n/\sim\right\} \wedge \mathbf{K}_m \to \left\{\coprod_{n\geq 0} \sigma^n \times \mathbf{K}^n{}_{i+m}/\sim\right\}. \tag{1.5}$$

Let $t \in \sigma^n$, $x = (x_1, \ldots, x_n) \in \mathbf{K}_i^n$ and $y \in \mathbf{K}_m$. The image of $x_s \wedge y \in \mathbf{K}_i \wedge \mathbf{K}_m$ under the map 1.3 is denoted by $x_s \circ y$ and $x \circ y$ means $(x_1 \circ y, \ldots, x_n \circ y)$. We define 1.5 by

$$\{(t, x)\} \circ y = \{(t, x \circ y)\}. \tag{1.6}$$

The $i = 0$ case of 1.3 is quite easy.

THEOREM 1.7. *The above construction is well defined and gives the cup product pairing* $\circ\colon \mathbf{K}_{i+1} \wedge \mathbf{K}_m \to \mathbf{K}_{i+m+1}$. □

A similar result holds for a much more general pairing of spectra and is presented in [31]. Before we prove 1.7 we want to observe some corollaries. The bar filtration is given by

$$pt = B_0\mathbf{K}_i \subset \cdots B_{s-1}\mathbf{K}_i \subset B_s\mathbf{K}_i \subset \cdots \subset B\mathbf{K}_i \equiv \mathbf{K}_{i+1} \tag{1.8}$$

where $B_s\mathbf{K}_i$ is the subspace generated by $\coprod_{s\geq n\geq 0} \sigma^n \times \mathbf{K}_i^n$. Working Hopf rings into the bar spectral sequence will follow from:

COROLLARY 1.9. *The cup product map* 1.5 *factors as follows:*

$$\begin{tikzcd} B_s\mathbf{K}_i \wedge \mathbf{K}_m \arrow[r, dashed] \arrow[d] & B_s\mathbf{K}_{i+m} \arrow[d] \\ \mathbf{K}_{i+1} \wedge \mathbf{K}_m \arrow[r, "\circ"] & \mathbf{K}_{i+m+1}. \end{tikzcd}$$ □

This is obvious from 1.6. This result can be proven easily using the simplicial model and this is enough to construct the spectral sequence of section 2. However, the explicit construction 1.6 is necessary for computations. For those purposes we need the standard:

LEMMA 1.10. *$B_s\mathbf{K}_i/B_{s-1}\mathbf{K}_i \simeq S^s \wedge \mathbf{K}_i^{(s)}$ where $X^{(s)}$ denotes the smash product of X with itself s times.* □

We will prove this later in the section. Combining 1.9 and 1.10 we have a map induced by $\circ$

$$(S^n \wedge \mathbf{K}_i^{(n)}) \wedge \mathbf{K}_m \to S^n \wedge \mathbf{K}_{i+m}^{(n)}. \tag{1.11}$$

This map is given by 1.6. The computations of our transpotence formulas need only the $n = 2$ case. With this in mind we record this map in a fashion accessible to homology calculations.

PROPOSITION 1.12. *The map 1.11 for $n = 2$ fits in the following commuting diagram*

$$\begin{tikzcd}
S^2 \times \mathbf{K}_i \times \mathbf{K}_i \times \mathbf{K}_m \arrow[r, "Id \times \Delta"] \arrow[ddd] & S^2 \times \mathbf{K}_i \times \mathbf{K}_i \times \mathbf{K}_m \times \mathbf{K}_m \arrow[d, "Id \times T \times Id"] \\
 & S^2 \times \mathbf{K}_i \times \mathbf{K}_m \times \mathbf{K}_i \times \mathbf{K}_m \arrow[d, "Id \times \circ \times \circ"] \\
 & S^2 \times \mathbf{K}_{i+m} \times \mathbf{K}_{i+m} \arrow[d] \\
(S^2 \wedge \mathbf{K}_i \wedge \mathbf{K}_i) \wedge \mathbf{K}_m \arrow[r] & S^2 \wedge \mathbf{K}_{i+m} \wedge \mathbf{K}_{i+m}
\end{tikzcd}$$

where Δ is the diagonal on $\mathbf{K}_m$ and T interchanges the two factors. □

The proof of 1.12 is immediate from 1.6.

We recall Milgram's construction [18]. Let σ^n denote the Euclidean n-simplex, i.e., all points $t = (t_1, \ldots, t_n) \in R^n$ with $0 \leq t_1 \leq \cdots \leq t_n \leq 1$. He inductively constructs $\mathbf{K}_m$ and its abelian H-space structure map $*_m = * : \mathbf{K}_m \times \mathbf{K}_m \to \mathbf{K}_m$. Our notation for the product of two points $x, y \in \mathbf{K}_m$ is $x * y$. The trivial case $\mathbf{K}_0$ is $Z/(p^j)$ with the discrete topology and $*$ is simply the group operation. Inductively assuming that $\mathbf{K}_{m-1}$ and the corresponding $*$ have been constructed, let $\mathbf{K}_m$ be the quotient of $\coprod_{n \geq 0} \sigma^n \times \mathbf{K}^n{}_{m-1}$ by the relations

$$
\begin{aligned}
(t, x) &\sim (t_2, \ldots, t_n; x_2, \ldots, x_n) && \text{if } t_1 = 0, \qquad (1.13)\\
&\sim (t_1, \ldots, t_{n-1}; x_1, \ldots, x_{n-1}) && \text{if } t_n = 1\\
& && \text{or } x_n = \cdot,\\
&\sim (t_1, \ldots, \hat{t}_q, \ldots t_n; x_1, \ldots, x_{q-1}, x_q * x_{q+1}, x_{q+2}, \ldots, x_n)\\
& && \text{if } 0 \leq q < n \text{ with } t_q = t_{q+1} \text{ or } x_q = \cdot
\end{aligned}
$$

where $\cdot$ is the basepoint, $\hat{}$ denotes deletion and the other notation is as before. The basepoint of $\mathbf{K}_m$ is $(0, \ldots, 0; \cdot, \ldots, \cdot)$.

Milgram defines $*$ by

$$
\begin{aligned}
(t; x) * (t_{n+1}, \ldots, t_{n+k}; x_{n+1}, \ldots, x_{n+k})\\
= (t_{\tau(1)}, \ldots, t_{\tau(n+k)}; x_{\tau(1)}, \ldots, x_{\tau(n+k)}) \qquad (1.14)
\end{aligned}
$$

where τ is any element of the summetric group on $m + n$ letters such that $t_{\tau(i)} \leq t_{\tau(i+1)}$.

LEMMA 1.15 (Milgram [18]). *The space $\mathbf{K}_m$ constructed above is the Eilenberg-MacLane space $\mathbf{K}(Z/(p^j), m)$ and $*$ makes $\mathbf{K}_m$ into an abelian topological group with identity element $\cdot$.* □

We can now begin our proofs.

Proof of Lemma 1.10. The points of $B_s\mathbf{K}_m$ which are identified to those in $B_{s-1}\mathbf{K}_m$ are those with t in a face of σ^s, i.e., $t_1 = 0$, $t_i = t_{i+1}$ or $t_s = 1$ or those with some $x_k = \cdot$. Collapsing out $\sigma^s \times \mathbf{K}_i^s$ by these points gives the result. □

In the rest of the section we prove Theorem 1.7. It is necessary to start our induction first.

LEMMA 1.16. *The map $\circ$: $\mathbf{K}_0 \wedge \mathbf{K}_m \to \mathbf{K}_m$ is given by $(q) \circ x = x^{*q} = x * \cdots * x$ q times where $q \in \mathbf{K}_0 = Z/(p^j)$.* □

Proof. The map $x \to x^{*q}$ is given by $\mathbf{K}_m \to \mathbf{K}_m{}^q \to \mathbf{K}_m$ where the first map is the diagonal and the second is the iterated $*$ product. This map multiplies $\pi_m(\mathbf{K}_m) \simeq Z/(p^j)$ by q which is just what $\circ$ should do restricted to $(q) \times \mathbf{K}_m \simeq \mathbf{K}_m$. To show this map factors through the smash product the only problem is to show that $x^{*p^j} = \cdot$. Represent x

by (t, y) with $y_i \in \mathbf{K}_{m-1}$ and assume inductively that $y_i{*}^{p^j} = \cdot$. This induction is trivial to begin. Then

$$\begin{aligned} x^{*p^j} &= (t, y)^{*p^j} \\ &= (t_1, \ldots, t_1, \ldots, t_n, \ldots, t_n; y_1, \ldots, y_1, \ldots, y_n, \ldots, y_n) \end{aligned}$$

where each coordinate is repeated p^j times. The identifications in $\mathbf{K}_m$ imply that this point is $(t; y_1{*}^{p^j}, \ldots, y_n{*}^{p^j})$. By induction this is $(t; \cdot, \ldots, \cdot)$ which is $\cdot$.

Proof of Theorem 1.7. We prove our result by induction on i. Assume we have proved 1.7 for $\mathbf{K}_i \wedge \mathbf{K}_m \to \mathbf{K}_{i+m}$ with 1.16 beginning our induction. We need it to satisfy:

$$(z_1 * z_2) \circ y = (z_1 \circ y) * (z_2 \circ y). \tag{1.17}$$

For $i = 0$, $z_i = q_i \in Z/(p^j) = \mathbf{K}_0$. So $(q_1 * q_2) \circ y = (q_1 + q_2) \circ y = y^{*(q_1+q_2)} = y^{*q_1} * y^{*q_2} = (q_1 \circ y) * (q_2 \circ y)$. For $i > 0$ we have:

$$\begin{aligned} [z_1 * z_2] \circ y &= [(t, x) * (t_{n+1}, \ldots, t_{n+k}; x_{n+1}, \ldots, x_{n+k})] \circ y \\ &= (t_{\tau(1)}, \ldots, t_{\tau(n+k)}; x_{\tau(1)}, \ldots, x_{(n+k)}) \circ y \qquad \text{by } 1.14 \\ &= (t_{\tau(1)}, \ldots, t_{\tau(n+k)}; x_{\tau(1)} \circ y, \ldots, x_{\tau(n+k)} \circ y) \qquad \text{by induction hypothesis and } 1.6 \\ &= (t; x \circ y) * (t_{n+1}, \ldots, t_{n+k}; x_{n+1} \circ y, \ldots, x_{n+k} \circ y) \qquad \text{by } 1.14 \\ &= (z_1 \circ y) * (z_2 \circ y). \end{aligned}$$

We now show that 1.6 gives a well defined map $\mathbf{K}_{i+1} \wedge \mathbf{K}_m \to \mathbf{K}_{i+m+1}$. We must show that the relations 1.13 are preserved. We will show the main case and leave the exceptional cases to the reader to check. Assume $0 \le q < n$ with $t_q = t_{q+1}$ or $x_q = \cdot$. Then

$$\begin{aligned} (t, x) \circ y &= (t, x \circ y) \\ &\sim (t_1, \ldots, \hat{t}_q, \ldots, t_n; x_1 \circ y, \ldots, (x_q \circ y) * (x_{q+1} \circ y), \ldots, x_n \circ y) \\ &= (t_1, \ldots, \hat{t}_q, \ldots, t_n; x_1 \circ y, \ldots, (x_q * x_{q+1}) \circ y, \ldots, x_n \circ y) \\ &= (t_1, \ldots, \hat{t}_q, \ldots, t_n; x_1, \ldots, x_q * x_{q+1}, \ldots, x_n) \circ y \end{aligned}$$

which is the necessary relation. That this map factors through the smash product is easy to verify using induction.

Our only remaining task is to show that this is the cup product pairing map. This follows by induction from the observation that $\circ$ commutes with suspension on the first factor since $S^1 \wedge \mathbf{K}_i = B_1\mathbf{K}_i$ and the following diagram commutes (1.9):

$$\begin{array}{ccc} S^1 \wedge \mathbf{K}_i \wedge \mathbf{K}_m & \longrightarrow & S^1 \wedge \mathbf{K}_{i+m} \\ \downarrow & & \downarrow \\ \mathbf{K}_{i+1} \wedge \mathbf{K}_m & \longrightarrow & \mathbf{K}_{i+m+1}. \end{array}$$ □

2. A bar spectral sequence of Hopf rings. In this section we show how the description of the cup product pairing given in Section One can be used in the bar spectral sequence. First we state the bar spectral sequence in the form we need it. Then we will develop its extra multiplicative properties. We continue the notation of Section One. Throughout this paper we assume $E_*(—)$ is a multiplicative homology theory with $E_*(\mathbf{K}_q{}^n) \simeq \otimes^n{}_{E_*} E_*\mathbf{K}_q$.

We denote the reduced homology theory for $E_*(—)$ by $\tilde{E}_*(—)$.

THEOREM 2.1. *If $E_*(—)$ is as above then there is a spectral sequence, $E^r_{*\,*}(\mathbf{K}_i)$ of E_*-Hopf algebras converging to $E_*\mathbf{K}_{i+1}$ which arises from the bar filtration* 1.8. *Then*

$$E^1{}_{s,*}(\mathbf{K}_i) \simeq \tilde{E}_*(\mathbf{K}_i{}^{(s)}) \simeq \otimes_{s\,E_*} \tilde{E}_*(K_i), \qquad s > 0,$$

is the bar resolution for

$$E^2{}_{s,t}(\mathbf{K}_i) \simeq \operatorname{Tor}_{s,t}{}^{E_*(\mathbf{K}_i)}(E_*, E_*) \equiv H_{s,t}E_*(\mathbf{K}_i).$$ □

The spectral sequence in this form has a complicated history. Milnor [19] had a spectral sequence of filtered spaces and Moore [20] had a spectral sequence with the E^2 term identified. Rothenberg and Steenrod [26] have both, and in such a way that generalized homologies could be used as initiated by Hodgkin [9]. Furthermore the appropriate Hopf algebra properties are proven in [26] in a way that allows generalized homologies to be used although they never mention them. Other papers which use such ideas with generalized homologies are [2], [29] and [32]. Certain aspects of the spectral sequence were only needed years after they could have been proven. In that sense it is fair to say that the above spectral sequence was completely understood in the mid-fifties by John Moore ([20] and unpublished).

This spectral sequence respects the $\circ$-product structure of $E_*\mathbf{K}_*$ in the sense of the following theorem which is a crucial tool in our calculations.

THEOREM 2.2. *For the spectral sequence of Theorem* 2.1 *there are maps*

$$\circ: E^r_{s,*}(\mathbf{K}_i) \otimes_{E_*} E_*(\mathbf{K}_m) \to E^r_{s,*}(\mathbf{K}_{i+m})$$

with $d^r(x \circ y) = (d^r x) \circ y$. *For the* $E^\infty_{*\,*}$*-term, this map is compatible with the map*

$$\circ: E_*(\mathbf{K}_{i+1}) \otimes_{E_*} E_*(K_m) \to E_*(\mathbf{K}_{i+m+1})$$

coming from $\circ: \mathbf{K}_{i+1} \wedge \mathbf{K}_m \to \mathbf{K}_{i+m+1}$. □

Proof. The spectral sequence of Theorem 3.1 is a spectral sequence of filtered spaces arising from the bar filtration. Hence the result is immediate from Corollary 1.9. □

This result has been generalized to all spectra pairings by R. W. Thomason and the second author [31]. Theorem 2.2 needs only 1.9 to prove it. This could be done simplicially. However, for computational purposes we need the explicit map given by 1.11 and 1.6 on $E^1_{*,*}$. We make precise the only case we need for our computations.

PROPOSITION 2.3. *For* $x \in E_*(\mathbf{K}_m)$, *let* $\tilde{\psi}(x) = \Sigma\, x' \otimes x''$ *where* $\tilde{\psi}$ *is the reduced coproduct. For* $y_1, y_2 \in \tilde{E}_*(\mathbf{K}_i)$, *the map*

$$\begin{array}{ccc}
\circ: E^1_{2,*}(\mathbf{K}_i) \otimes_{E_*} E_*(\mathbf{K}_m) & \longrightarrow & E^1_{2,*}(\mathbf{K}_{i+m}) \\
\simeq \big\downarrow & & \big\downarrow \simeq \\
(\tilde{E}_*(\mathbf{K}_i) \otimes_{E_*} \tilde{E}_*(\mathbf{K}_i)) \otimes_{E_*} E_*(\mathbf{K}_m) & \longrightarrow & (\tilde{E}_*(\mathbf{K}_{i+m}) \otimes_{E_*} \tilde{E}_*(\mathbf{K}_{i+m}))
\end{array}$$

of Theorem 2.2 *is given by*

$$(y_1 \otimes y_2) \circ x = \Sigma\, (-1)^{\deg y_2 \deg x'} (y_1 \circ x') \otimes (y_2 \circ x''). \qquad \square$$

Proof. This follows from Proposition 1.12 by applying E_* homology. □

3. Hopf ring review. The relevant properties of Hopf rings were developed in [25, Section 1], which was written with this paper in mind. We will assume the reader is familiar with that material, most of which will be used here.

A (graded) *Hopf ring* is a collection of graded R-Hopf algebras $H_*(*) = \{H_*(k)\}_{k \in Z}$ with a pairing $\circ: H_*(n) \otimes_R H_*(k) \to H_*(n+k)$, which turns $H_*(*)$ into a graded ring object in the category of graded R-coalgebras. We assume everything in sight is commutative and associative. Let $E_*(—)$ be a multiplicative homology theory and $\mathbf{G}_* = \{\mathbf{G}_k\}_{k \in Z}$ a multiplicative Ω-spectrum. If

$$E_*(\mathbf{G}_k \times \mathbf{G}_k) \simeq E_*\mathbf{G}_k \otimes_{E_*} E_*\mathbf{G}_k$$

for all k, then

$$E_*\mathbf{G}_* = \{E_*\mathbf{G}_k\}_{k \in Z}$$

is a Hopf ring. In this paper, since our $\mathbf{G}_* = \mathbf{K}_* = \{\mathbf{K}(Z/(p^j), k)\}_{k \geq 0}$, we will only look at nonnegatively graded Hopf rings, i.e., $H_*(*) = \{H_*(k)\}_{k \geq 0}$. We caution the reader that $H_*(k)$ need not be nonnegatively graded and in fact never will be in this paper.

If G^* is the coefficient ring for the cohomology theory coming from $\mathbf{G}_*$, the "ring-ring", $E_*[G^*]$, is a Hopf ring and there are maps of Hopf rings

$$E_*[G^*] \to E_*\mathbf{G}_* \to E_*[G^*]$$

whose composition is the identity. We say H is an *$R[S]$-Hopf ring* if there is a given map of Hopf rings $R[S] \to H$. In particular $E_*\mathbf{G}_*$ is an $E_*[G^*]$-Hopf ring. In the case of interest in this paper, $\mathbf{G}_* = \mathbf{K}_*$, G^* is just $Z/(p^j)$ concentrated in degree zero so $E_*[G^*]$ is just $E_*\mathbf{K}_0 = E_*[Z/(p^j)]$.

Also in [25] we defined a *free Hopf ring functor* which assigns an $R[S]$ Hopf ring to every collection of supplemented R coalgebras or R-Hopf algebras $C_*(*)$. This Hopf ring is generated as an R-module by all possible finite $*$ products of all possible finite $\circ$ products of elements of $R[S]$ and $C_*(*)$ and has only those relations which follow formally from the definition of a Hopf ring.

4. Morava's Extraordinary K-theories. We wish to apply our spectral sequence of Hopf rings to compute the generalized homology of the Eilenberg-MacLane spaces; $E_*\mathbf{K}_*$. The homology theories we intend to use are the extraordinary K-theories introduced by Jack Morava in his study of complex cobordism.

Recall that for every prime there is a Brown-Peterson spectrum BP and related homology theory $\mathrm{BP}_*(—)$. The references are [1], [4], and [22]. The coefficient ring, BP_*, is isomorphic to $Z_{(p)}[v_1, v_2, \ldots]$ where the degree of v_n is $2(p^n - 1)$. We collect the basic facts:

PROPOSITION 4.1. *Let p be an odd prime and let $n > 0$.*

(a) *There is a multiplicative homology theory, $K(n)_*(—)$, with coefficient ring $K(n)_* \simeq F_p[v_n, v_n^{-1}]$.*
(b) *There is a map of multiplicative homology theories*

$$BP_*(—) \to K(n)_*(—)$$

which takes v_n to v_n and v_i to zero for $i \neq n$, on the coefficient ring.
(c) *The exterior product induces a Künneth isomorphism:*

$$K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y) \xrightarrow{\simeq} K(n)_*(X \times Y). \qquad \square$$

The construction is dealt with in [11]. The multiplicative properties are proven in [21], [27] and [33]. In [33], the Künneth isomorphism is made explicit. It is automatic because $K(n)_*$ is a graded field. Note that (c) implies $K(n)_*(X)$ is always a coalgebra over $K(n)_*$ and if X is an H-space, then $K(n)_*(X)$ is actually always a Hopf algebra over $K(n)_*$. The K-theories exist at $p = 2$ but the nice multiplicative properties which are so essential to us do not exist. There are multiplications at $p = 2$, but they are never nicely commutative; see [33].

We call $K(n)_*(—)$ the "nth Morava K-theory." It is consistent to consider $K(0)_*(—)$ as rational homology. Morava has fit these K-theories into a beautiful structure theorem for complex cobordism. Morava's papers are not generally available. However, part of his theorem is presented in [11]. The theories $K(n)_*(-)$ are called K-theories because $K(1)_*(-)$ is one of the $p-1$ isomorphic summands of the mod p homology theory obtained from BU, and because $K(n)_*(-)$ is periodic of period $2(p^n-1)$. Furthermore, in Morava's structure theorem for complex cobordism the $K(n)_*(-)$ all arise inductively in the same fashion.

In an unpublished preprint [14], Margolis constructed the $p = 2$ connective versions of these spectra. His work goes through at p odd as well. He also gives a partial description of the operations. Morava always works with the periodic (non-connected) theories. He constructed them using the geometric techniques of Sullivan [30]; see [3]. Morava computes the operations for $K(n)_*(-)$; see [34] and [35]. Although we do not use the operations in this paper, they have been very useful in our study of stable homotopy [15], [16], [17], [23], and [24].

Since any graded $K(n)_*$ module is periodic, no information is lost by passing to the corresponding cyclically graded (over $Z/(2p^n-2)$) $\mathbf{F}_p$ vector space.

DEFINITION 4.2. *$\overline{K(n)}_{\bar{t}}(X) \equiv K(n)_t(X)$ where $\bar{t} \in Z/(2p^n-2)$ is the reduction of $t \in Z$.*

This is well defined because multiplication by v_n induces an isomorphism $K(n)_t(X) \simeq K(n)_{t+2(p^n-1)}(X)$ with inverse multiplication by v_n^{-1}. Another approach is to take the ring map

$$K(n)_* \to \mathbf{F}_p$$

given by taking v_n to 1. Then $\overline{K(n)}_*(X) \simeq \mathbf{F}_p \otimes_{K(n)_*} K(n)_*(X)$.

Since $K(n)_*(X)$ is always a free $K(n)_*$ module, it can be recovered from $\overline{K(n)}_*(X)$. The advantages of using $\overline{K(n)}_*(X)$ are two-fold. First, we avoid the necessity of keeping track of powers of v_n, and second, $\overline{K(n)}_* \simeq \mathbf{F}_p$ is a perfect field whereas $K(n)_* \simeq \mathbf{F}_p[v_n, v_n^{-1}]$ is not. This allows us to define the Verschiebung map for $\overline{K(n)}_*(X)$, but not for $K(n)_*(X)$. There are no difficulties in computing with spectral sequences and other gadgets in a cyclically graded context such as $\overline{K(n)}_*(-)$. We will use the same notation for elements in $K(n)_*(-)$ and their images in $\overline{K(n)}_*(-)$.

5. $K(n)_*\mathbf{K}_1$. In this section we compute $K(n)_*\mathbf{K}_1$ where $\mathbf{K}_1 = \mathbf{K}(Z/(p^j), 1)$. The case $\mathbf{K}(Z, 1) = S^1$ is trivial. Because of the problems with $p = 2$ in Section 4 we will always assume the prime p is odd. Our approach will be to study the fibration

$$\mathbf{K}_1 \xrightarrow{\delta} \mathbf{C}P^\infty \xrightarrow{p^j} \mathbf{C}P^\infty. \tag{5.1}$$

We begin with a description of $K(n)_*\mathbf{C}P^\infty$. The cohomology theory $K(n)*(—)$ inherits a complex orientation from $\mathrm{BP}*(—)$ by the map of 4.1(b). For such a theory we have the following lemma; see Section 3 of [25] and [1].

LEMMA 5.2. *Let E be a multiplicative module spectrum over MU (or BP). Then*

(a) $E^*(\mathbf{C}P^\infty) \simeq E^*[[x^E]]$ *the power series on* $x^E \in E^2\mathbf{C}P^\infty$ *over* E^*.
(b) $E^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) \simeq E^*(\mathbf{C}P^\infty) \hat{\otimes}_{E^*} E^*(\mathbf{C}P^\infty)$.
(c) $E_*(\mathbf{C}P^\infty)$ *is* E_* *free on* $\beta_i \in E_{2i}(\mathbf{C}P^\infty)$, $i \geq 0$, *dual to* x^i, *i.e.*, $\langle x^i, \beta_j \rangle = \delta_{ij}$.
(d) $E_*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) \simeq E_*(\mathbf{C}P^\infty) \otimes_{E_*} E_*(\mathbf{C}P^\infty)$.
(e) *The diagonal* $\mathbf{C}P^\infty \to \mathbf{C}P^\infty \times \mathbf{C}P^\infty$ *induces a coproduct* ψ *on* $E_*(\mathbf{C}P^\infty)$ *with* $\psi(\beta_n) = \Sigma^n{}_{i=0}\, \beta_i \otimes \beta_{n-i}$.
(f) *The H-space product* m: $\mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$ *induces a coproduct* m^* *on* $E^*(\mathbf{C}P^\infty)$ *with* $m^*(x) = \Sigma_{i,j\geq 0}\, a_{ij} x^i \otimes x^j$, *and* $a_{ij} \in E^{-2(i+j-1)} = E_{2(i+j-1)}$.
(g) $F(y, z) = y +_F z = \Sigma_{i,j\geq 0}\, a_{ij} y^i z^j$ *is a commutative associative formal group law over* E^*, *i.e.*,

$$y +_F z = z +_F y, \qquad y +_F 0 = y$$

and

$$y +_F (z +_F w) = (y +_F z) +_F w. \qquad \square$$

Definition 5.3. Let $[1]_F(s) = s$ and inductively let $[n]_F(s) = [n-1]_F(s) +_F s$.

THEOREM 5.4. (3.4 and 3.6 of [25]). *Let* $\beta(r) = \Sigma_{i\geq 0}\, \beta_i r^i$. *Let the product,* $*$, *be that induced by the H-space structure of* $\mathbf{C}P^\infty$. *In the power series ring* $E_*(\mathbf{C}P^\infty)[[s, t]]$:

(a) $\beta(s) * \beta(t) = \beta(s +_F t)$ *and*
(b) $\beta(s)^{*m} = \beta([m]_F(s))$. $\square$

The above theorem gives a complete description of the multiplicative structure of $E_*(\mathbf{C}P^\infty)$ in terms of the formal group law. Specializing to the case $E = K(n)$ we can make this description very explicit.

THEOREM 5.5. $[p]_F(s) = v_n s^{p^n}$ *for* $F = F_{K(n)}$. □

Proof. In 2.11 (b) of [25] we show that for Hazewinkel's generators of BP_* we have

$$[p]_{F_{\mathrm{BP}}}(s) = \sum_{n>0}^{F_{\mathrm{BP}}} v_n s^{p^n} \qquad \operatorname{mod}(p).$$

This reduces by the map of 4.1(b) to $[p]_{F_{K(n)}}(s) = v_n s^{p^n}$. □

THEOREM 5.6. *As an algebra* $K(n)_*(\mathbf{C}P^\infty)$ $(n > 0)$ *is generated by the elements* $\beta_{(i)} = \beta_{p^i}$ *with relations* $\beta^{*p}{}_{(n+i-1)} = v_n{}^{p^i}\beta_{(i)}$, *where* $\beta_{(i)} = 0$ *for* $i < 0$. □

Proof. The assertion that the $\beta_{(i)}$ generate is actually true for $E_*(\mathbf{C}P^\infty)$ if E_* is a $Z_{(p)}$ algebra. We will show that β_k is decomposable if k is not a power of p. Write $k = ap^j$ where $a \neq 0(p)$ and also $a \neq 1$. If we work modulo the β_i with $i < k$ and equate the coefficients of $s^{(a-1)p^i}t^{p^i}$ in 5.4(a) we obtain

$$0 = ((a-1)p^i,\ p^i)\beta_k.$$

Since the binomial coefficient $((a-1)p^i,\ p^i) \not\equiv 0(p)$ the result follows. To obtain the relations, we use 5.4(b) with $m = p$ and Theorem 5.5. We have

$$\beta(s)^{*p} = \beta([p]_F(s)) = \beta(v_n s^{p^n}).$$

Equating coefficients of $s^{p^{n+i}}$ we have

$$\beta^{*p}{}_{(n+i-1)} = v_n{}^{p^i}\beta_{(i)}.$$ □

We are now ready to describe and compute $K(n)_*(\mathbf{K}_1)$ as a Hopf algebra.

THEOREM 5.7. *Let* $\mathbf{K}_1 = \mathbf{K}(Z/(p^j), 1)$

(a) *The map* δ: $\mathbf{K}_1 \to \mathbf{C}P^\infty$ *of* 5.1 *induces a Hopf algebra monomorphism*

$$\delta_*: K(n)_*(\mathbf{K}_1) \to K(n)_*(\mathbf{C}P^\infty).$$

(b) *As a* $K(n)_*$ *module,* $K(n)_*(\mathbf{K}_1')$ *is free on* $a_m \in K(n)_{2m}(\mathbf{K}_1)$, $0 \le m < p^{nj}$ *with* $\delta_*(a_m) = \beta_m$.

(c) *The coproduct* ψ *is given by*

$$\psi(a_m) = \sum_{i=0}^{m} a_i \otimes a_{m-i}.$$

(d) *The algebra is generated by* $a_{(i)} = a_{p^i}$ *for* $0 \le i < nj$ *with relations* $a^{*p}{}_{(n+i-1)} = v_n{}^{p^i} a_{(i)}$ *where* $a_{(i)} = 0$ *for* $i < 0$. □

Proof. The map δ is an H-space map so δ_* is a map of Hopf algebras. Once (a) and (b) have been established, (c) and (d) follow from the description of $K(n)_*(\mathbf{C}P^\infty)$ given in 5.2 and 5.6. The fibre of the map δ is S^1 so there is a Gysin sequence ([28])

$$\begin{tikzcd} \overline{K(n)_*}(\mathbf{K}_1) \arrow[r, "\delta_*"] & \overline{K(n)}_*(\mathbf{C}P^\infty) \arrow[r, "\Phi"] & \overline{K(n)}_{*-2}(\mathbf{C}P^\infty). \arrow[ll, bend left, "\partial"] \end{tikzcd}$$

The map Φ is $\Phi(y) = y \cap [p^j]_F(x)$. By iterating 5.5 we see $[p^j]_F(s) = s^{p^{nj}}$ (recalling that $v_n = 1$ in $\overline{K(n)}_*$). Thus, $\Phi(\beta_{nj+i}) = \beta_{nj+i} \cap x^{p^{nj}} = \beta_i$. The Gysin sequence is therefore a short exact sequence and the kernel of Φ is given by β_i, $i < nj$. The result follows. □

Remark 5.8. Notice that for $n > 1$, $K(n)_*(\mathbf{K}_1)$ is a truncated polynomial algebra on generators $a_{(i)}$, $n(j-1) < i < nj$.

We need to compare $K(n)_*(\mathbf{K}_1)$ for various j. Let

$$\alpha: \mathbf{K}(Z/(p^j), 1) \to \mathbf{K}(Z/(p^{j+1}), 1)$$

be the map which comes from the inclusion given by multiplication by p, $Z/(p^j) \hookrightarrow Z/(p^{j+1})$. For $K(n)_*(—)$, the induced map is denoted α_*. The standard reduction $Z/(p^{j+1}) \to Z/(p^j)$ induces a map β: $\mathbf{K}(Z/(p^{j+1}), 1) \to \mathbf{K}(Z/(p^j), 1)$.

LEMMA 5.9.

(a) $\alpha_*(a_i) = a_i$

(b) $\beta_*(a_{(n+i)}) = v_n{}^{p^i} a_{(i)}$ *where* $a_{(i)} = 0$ *if* $i < 0$. □

Proof. Let $\delta_j\colon \mathbf{K}(Z/(p^j),\,1) \to \mathbf{C}P^\infty$ be the map of 5.1. Then we have $\delta_{j+1}\alpha = \delta_j$ and (a) follows. To compute $\beta_*(a_{(n+i)})$, consider the commuting diagram

$$\begin{tikzcd}
\mathbf{K}(Z/(p^{j+1}),\,1) \arrow[r, "\delta_{j+1}"] \arrow[d, "\beta"'] & \mathbf{C}P^\infty \arrow[d, "p"] \\
\mathbf{K}(Z/(p^{j}),\,1) \arrow[r, "\delta_{j}"] & \mathbf{C}P^\infty
\end{tikzcd}$$

Then a routine argument shows that in $K(n)_*(\mathbf{C}P^\infty)$,

$$p_*\beta_{(n+i)} = \beta^{*p}{}_{(n+i-1)} = v_n{}^{p^i}\beta_{(i)}$$

by 5.7(d) so $\beta_*(a_{(n+i)}) = v_n{}^{p^i}a_{(i)}$. □

Let χ denote the conjugation on $K(n)_*$ Hopf algebras.

THEOREM 5.10.

(a) $\chi(\beta_i) = (-1)^i\beta_i$
(b) $\chi(\beta_{(i)}) = -\beta_{(i)}$
(c) $\chi(a_i) = (-1)^i a_i$
(d) $\chi(a_{(i)}) = -a_{(i)}$ □

Proof. Since p is odd (b), (c) and (d) follow from (a). We need $s +_F (-s) = 0$. It is enough to prove this for BP. There,

$$s +_F (-s) = \exp^{\mathrm{BP}}(\log^{\mathrm{BP}} s + \log^{\mathrm{BP}}(-s)).$$

For this see [1] or 3.10 or [25]. Now $\log^{\mathrm{BP}} s = \sum_{i\ge 0} m_i s^{p^i}$ so $\log^{\mathrm{BP}}(-s) = -\log^{\mathrm{BP}} s$ and $s +_F (-s) = \exp^{\mathrm{BP}}(0) = 0$. Thus by 5.4(a), $\beta(s)*\beta(-s) = \beta(s +_F (-s)) = \beta(0) = \beta_0$. By definition, $\chi(\beta(s))*\beta(s) = \beta_0$. This gives:

$$\chi(\beta(s)) = \chi(\beta(s))*\beta_0 = \chi(\beta(s))*\beta(s)*\beta(-s) = \beta_0*\beta(-s) = \beta(-s).$$

The result follows. □

6. Tor and differentials. In this section we compute various Tor groups that will be needed later. We also discuss the typical behavior of differentials in the spectral sequences we use.

We will be interested in computing

$$\mathrm{Tor}_{**}^{\overline{K(n)}_*\mathbf{K}_i}(\overline{K(n)}_*, \overline{K(n)}_*)$$

$$= \mathrm{Tor}_{**}^{\overline{K(n)}_*\mathbf{K}_i}(\mathbf{F}_p, \mathbf{F}_p) \equiv H_{**}\overline{K(n)}_*\mathbf{K}_i$$

This has a Hopf algebra structure but depends only on the algebra structure of $\overline{K(n)}_*\mathbf{K}_i$, not on the coalgebra structure. We recall the following standard fact.

LEMMA 6.1. *If A is a $\overline{K(n)}_* \simeq \mathbf{F}_p$ Hopf algebra and $A \simeq B \otimes_{\mathbf{F}_p} C$ as algebras (but not necessarily as Hopf algebras), then*

$$H_{**}A \simeq H_{**}B \otimes_{\mathbf{F}_p} H_{**}C$$

as Hopf algebras. □

This is just the Künneth theorem over a field.

We will have occasion to compute Tor for three different algebras which we now define. Recall that we are working with cyclically graded objects over $Z/(2p^n - 2)$.

$$T_k(x) = \mathbf{F}_p[x]/(x^{p^k}) \text{ where } x \text{ is of even degree;} \tag{6.2}$$

$$R_\pm(x) = \mathbf{F}_p[x]/(x^p \pm x) \tag{6.3}$$

where $\deg x$ is even with

$$p \deg x \equiv \deg x \, (2p^n - 2);$$

and

$$S(x_0, x_1, x_2, \ldots)$$

$$= \mathbf{F}_p[x_0, x_1, \ldots]/(x_0^p, x_1^p - x_0, \ldots, x_i^p - x_{i-1}, \ldots), \tag{6.4}$$

where $\deg x_0$ is even and $p \deg x_i \equiv \deg x_{i-1}(2p^n - 2)$.

Remark 6.5. Theorem 5.7 implies that $\overline{K(n)}_* \mathbf{K}(Z/(p^j), 1)$ is a tensor product of algebras of type $R_-(x)$ for $n = 1$ and $T_k(x)$ for $n > 1$. Theorem 5.6 implies that $\overline{K(n)}_* \mathbf{C}P^\infty$ is the tensor product of algebras of the form $R_-(x)$ for $n = 1$, and $S(x_0, x_1, \ldots)$ for $n > 1$.

Let $E(y)$ denote the exterior algebra on y and $\Gamma(y)$ the divided power Hopf algebra on y, i.e., $\Gamma(y)$ is free over $\gamma_i(y) \in \Gamma_{2i}(y)$ $(\gamma_1(y) = y)$ with coproduct

$$\psi(\gamma_n(y)) = \sum_{i=0}^{n} \gamma_{n-i}(y) \otimes \gamma_i(y)$$

and product

$$\gamma_i(y)\gamma_j(y) = (i, j)\gamma_{i+j}(y).$$

Thus $\Gamma(y)$ has algebra generators $\gamma_{p^i}(y)$.

The following is a standard calculation.

LEMMA 6.6.

$$H_{*\,*}T_k(x) = E(\sigma x) \otimes \Gamma(\phi(x^{p^{k-1}}))$$

where $\sigma(x) \in H_{1,\deg x}T_k(x)$ *is the "suspension" of* x *and* $\phi(x^{p^{k-1}}) \in H_{2,p^k \deg x}T_k(x)$ *is the "transpotence" of* $x^{p^{k-1}}$ [5]. *In the bar construction* σx *is represented by* $[x]$ *and* $\phi(x^{p^{k-1}})$ *is represented by* $[x^{(p-1)p^{k-1}} | x^{p^{k-1}}]$ *which is homologous to* $[x^{p^k-i} | x^i]$ *for* $0 < i < p^k$. □

LEMMA 6.7.

$$H_{*\,*}R_\pm(x) \simeq \mathbf{F}_p \qquad \square$$

Proof. To compute this Tor we construct a free $R_\pm(x)$ resolution of $\mathbf{F}_p$

$$0 \longleftarrow \mathbf{F}_p \xleftarrow{\epsilon} F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \longleftarrow \cdots$$

where each F_i is a free $R_\pm(x)$ module on a single generator f_i with $\epsilon(f_0) = 1$, $d_{2i+1}f_{2i+1} = xf_{2i}$ and $d_{2i+2}f_{2i+2} = (x^{p-1} \pm 1)f_{2i+1}$. Tensoring this resolution with $\mathbf{F}_p$ over $R_{\pm}(x)$ is easily seen to preserve exactness. $\square$

LEMMA 6.8. *$H_{**}S(x_0, x_1, \ldots) = \Gamma(\phi(x_0))$, where $\phi(x_0) \in H_{2,p \deg x_0}$, the transpotence of x_0, is represented by $[x_0{}^{p-1} | x_0]$ in the bar construction.* $\square$

Proof. The element $x_i \in S(x_0, x_1, \ldots)$ generates a subalgebra of the form $T_{i+1}(x_i)$ and $S(x_0, x_1, \ldots)$ is the direct limit of these subalgebras. Tor commutes with direct limits so $H_{**}S(x_0, x_1, \ldots) \simeq \lim_{\rightarrow} H_{**}T_{i+1}(x_i)$. By Lemma 6.6, $H_{**}T_{i+1}(x_i) \simeq E(\sigma x_i) \otimes \Gamma(\phi(x_0))$. The map in Tor induced by the inclusion $T_{i+1}(x_i) \hookrightarrow T_{i+2}(x_{i+1})$ sends σx_i to 0 and $\phi(x_0)$ to $\phi(x_0)$ so the result follows. $\square$

All of the differentials we must compute in spectral sequences are of the form in the following lemma.

LEMMA 6.9. *If*

$$E(x) \otimes \Gamma(y)$$

is a differential Hopf algebra with differential determined by $d(\gamma_{p^k}(y)) = x$, then the homology is the sub-Hopf algebra of $\Gamma(y)$ given by the $\gamma_i(y)$, $0 \le i < p^k$. $\square$

Proof. For degree reasons the $\gamma_i(y)$, $0 \le i < p^k$, are cycles. It will now suffice to show inductively that $d(\gamma_n(y)) = x\gamma_{n-p^k}(y)$. Since d is a differential of coalgebras we must have $(d \otimes 1 + 1 \otimes d)\psi = \psi d$. So inductively,

$$\begin{aligned}(d \otimes 1 + 1 \otimes d)\psi(\gamma_n(y)) &= (d \otimes 1 + 1 \otimes d) \textstyle\sum \gamma_{n-i}(y) \otimes \gamma_i(y) \\ &= d\gamma_n(y) \otimes 1 + 1 \otimes d\gamma_n(y) \\ &\quad + \textstyle\sum x\gamma_{n-i-p^k}(y) \otimes \gamma_i(y) \\ &\quad + \textstyle\sum \gamma_{n-i}(y) \otimes x\gamma_{i-p^k}(y)\end{aligned}$$

$d(\gamma_n(y)) = x\gamma_{n-p^k}(y)$ is the only element with the appropriate reduced coproduct. $\square$

The bar spectral sequence computations we make will usually have many different non-trivial differentials. However, all of our situations, are covered by the following description.

LEMMA 6.10. *Let $A = \otimes^i T_{k_i}(x_i)$ and let $H_{*\,*}A$ be the E^2 term of an homology spectral sequence of Hopf algebras. Define $y_i = \pm x_i^{p^{k_i-1}}$, so $H_{*\,*}A \simeq \otimes^i (E(\sigma x_i) \otimes \Gamma(\phi(y_i)))$. Assume further that the degrees of the x_i's are distinct, and also the degrees of the y_i's.*

(a) *Then the only possible non-trivial differentials are of the form*

$$d^{2p^{n_i}-1}\gamma_{p^{n_i}}(\phi(y_i)) = a\sigma(x_j)$$

where $a \neq 0$, some i, j and $n_i > 0$.

(b) *If there is a permutation τ, and n_i's such that*

$$d^{2p^{n_i}-1}\gamma_{p^{n_i}}(\phi(y_i)) = a_i\sigma(x_{\tau(i)})$$

for all i, $a_i \neq 0$, then the E^∞ term of the spectral sequence is the subalgebra of $H_{*\,*}A$ generated by the $\gamma_{p^j}(\phi(y_i))$, $0 \leq j < n_i$. □

Proof. Part (b) follows from Lemma 6.9 and (a). An element of lowest homological degree with a non-trivial differential acting on it must be a generator. Furthermore, the target must be a primitive. This is a standard Hopf algebra spectral sequence computation. For degree reasons the elements $\sigma(x_j)$ and $\phi(y_i)$ must be permanent cycles. Thus the first non-trivial differential must begin with $\gamma_{p^{n_i}}(\phi(y_i))$ for some y_i and $n_i > 0$. Since this has even total degree in the spectral sequence, the target of the differential must be an odd degree primitive. The only odd degree primitives are the $\sigma(x_j)$. Lemma 6.9 can now be used to compute the homology. Since the degrees of the $\sigma(x_j)$ are distinct, there can be no complications in this description. Moving on to the next differential, we see that our Hopf algebra is of the form of $H_{*\,*}A$ tensored with sub-Hopf algebras of Γ of the type in 6.9. The same argument applies again to this differential Hopf algebra. We only need to point out that the sub-Hopf algebra of Γ mentioned above is made up of permanent cycles for degree reasons. Now the argument proceeds as before. □

7. The Frobenius and Verschiebung maps. Let A be a bicommutative Hopf algebra over F_p for $p > 2$. The example we have in mind is of course $\overline{K(n)}_*\mathbf{K}_q$, which is cyclically graded. The *Frobenius map* $F: A \to A$ is defined by $F(x) = x^p$. Let $A^* = \mathrm{Hom}_{\mathbf{F}_p}(A, \mathbf{F}_p)$. We would like to define the *Verschiebung map* $V: A \to A$ as the dual of the Frobenius on A^*. However, if A does not have finite type, which is usually the case in our examples, A^* is not a Hopf algebra (it lacks a coproduct because $(A \otimes A)^*$ properly contains $A^* \otimes A^*$) and its dual properly contains A. This difficulty can be surmounted if A^* has a topology under which it is compact. In our case such a topology is given by the skeletal filtration since the skeleta are all finite. Then $(A \otimes A)^*$ is the completion of $A^* \otimes A^*$ and A is the continuous linear dual of A^*.

We will also consider the Verschiebung on $\overline{K(n)}^*\mathbf{K}_q$. This has a topology under which it is compact. Its continuous linear dual $\overline{K(n)}_*\mathbf{K}_q$ is a Hopf algebra in the usual sense so V can be defined as the dual of F as above.

LEMMA 7.1.

(a) *Suppose A is graded over Z or $Z/(m)$ with m even and prime to p. Then $F(A_n) \subset A_{pn}$ if n is even, and $F(A_n) = 0$ if n is odd. $V(A_n) \subset A_q$ if $n = 2pq$ and $V(A_n) = 0$ otherwise.*

(b) *With a shift of grading, V and F are Hopf algebra maps with $VF = FV$.*

(c) *If A is a Hopf ring,*

$$\begin{aligned} VF(x) &= [p] \circ x, \\ V(x \circ y) &= V(x) \circ V(y), \text{ and} \\ F(V(x) \circ y) &= x \circ F(y). \end{aligned}$$

(d) *For the coalgebra $\Gamma(x)$,*

$$\begin{aligned} V(\gamma_{pq}(x)) &= \gamma_q(x) \text{ and} \\ V(\gamma_q(x)) &= 0 \text{ if } p \text{ does not divide } q. \end{aligned}$$

(e) *For $a_{(i+1)} \in \overline{K(n)}_*\mathbf{K}_1$,*

$$\begin{aligned} V(a_{(i+1)}) &= a_{(i)}, \text{ and} \\ V(a_{(0)}) &= 0 \end{aligned} \qquad \square$$

Proof. Part (a) is trivial. Part (b) follows from the fact that the coproduct is an algebra map and the product map is a coalgebra map.

For (c) we compute $x \circ F(y) = x \circ y^{*p}$ with the Hopf ring distributive law, [25, 1.12(c)(vi)]. We have

$$x \circ y^{*p} = \sum (x' \circ y) * (x'' \circ y) * \cdots * (x^{(p)} \circ y).$$

Since we are working mod p in a bicommutative Hopf algebra, all of the terms in this sum will cancel except for those with $x' = x'' = \cdots = x^{(p)}$. It follows from the definition of V that the iterated coproduct of x is $V(x) \otimes \cdots \otimes V(x)$ plus asymmetric terms, so

$$\begin{aligned} x \circ F(y) = x \circ y^{*p} &= (V(x) \circ y) * (V(x) \circ y) * \cdots * (V(x) \circ y) \\ &= (V(x) \circ y)^{*p} = F(V(x) \circ y). \end{aligned}$$

If we set $y = [1]$ we get $FV(x) = x \circ [p] = [p] \circ x$. Finally, $V(x \circ y) = V(x) \circ V(y)$ because the $\circ$ product respects the coalgebra structure. Part (d) follows from the fact that the iterated coproduct of $\gamma_i(x)$ is

$$\sum_{i_1 + \cdots i_p = i} \gamma_{i_1}(x) \otimes \cdots \otimes \gamma_{i_p}(x) = \gamma_{i/p}(x) \otimes \cdots \otimes \gamma_{i/p}(x)$$

plus asymmetric terms. Part (e) now follows from (d) and the definition of $a_{(i+1)} = a_{p^i+1}$. □

8. The spectral sequence for $\mathbf{K}_1$. We have $\mathbf{K}_0 = Z/(p^j)$ and $\mathbf{K}_1 = \mathbf{K}(Z/(p^j), 1)$. Thus $K(n)_*\mathbf{K}_0 \simeq K(n)_*[Z/(p^j)]$, the "ring-ring" on $Z/(p^j)$ over $K(n)_*$. We use the computation of $K(n)_*\mathbf{K}_1$ in Section 5 to describe the behavior of the bar spectral sequence

$$E^r_{**}(\mathbf{K}_0) \Rightarrow \overline{K(n)}_*\mathbf{K}_1.$$

THEOREM 8.1. *In the bar spectral sequence* $E^r_{**}(\mathbf{K}_0) \Rightarrow \overline{K(n)}_*\mathbf{K}_1$,

(a) $E^2_{**}(\mathbf{K}_0) \simeq E(\sigma([1] - [0])) \otimes \Gamma(\phi(([p^{j-1}] - [0]) = ([1] - [0])^{p^{j-1}}))$;

(b) *the only nontrivial differential is given by*

$$d^{2p^{nj}-1}(\gamma_{p^{nj}}(\phi([p^{j-1}] - [0]))) = c\sigma([1] - [0])$$

where $c \neq 0$, *as in* 6.9;

(c) $E^{2p^{nj}}_{**}(\mathbf{K}_0) = E^{\infty}_{**}(\mathbf{K}_0)$ *is the sub-Hopf algebra of*

$$\Gamma(\phi([p^{j-1}] - [0]))$$

given by $\gamma_i(\phi([p^{j-1}] - [0]))$, $0 \leq i < p^{nj}$;

(d) $\gamma_i(\phi([p^{j-1}] - [0]))$ *converges to* $a_i \in \overline{K(n)}_*\mathbf{K}_1$, *and this solves all of the non-trivial extension problems.*

Proof. Part (a) follows from the facts that $\overline{K(n)}_*\mathbf{K}_0$ is a truncated polynomial algebra on $\underline{[1] - [0]} \in \widetilde{K(n)}_*\mathbf{K}_0$, $E^2_{**}(\mathbf{K}_0) = H_{**}\overline{K(n)}_*\mathbf{K}_0$, and 6.6. Parts (b), (c), and (d) follow from Lemma 6.10, inspection and comparison with the known structure of $\overline{K(n)}_*\mathbf{K}_1$ from Theorem 5.7. □

9. The mod p Eilenberg-MacLane spaces. Recall that p is an odd prime. In this section we assume that $\mathbf{K}_* = \mathbf{K}(Z/(p), *)$, and compute $K(n)_*\mathbf{K}_*$ for all n. For

$$I = (i_1, i_2, \ldots, i_q), \qquad 0 \leq i_k < n,$$

we define

$$a_I \in K(n)_*\mathbf{K}_q$$

by

$$a_I = a_{(i_1)} \circ a_{(i_2)} \circ \cdots \circ a_{(i_q)}.$$

Before we state our main results, we need the following important lemma.

LEMMA 9.1. (a) $a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)}$,
(b) $a_{(i)} \circ a_{(i)} = 0$. □

Proof. By Hopf-ring commutativity [25, 1.12(c) (iv) and (v)],

$$a_{(i)} \circ a_{(j)} = \chi(a_{(j)}) \circ a_{(i)} = -a_{(j)} \circ a_{(i)} \qquad \text{by 5.10(d).}$$

Part (b) follows from (a). □

Let $T_k^{K(n)_*}(x) = K(n)_*[x]/(x^{p^k})$. The purpose of this section is to prove the following theorem.

THEOREM 9.2. *For p an odd prime and $\mathbf{K}_* = \mathbf{K}(Z/(p), *)$:*

(a) *As algebras,*

$K(n)_*\mathbf{K}_0 \simeq K(n)_*[Z/(p)]$, *the group-ring of $Z/(p)$ over $K(n)_*$;*

$$K(n)_*\mathbf{K}_q \simeq \bigotimes_{\{I:0<i_1<i_2<\cdots<i_q<n\}} T_{\rho(I)}^{K(n)_*}(a_I), \qquad 0 < q < n,$$

where $\rho(I) = 1 + \max(\{0\} \cup \{s+1 \mid i_{q-s} = n-1-s\})$;

$$K(n)_*\mathbf{K}_n \simeq K(n)_*[a_I]/(a_I{}^{*p} + (-1)^q v_n a_I), \qquad q = n,$$
$$I = (0, 1, \ldots, n-1);$$
$$K(n)_*\mathbf{K}_q \simeq K(n)_*, \qquad q > n.$$

(b) *The a_I, $0 \le i_1 < i_2 < \cdots < i_q < n$ are all non-zero and distinct. In $\overline{K(n)}_*\mathbf{K}_q$, each one is $\pm$ the p^i-th power of some generator for some $i \ge 0$.*

(c) *The p-th powers are computed as*

$$a_I{}^{*p} = 0 \qquad \text{if } i_q \ne n-1$$
$$a_I{}^{*p} = (-1)^{q-1} v_n a_{(0)} \circ a_{(i_1+1)} \circ \cdots \circ a_{(i_{q-1}+1)} \qquad \text{if } i_q = n-1$$
$$0 \le i_1 < i_2 < \cdots < i_q < n.$$

(d) *In $\overline{K(n)}_*\mathbf{K}_q$ we have, up to signs, an equality of sets*

$$\{a_I{}^{*p^{\rho(I)-1}} \mid 0 < i_1 < i_2 < \cdots < i_q < n\} \cdot$$
$$= \{a_J \mid 0 \le j_1 < j_2 < \cdots < j_q < n-1\}.$$

(e) *The coalgebra structure follows from the Hopf ring properties and the structure of $K(n)_*\mathbf{K}_1$ given in Theorem 5.7.*

(f) *The Verschiebung map for $\overline{K(n)}_*\mathbf{K}_q$ is computed on a_I as follows:*

$$V(a_I) = 0, \qquad i_1 = 0,$$
$$= a_{(i_1-1)} \circ a_{(i_2-1)} \circ \cdots \circ a_{(i_q-1)}, \qquad i_1 \ne 0$$
$$0 \le i_1 < i_2 < \cdots < i_q < n. \qquad \square$$

COROLLARY 9.3. *The Hopf ring $K(n)_*\mathbf{K}_*$ is the free $K(n)_*[Z/(p)]$-Hopf ring on the Hopf algebra $K(n)_*\mathbf{K}_1$.* □

Proof. We know that $K(n)_*\mathbf{K}_*$ is a $K(n)_*[Z/(p)] = K(n)_*\mathbf{K}_0$ Hopf ring. Theorem 9.2(a) tells us that this Hopf ring is generated by $K(n)_*\mathbf{K}_1$. The relations in the Hopf algebras of Theorem 9.2(a) are all derived from 9.1 and 9.2(c). The first was a consequence of the Hopf ring structure and $K(n)_*\mathbf{K}_1$. In the proof of Theorem 9.2 we will show that the second is too. Since there are no relations other than those forced by $K(n)_*\mathbf{K}_1$ and the Hopf ring structure, the result follows. □

We will prove Theorem 9.2 inductively on q using the bar spectral sequence which we now describe.

THEOREM 9.4. *In the bar spectral sequence, $E^r_{*\,*}(\overline{K(n)}_*\mathbf{K}_{q-1}) \Rightarrow \overline{K(n)}_*\mathbf{K}_q$, we have*

(a) $E^2_{*\,*}\overline{K(n)}_*\mathbf{K}_{q-1} \simeq H_{*\,*}\overline{K(n)}_*\mathbf{K}_{q-1}$

$$\simeq \begin{cases} E(\sigma([1]-[0])) \otimes \Gamma(\phi([1]-[0])), & q=1, \\ \otimes_I (E(\sigma a_I) \otimes \Gamma(\phi(a_{(i_1-1,i_2-1,\ldots,i_{q-1}-1)}))) & \\ 0 < i_1 < i_2 < \cdots < i_{q-1} < n, & n \geq q > 1, \\ \overline{K(n)}_* & q > n; \end{cases}$$

(b) *the differentials are given by*

$$d^{2p^{i_1+1}-1}\gamma_{p^{i_1+1}}\phi a_{(i_2-i_1-1,i_3-i_1-1,\ldots,i_{q-1}-i_1-1,n-2-i_1)} = r_I\sigma a_{(i_1+1,i_2+1,\ldots,i_{q-1}+1)},$$

$$r_I \neq 0, \qquad 0 \leq i_1 < i_2 < \cdots < i_{q-1} < n-1, \qquad q > 1;$$

(c) *modulo decomposables in $E^\infty_{*\,*}$, a_I is represented by*

$$\gamma_{p^{i_1}}\phi a_{(i_2-i_1-1,i_3-i_1-1,\ldots,i_q-i_1-1)},$$

for $0 \leq i_1 < i_2 < \cdots < i_q < n$, where by convention $a_\phi = [1]-[0]$. □

Proof of Theorem 9.2. Part (f) follows from 7.1(c) and (e). Part (e) follows from Hopf ring properties and (a). Part (d) follows from (a) and (c). Part (c) follows from the Hopf ring structure and $K(n)_*\mathbf{K}_1$. In particular, using 7.1(c) and (e),

$$\begin{aligned}
a_I{}^{*p} &= F(a_I) \\
&= F(a_{(i_1,\ldots,i_{q-1})} \circ a_{(i_q)}) \\
&= F(V(a_{(i_1+1,\ldots,i_{q-1}+1)}) \circ a_{(i_q)}) \\
&= a_{(i_1+1,\ldots,i_{q-1}+1)} \circ F(a_{(i_q)}) \\
&= \begin{cases} 0 & \text{if } i_q \neq n-1 \\ a_{(i_1+1,\ldots,i_{q-1}+1)} \circ v_n a_{(0)}, & \text{if } i_q = n-1. \end{cases}
\end{aligned}$$

Use 9.1(a) to finish the proof. Part (b) follows from (a) and (c). All we need now is to show (a). We know the $q = 1$ case from Theorem 5.7 ($q = 0$ is trivial). The $q > n$ case follows immediately from 9.4(a) for $q > n$. A little checking verifies that the differentials of 9.4(b) fit the description of 6.10(b). This computes the $E^\infty_{*\,*}$ term of the spectral sequence. Another simple check using 9.4(c) shows that the generators of E^∞ are in 1-1 correspondence with the a_I, $a \leq i_1 < i_2 < \cdots < i_q < n$. Using 9.2(c) we can solve all of the extension problems to obtain Theorem 9.2(a). $\square$

Proof of Theorem 9.4. The proof is by induction on q. We know $q = 0, 1$ already. We now prove 9.4 for $E^r_{*\,*}(\overline{K(n)}_*\mathbf{K}_q) \Rightarrow \overline{K(n)}_*\mathbf{K}_{q+1}$. Part (a) follows from the q version of Theorem 9.2(a) and (d) together with 6.1, 6.6, 6.7, and 8.1. Part (c) is 8.1 for $1 + q = 1$. To prove the rest of (c) for $q + 1 \geq 2$, we use the pairing which comes from

$$\circ\colon \overline{K(n)}_*\mathbf{K}_q \otimes_{\overline{K(n)}_*} \overline{K(n)}_*\mathbf{K}_1 \to \overline{K(n)}_*\mathbf{K}_{q+1}.$$

On the E^2 level this is

$$\circ\colon H_{*\,*}\overline{K(n)}_*\mathbf{K}_{q-1} \otimes_{\overline{K(n)}_*} \overline{K(n)}_*\mathbf{K}_1 \to H_{*\,*}\overline{K(n)}_*\mathbf{K}_q.$$

We compute this map as follows.

LEMMA 9.5. *For* $0 \le i_1 < i_2 < \cdots < i_q < n$, $q > 0$, $\tilde{I} = (i_2, \ldots, i_q)$, $\phi(a_{\tilde{I}}) \circ a_{(i_1+1)} = (-1)^{q-1}\phi a_I$, *where if* $q = 1$, $a_\phi = [1] - [0]$ *and* $i_1 < n - 1$. □

Proof. We use the spectral sequence pairing computation as in 2.3. We also use Hopf ring distributivity and the fact that $a_J \circ a_0 = 0$. We have, for $i_1 = 0$,

$$\begin{aligned}\phi(a_{\tilde{I}}) \circ a_{(1)} &= (a_{\tilde{I}}{}^{*p-1} \,|\, a_{\tilde{I}}) \circ a_{(1)} \\ &= ((a_{\tilde{I}} \circ a_{(0)})^{*p-1} \,|\, a_{\tilde{I}} \circ a_{(0)})\end{aligned}$$

which, by 9.1, is

$$\begin{aligned}&= (-1)^{p(q-1)} a_I{}^{*p-1} \,|\, a_I = (-1)^{q-1} a_I{}^{*p-1} \,|\, a_I \\ &= (-1)^{q-1}\phi(a_I)\end{aligned}$$

For the $i_1 > 0$ case a similar direct computation should work. However, such a proof has eluded us. To carry out our present proof we use degree arguments which we feel should be unnecessary. The internal degree of ϕa_I is $2(p^{i_1} + p^{i_2} + \cdots + p^{i_q})$. For the various I, these are distinct modulo $2(p^n - 1)$. Furthermore, there are no decomposables in these bidegrees as the only possibilities are $\sigma a_J * \sigma a_{J'}$, $J \ne J'$. The internal degree of this is

$$2(p^{j_1} + \cdots + p^{j_q}) + 2(p^{j_1'} + \cdots + p^{j_q'}), \qquad p > 2,$$

which is clearly distinct mod $2(p^n - 1)$ from any $|\phi a_I|$.

Recall $\tilde{I} = (i_2, i_3, \ldots, i_q)$. We compute

$$\begin{aligned}\phi a_{\tilde{I}} \circ a_{(i_1+1)} &= (a_{\tilde{I}}{}^{*p-1} \,|\, a_{\tilde{I}}) \circ a_{(i_1+1)} \\ &= \sum_{\substack{c_i > 0 \\ c_1 + \cdots + c_p = p^{i_1+1}}} (a_{\tilde{I}} \circ a_{c_1}) * \cdots * (a_{\tilde{I}} \circ a_{c_{p-1}}) \,|\, a_{\tilde{I}} \circ a_{c_p}.\end{aligned}$$

For the degree reasons discussed above this must be homologous to a multiple of ϕa_I. Therefore, we lose no information by projecting onto the bar complex for the quotient algebra $T_1(a_I)$, which is a factor of the algebra $\overline{K(n)}_* \mathbf{K}_q$ (see 9.2(a)) because $i_1 > 0$ and so a_I is a generator. In our new complex the space of cycles with bidegree $(2, 2p(p^{i_1} + \cdots + p^{i_q}))$ has basis $\{a_I{}^{*p-s} \mid a_I{}^{*s}\colon 0 < s < p\}$, and these elements are all homologous. The internal degree of $a_I \circ a_{c_i}$ can equal that of $a_I{}^{*s}$ only if $c_i = p^{i_1}$ and $s = 1$, so all terms in our expression for $\phi(a_{\tilde I}) \circ a_{(i_1+1)}$ project to zero except

$$\begin{aligned}(a_{\tilde I} \circ a_{(i_1)})^{*p-1} \mid (a_{\tilde I} \circ a_{(i_1)}) &= (-1)^{p(q-1)}(a_{(i_1)} \circ a_{\tilde I})^{*p-1} \mid a_{(i_1)} \circ a_{\tilde I} \\ &= (-1)^{q-1} a_I{}^{*p-1} \mid a_I = (-1)^{q-1}\phi a_I.\end{aligned}$$

The $q = 1$ case is easy. □

LEMMA 9.6. *For $i < n - 1 - i_1$, $0 \le i_1 < i_2 < \cdots < i_q < n$, $\tilde I = (i_2, \ldots, i_q)$, $\gamma_{p^i}(\phi a_{\tilde I}) \circ a_{(i_1+i+1)} = (-1)^{q-1}\gamma_{p^i}(\phi a_I)$, modulo decomposables.* □

Proof. Let V^i be the iterated Verschiebung map. Then, by 7.1,

$$\begin{aligned}V^i(\gamma_{p^i}(\phi a_{\tilde I}) \circ a_{(i_1+i+1)}) &= V^i(\gamma_{p^i}(\phi a_{\tilde I})) \circ V^i a_{(i_1+i+1)} \\ &= \phi a_{\tilde I} \circ a_{(i_1+1)} \\ &= (-1)^{q-1}\phi a_I \\ &= (-1)^{q-1} V^i \gamma_{p^i}(\phi a_I).\end{aligned}$$

Thus the lemma is true modulo the kernel of V^i on elements of external degree $2p^i$. Since $V^i\gamma_{p^i} = \gamma_{p^0}$ and all other elements in external degree $2p^i$ are decomposable, the kernel of V^i on degree $2p^i$ is contained in the decomposables (actually they are equal). □

We are now prepared to prove part (c) of 9.4.

LEMMA 9.7. *Let $0 \le i_1 < i_2 < \cdots < i_{q+1} < n$, $q + 1 \ge 1$ with $\tilde I^{-j} = (i_2 - j, i_3 - j, \ldots, i_{q+1} - j)$. Then $\gamma_{p^{i_1}}\phi(a_{\tilde I^{-(i_1+1)}})$ is a permanent cycle representing a_I modulo decomposables in $E^\infty_{**}\overline{K(n)}_*\mathbf{K}_q$.* □

Proof. This has been done for $q + 1 = 1$, so we assume $q + 1 \ge 2$. Let $J = (i_3 - i_1 - 1, \ldots, i_{q+1} - i_1 - 1)$, recall that $a_J = [1] - [0]$ if $J = \emptyset$. By induction, $\gamma_{p^{i_1}}(\phi a_J)$ is a permanent cycle representing $a_{(i_1, i_3, \ldots, i_{q+1})}$ modulo decomposables. By Theorem 2.2, $d^r(x \circ y) = d^r(x) \circ y$. So we have

$$
\begin{aligned}
d^r((-1)^{q-1}\gamma_{p^{i_1}}(\phi a_{\bar{I}-(i_1+1)})) &= d^r(\gamma_{p^{i_1}}(\phi a_J) \circ a_{(i_2)}) && \text{by 9.6} \\
&= d^r(\gamma_{p^{i_1}}(\phi a_J)) \circ a_{(i_2)} && \text{by 2.2} \\
&= 0 \circ a_{(i_2)} && \text{by induction} \\
&= 0
\end{aligned}
$$

Thus we have a permanent cycle and since the pairing on the spectral sequence level preserves the Hopf ring structures, we have

$$
\begin{aligned}
&(-1)^{q-1}\gamma_{p^{i_1}}\phi a_{I-(i_1+1)} && \text{represents, by induction,} \\
&a_{(i_1 i_3,\ldots,i_{q+1})} \circ a_{(i_2)} = (-1)^{q-1}a_I, && \text{modulo decomposables.} \quad \square
\end{aligned}
$$

By Lemma 6.10(a) we know that not only are the elements in 9.7 permanent cycles but they survive non-trivially.

We must now prove 9.4(b) for $E^r_{**} \quad (\overline{K(n)}_*\mathbf{K}_q) \Rightarrow \overline{K(n)}_*\mathbf{K}_{q+1}$. We will do this by induction on q. To prove this case we will first assume $q+1>2$ and come back to the $q+1=2$ case later.

Let $J=(i_3, i_4, \ldots, i_q, n-1)$, $0 \le i_1 < i_2 < \cdots < i_q < n-1$. We know by induction that

$$
d^{2p^{i_1+1}-1}\gamma_{p^{i_1+1}}(\phi a_{J-(i_1+1)}) = r\sigma a_{(i_1+1, i_3+1, \ldots, i_q+1)}, \qquad r \neq 0.
$$

By 9.6, with $K=(i_2, i_3, \ldots, i_q, n-1)$,

$$
\gamma_{p^{i_1+1}}(\phi a_{J-(i_1+1)}) \circ a_{(i_2+1)} = (-1)^{q-1}\gamma_{p^{i_1+1}}(\phi a_{K-(i_1+1)})
$$

when $i_2+1<n$, which is always true if $q+1>2$, but not if $q+1=2$ which is why $q+1=2$ is a special case. So, for $q+1>2$,

$$
\begin{aligned}
&d^{2p^{i_1+1}-1}\gamma_{p^{i_1+1}}(\phi a_{K-(i_1+1)}) \\
&\quad = d^{2p^{i_1+1}-1}((-1)^{q-1}\gamma_{p^{i_1+1}}(\phi a_{J-(i_1+1)}) \circ a_{(i_2+1)}) \\
&\quad = (-1)^{q-1}d^{2p^{i_1+1}-1}(\gamma_{p^{i_1+1}}(\phi a_{J-(i_1+1)})) \circ a_{(i_2+1)} \\
&\quad = (-1)^{q-1}r\sigma a_{(i_1+1, i_3+1, \ldots, i_q+1)} \circ a_{(i_2+1)} \\
&\quad = r\sigma a_{(i_1+1, i_2+1, \ldots, i_q+1)}.
\end{aligned}
$$

It remains to prove part (b) for $q+1=2$. We have

$$E^2_{**}\overline{K(n)}_*\mathbf{K}_1 \simeq \otimes_{0<i<n} E(\sigma a_{(i)}) \otimes_{0<i<n} \Gamma(\phi a_{(i-1)})$$

with $\gamma_{p^{i_1}} \phi a_{(i_2-i_1-1)}$ a permanent cycle representing $a_{(i_1,i_2)}$ for $0 \leq i_1 < i_2 < n$. We need to show that

$$d^{2p^{i_1+1}-1}\gamma_{p^{i_1+1}}(\phi a_{(n-2-i_1)}) = r\sigma a_{(i_1+1)}$$

where $0 \leq i_1 < n - 1$, $r \neq 0$. First note that if $\gamma_{p^{i_1+1}}(\phi a_{(n-2-i_1)})$ fails to survive to E^∞, then it must have a differential of the form of 6.10(a) on it since $\gamma_{p^{i_1}}(\phi a_{(n-2-i_1)})$ is known to survive. Such a differential, $d^{2p^{i_1+1}-1}$, sends $\gamma_{p^{i_1+1}}(\phi(a_{(n-2-i_1)})$ to the bidegree of $\sigma a_{(i_1+1)}$, which is the only nontrivial element in this bidegree. Therefore, we are done if we show that $\gamma_{p^{i_1+1}}(\phi a_{(n-2-i_1)})$ does not survive. Assume that $\gamma_{p^{i_1+1}}(\phi a_{(n-2-i_1)})$ is a permanent cycle which survives to $x \in \overline{K(n)}_*\mathbf{K}_2$. Since $p = 0$ in $\mathbf{K}_0$, $\circ$ multiplication by $[p] = [0]$ is trivial [25, 1.12(c) (ii)]. Thus $[p] \circ x = 0$. By 7.1(c), $[p] \circ x = FV(x)$. From the spectral sequence we see that $V(x) = a_{(i_1,n-1)}$ which is represented by $\gamma_{p^{i_1}}(\phi a_{(n-2-i_1)})$, so $0 = [p] \circ x = FV(x) = F(a_{(i_1,n-1)}) = (-1)a_{(0)} \circ a_{(i_1+1)}$ by 9.2(c). This is non-zero because it is represented by $-\phi a_{(i_1)}$ which is known to be a non-zero permanent cycle. This contradicts our assumption that $\gamma_{p^{i_1+1}}(\phi a_{(n-2-i_1)})$ survives. This concludes the proof. $\square$

For $\overline{K(n)}_*\mathbf{K}(Z/(p^j), q)$, $j > 1$, such problems as just discussed above are rather commonplace. A similar solution not involving the spectral sequence pairing will work there too.

10. The Conner-Floyd Conjecture. Conner and Floyd conclude their book on differentiable periodic maps [6] with a conjecture which would solve many of the questions they were interested in. Since then, Floyd [8] and Tom Dieck [7] have answered the original questions leaving the conjecture unsolved. The purpose of this section is to prove the Conner-Floyd conjecture.

Let $\mathrm{MSO}_*(X)$ be the oriented bordism of X (denoted $\Omega_*(X)$ in [6]). The annihilator ideal of $x \in \mathrm{MSO}_*(X)$ is the ideal $A(x) \subset \mathrm{MSO}_*$ of elements $y \in \mathrm{MSO}_*$ such that $0 = yx \in \mathrm{MSO}_*(X)$. Let $BZ/(p)$ be the classifying space for the group $Z/(p)$. Recall that p is an odd prime throughout this section. There is a canonical bordism element given by the obvious map $S^1 \to BZ/(p) \simeq \mathbf{K}(Z/(p), 1)$. Taking the exterior product we have

$$\gamma_n \in \mathrm{MSO}_*(\underbrace{BZ/(p) \times \cdots \times BZ/(p))}_{n \text{ copies}}$$

given by

$$\underbrace{S^1 \times \cdots \times S^1}_{n \text{ copies}} \to \underbrace{BZ/(p) \times \cdots \times BZ/(p)}_{n \text{ copies}}.$$

Conner and Floyd [6, p. 145] give manifolds $M^{2(p^k-1)}$ which are Milnor basis elements of $\mathrm{MSO}_*/(p)$.

THEOREM 10.1. (The Conner-Floyd Conjecture [6, p. 146]). *For p an odd prime and*

$$\gamma_n \in MSO_n(\underbrace{BZ/(p) \times \cdots \times BZ/(p)}_{n \text{ copies}}),$$

the annihilator ideal

$$A(\gamma_n) = (p, M^{2(p-1)}, \ldots, M^{2(p^{n-1}-1)}) \subset MSO_*. \qquad \square$$

Conner and Floyd proved that $A(\gamma_n) \supset (p, M^{2(p-1)}, \ldots, M^{2(p^{n-1}-1)})$ [6, p. 146]. This is purely a p-primary problem because $BZ/(p)$ is a p-local space. By [22], $\mathrm{MSO}_*(X)_{(p)} \simeq \mathrm{MSO}_{*(p)} \otimes_{\mathrm{BP}_*} \mathrm{BP}_*(X)$ and, by construction, the Milnor basis manifolds $M^{2(p^k-1)}$ project to generators of BP_*. Thus, the Conner-Floyd conjecture is equivalent to the following.

THEOREM 10.2. *Let p be an odd prime. For*

$$\gamma_n \in BP_n(BZ/(p) \times \cdots \times BZ/(p)),$$

the annihilator ideal

$$A(\gamma_n) = (p, v_1, v_2, \ldots, v_{n-1}) \subset BP_*. \qquad \square$$

To prove this we can go to the n-th Eilenberg-MacLane space.

THEOREM 10.3. *Let p be an odd prime. For the canonical element $\iota_n \in BP_n(\mathbf{K}(Z/(p), n))$; the annihilator ideal*

$$A(\iota_n) = (p, v_1, \ldots, v_{n-1}) \subset BP_*. \qquad \square$$

Proof of 10.2. Using the obvious map

$$\underbrace{BZ/(p) \times \cdots \times BZ/(p)}_{n \text{ copies}} \simeq \underbrace{\mathbf{K}_1 \times \cdots \times \mathbf{K}_1}_{n \text{ copies}} \to \underbrace{\mathbf{K}_1 \wedge \cdots \wedge \mathbf{K}_1}_{n \text{ copies}} \to \mathbf{K}_n,$$

We see that γ_n goes to ι_n and thus $A(\gamma_n) \subset A(\iota_n)$. By Conner and Floyd we have

$$(p, v_1, \ldots, v_{n-1}) \subset A(\gamma_n)$$

$$A(\gamma_n) \subset A(\iota_n) \qquad \text{by naturality,}$$

and

$$A(\iota_n) \subset (p, v_1, \ldots, v_{n-1}) \qquad \text{by 10.3,}$$

so 10.2 follows. $\square$

Let $T_n \subset \mathrm{BP}_*$ be the multiplicative set $\{1, v_n, v_n^2, \ldots\}$. Since localization preserves exactness, $v_n^{-1}\mathrm{BP}_*(X) \equiv T_n^{-1}\mathrm{BP}_*(X)$ is an homology theory (see [10]). Let

$$\rho: \mathrm{BP}_*(X) \to v_n^{-1}\mathrm{BP}_*(X)$$

be the localization map.

THEOREM 10.4. *Let ρ be an odd prime. The annihilator ideal for*

$$\rho(\iota_n) \in v_n^{-1}BP_n(K(Z/(p), n)), A(\rho(\iota_n)) = (p, v_1, \ldots, v_{n-1}) \subset BP_*. \qquad \square$$

Theorem 10.3 follows immediately from 10.4 and Conner and Floyd's result that $(p, v_1, \ldots, v_{n-1}) \subset A(\gamma_n)$.

Peter Landweber [13] shows that if $0 \neq x$ in $v_n^{-1}\mathrm{BP}_*X$, then the annihilator ideal of x is contained in the prime ideal $(p, v_1, \ldots, v_{n-1})$.

Actually, we do not need to appeal to Landweber's result since we compute the entire structure of $v_n^{-1}\mathrm{BP}_*\mathbf{K}(Z/(p), n)$.

Let $M_k^{n-k} \equiv v_n^{-1}N_k^{n-k}$ be defined inductively by

$$N_k^0 \equiv \mathrm{BP}_*/(p, v_1, \ldots, v_{k-1})$$

and $0 \to N_k^{n-k-1} \to M_k^{n-k-1} \to N_k^{n-k} \to 0$; see [16].

THEOREM 10.5. *Let p be an odd prime.*

(a) $v_k^{-1}\widetilde{BP}_*(\mathbf{K}(Z/(p), n)) = 0, \qquad 0 \leq k < n.$

(b) *As a BP_* module,*

$$v_n^{-1}\widetilde{BP}_*(\mathbf{K}(Z/(p), n)) \simeq \oplus_{i=1}^{p-1} M_0{}^n(i) \qquad \textit{where } (v_0 = p)$$

$$M_0{}^n(i) \equiv M_0{}^n \quad \textit{with } v_0{}^{-1}v_1{}^{-1} \cdots v_{n-1}{}^{-1} \in M_0{}^n(i)$$

in degree $n + 2(i-1)(1 + p^1 + \cdots + p^{n-1})$ *and*

$$\rho(\iota_n) = v_0{}^{-1}v_1{}^{-1} \cdots v_{n-1}{}^{-1} \in M_0{}^n(1). \qquad \square$$

Theorem 10.4 follows immediately from 10.5. We will need the rest of the section to prove 10.5.

Proof of Theorem 10.5. We only prove part (b). Part (a) is similar but easier. We wait until the end to identify $\rho(\iota_n)$. We use Morava's structure theorem for complex cobordism as developed in [11, Section 5]. Following [11], there is a sequence of homology theories, $P(n)_*(—)$, with $P(n)_* \simeq \mathrm{BP}_*/I_n$, $I_n = (p, v_1, \ldots, v_{n-1})$. There are exact sequences

$$\begin{tikzcd} P(k)_*(X) \arrow[rr, "v_k"] & & P(k)_*(X) \arrow[dl, "\rho_k"] \\ & P(k+1)_*(X) \arrow[ul, "\partial_k"] & \end{tikzcd} \tag{10.6}$$

where the degree of ∂_k is $-2p^k + 1$.

LEMMA 10.7. *Let p be an odd prime. For $0 \le k \le n$, as BP_* modules*

$$v_n^{-1}\widetilde{P(k)}_*(\mathbf{K}(Z/(p), n)) \simeq \oplus_{i=1}^{p-1} M_k^{n-k}(i)$$

$$M_k^{n-k}(i) \equiv M_k^{n-k}$$

with $v_k^{-1}v_{k+1}^{-1} \cdots v_{n-1}^{-1} \in M_k^{n-k}(i)$ in degree

$$2(i-1)(1+p^1+\cdots+p^{n-1})+2(1+p^1+\cdots+p^{k-1})+n-k. \quad \square$$

Most of 10.5(b) follows from this because $P(0)_*(-) = BP_*(-)$.

Proof of Lemma 10.7. From [11], $v_n^{-1}P(n)_*(X)$ is a free $v_n^{-1}P(n)_*$ module and $v_n^{-1}P(n)_*(X)$ is unnaturally isomorphic to $K(n)_*(X) \otimes \mathbf{F}_p[v_{n+1}, \ldots]$. This plus our computation of $K(n)_*\mathbf{K}(Z/(p), n)$ in the previous section proves the lemma for $k = n$. We now use downward induction on k. Assume the result for $k + 1 \le n$. Localize the exact sequence 10.6 to obtain the exact sequence

$$\begin{tikzcd}[column sep=small]
v_n^{-1}P(k)_*(X) \arrow[rr, "v_k"] & & v_n^{-1}P(k)_*(X) \arrow[dl, "\rho_k"] \\
& v_n^{-1}P(k+1)_*(X) \arrow[ul, "\partial_k"] &
\end{tikzcd} \tag{10.8}$$

We will show that for

$$X = \mathbf{K}(Z/(p), n), \qquad \rho_k \text{ is } \equiv 0$$

and all elements in $v_n^{-1}\widetilde{P(k)}_*(X)$ are v_k torsion. If there are elements which are not v_k torsion, then

$$v_k^{-1}v_n^{-1}\widetilde{P(k)}_*(X) \simeq v_n^{-1}v_k^{-1}\widetilde{P(k)}_*(X)$$

would be nontrivial. However, by [11],

$$v_k^{-1}P(k)_*(X) \simeq K(k)_* \, K(k)_*(X) \otimes F_p[v_{k+1}, \ldots],$$

and we know that

$$\widetilde{K(k)}_*(\mathbf{K}(Z/(p), n)) = 0 \qquad \text{for } k < n,$$

so $v_n^{-1}\widetilde{P(k)}_*(X)$ is v_k torsion for $X = \mathbf{K}(Z/(p), n)$. To show that $\rho_k \equiv 0$, observe that $v_n^{-1}\widetilde{P(k+1)}_*(X)$ is concentrated in even degrees if $n - k - 1$ is even. If $\rho_k(y) \neq 0$ for some y then it must be even degree. By the above, $v_k{}^j y = 0$ for some j.

Thus there is an element in the kernel of v_k multiplication which has degree even if $n - k - 1$ is even. By exactness and the fact that the boundary homomorphism ∂_k has odd degree, there must be an element of

$$v_n^{-1}\widetilde{P(k+1)}_*(X) \quad \text{of odd degree.}$$

This contradicts our computation of

$$v_n^{-1}\widetilde{P(k+1)}_*(X), \qquad X = \mathbf{K}(Z/(p), n).$$

If $n - k - 1$ is odd, do the same argument interchanging "even" and "odd". Thus 10.8 reduces to a short exact sequence coming from

$$0 \longrightarrow M_{k+1}^{n-k-1} \longrightarrow M_k{}^{n-k} \xrightarrow{v_k} M_k{}^{n-k} \longrightarrow 0,$$

see [16], and the result follows easily. □

Our only remaining problem is to identify $\rho(\iota_n)$. It is enough, from 10.7, to show that it is non-zero because we know that $(p, v_1, \ldots, v_{n-1}) \subset A(\rho(\iota_n))$ from Conner and Floyd.

LEMMA 10.9. *Let X be $n - 1$ connected. Then*

$$\widetilde{P(k)}_* X \simeq \tilde{H}_*(X, Z/(p))$$

for degrees less than $n + 2(p^k - 1)$. □

Proof. This is a standard argument using the fact that $P(k)_* \simeq H_*$ for degrees less than $2(p^k - 1)$. □

Thus, for $0 \le i < k$, we have $a_{(i)} \in P(k)_{2p^i}\mathbf{K}_1$, just as for $K(n)_*(—)$ in 5.7. Also by 10.9 we have a $\iota_1 \in P(k)_1\mathbf{K}_1$ which gives rise to $\iota_j = \iota_1 \circ \cdots \circ \iota_1 \in P(k)_j\mathbf{K}_j$.

LEMMA 10.10. *Let p be an odd prime. In* 10.6 *we have*

$$\partial_k(\iota_{n-k-1} \circ a_{(0)} \circ a_{(1)} \circ \cdots \circ a_{(k)}) = \pm\iota_{n-k} \circ a_{(0)} \circ \cdots \circ a_{(k-1)} \qquad 0 \le k < n. \qquad \square$$

Proof. The map ∂_k is induced by a map of spectra

$$P(k+1) \xrightarrow{\partial_k} \Sigma^{2p^k-1} P(k)$$

The stable mod p cohomology of $P(k+1)$ is $A/A(Q_{k+1}, Q_{k+2}, \ldots)$ by 2.5(b) of [11]. In cohomology we have $\partial_k{}^*(1) = Q_k 1$. Thus in the range we are working in where $P(k+1)_*X$ and $P(k)_*X$ are really just the mod p homology, we have that ∂_k is just the operation dual to Q_k on homology. Thus it behaves like a differential with respect to $\circ$ product. Since ∂_k lowers degrees by $2p^k - 1$, we see that $\partial_k(a_{(i)}) = 0$, $i < k$, and $\partial_k(\iota_1) = 0$ so $\partial_k(\iota_{n-k}) = 0$. A simple check reveals that $\partial_k(a_{(k)}) = \pm\iota_1$, and our result follows. This check is carried out by using the Thom map $\mu: P(k)_*X \to H_*(X; Z/(p))$ and seeing that

$$\mu\partial_k(a_{(k)}) = Q_k{}^*\mu a_{(k)} = \mu(\iota_1).$$

By 10.9, μ is an isomorphism in this degree. $\square$

We can now complete the proof of 10.5(b) by proving that $\rho(\iota_n) \neq 0$. The element $a_{(0)} \circ a_{(1)} \circ \cdots \circ a_{(n-1)} \in P(n)_*\mathbf{K}_n$ is non-zero because it reduces to $0 \neq a_{(0)} \circ \cdots \circ a_{(n-1)} \in K(n)_*\mathbf{K}_n$. Thus it is non-zero in the intermediate stage in $v_n{}^{-1}P(n)_*\mathbf{K}_n$. By the proof of 10.7 we know that $0 \neq \partial_0 \cdot \partial_1 \cdots \partial_{n-1}(a_{(0)} \circ \cdots \circ a_{(n-1)}) \in v_n{}^{-1}\mathrm{BP}_n\mathbf{K}_n$. By 10.10, in $\mathrm{BP}_n\mathbf{K}_n$,

$$\iota_n = \partial_0 \cdots \partial_{n-1}(a_{(0)} \circ \cdots \circ a_{(n-1)}).$$

Thus when we localize we have $\rho(\iota_n) \neq 0$. $\square$

11. The mod p^j Eilenberg-MacLane spaces. In this section we assume that p is an odd prime and $\mathbf{K}_* = \mathbf{K}(Z/(p^j), *)$, and compute $K(n)_*\mathbf{K}_*$ for all n. For

$$I = (i_1, i_2, \ldots, i_q), \qquad 0 \leq i_k < nj,$$

we define

$$a_I \in K(n)_*\mathbf{K}_q$$

by

$$a_I = a_{(i_1)} \circ a_{(i_2)} \circ \cdots \circ a_{(i_q)}.$$

Let $[r]$ be the greatest integer function, i.e., $[r]$ is the largest integer less than or equal to r. Let $\bar{i}$ be the unique integer such that $i \equiv \bar{i} \bmod (n)$ and $0 \leq \bar{i} < n$. Let $T_k^{K(n)_*}(x) = K(n)_*[x]/(x^{p^k})$. The purpose of this section is to prove the following theorem.

THEOREM 11.1. *For p an odd prime and $\mathbf{K}_* = \mathbf{K}(Z/(p^j), *)$, $j > 0$:*

(a) *As algebras*

$$K(n)_*\mathbf{K}_0 \simeq K(n)_*[Z/(p^j)],$$

the group ring of $Z/(p^j)$ over $K(n)_*$;

$$K(n)_*\mathbf{K}_q \simeq \otimes_I T_{\rho(I)}^{K(n)_*}(a_I), \qquad 0 < q < n,$$

where the tensor product is over the set

$$\{I \mid n(j-1) < i_1 < i_2 < \cdots < i_q < nj\}$$

with $\rho(I) = mn + \hat{\bar{i}}_c + 1$ where m and $\hat{\bar{i}}_c$ are defined as follows. Given I as above, let $\hat{\bar{I}} = (\hat{\bar{i}}_1, \ldots, \hat{\bar{i}}_{n-q})$ with $\{\bar{i}_k\} \cup \{\hat{\bar{i}}_k\} = \{0, 1, \ldots, n-1\}$, $\hat{i}_1 < \hat{i}_2 < \cdots < \hat{i}_{n-q}$. Let $j - 1 = m(n-q) + c - 1$, $0 < c \leq n - q$.

$$K(n)_*\mathbf{K}_n \simeq \otimes_I K(n)_*[a_I]/(a_I{}^{*p} + (-1)^q v_n{}^{c(I)} a_I),$$
$$I = (nk, n(j-1)+1, n(j-1)+2, \ldots, nj-1),$$
$$0 \leq k < j.$$

$$c(I) = p^{n(j-1)} - (p-1)(p^{n(j-2)} + p^{n(j-3)} + \cdots + p^{nk});$$

$$K(n)_*\mathbf{K}_q \simeq K(n)_*, \qquad q > n.$$

(b) *The* a_I *with* $0 \le \bar{i}_1 < \bar{i}_2 < \cdots < \bar{i}_q < n$,

$$0 \le i_1 < i_2 < \cdots < i_q < nj,$$

and

$$n(j-1) < i_2$$

are all non-zero and distinct. In $\overline{K(n)}_*\mathbf{K}_q$, $q < n$, *each one is* $\pm$ *the* p^i*-th power of some generator for some* $i \ge 0$. *Every non-zero* a_I *can be written in this form.*

(c) *In* $\overline{K(n)}_*\mathbf{K}_q$, *the* p*-th powers of the* a_I *of* (b) *are computed as*

$$a_I{}^{*p} = 0 \qquad \text{if } i_1 < n-1,\ i_q < nj-1;$$

$$a_I{}^{*p} = a_{(i_1-n+1, i_2+1, \ldots, i_q+1)} \qquad \text{if } n-1 \le i_1,\ i_q < nj-1;$$

$$a_I{}^{*p} = (-1)^{q-1} a_{(n[i_1/n], n(j-1)+\bar{i}_1+1, i_2+1, \ldots, i_{q-1}+1)} \qquad \text{if } i_q = nj-1.$$

(d) *In* $\overline{K(n)}_*\mathbf{K}_q$, $q < n$, *we have up to signs, an equality of sets:*

$$\{a_I{}^{*p^{\rho(I)-1}} \mid n(j-1) < i_1 < i_2 < \cdots < i_q < nj\}$$
$$= \{a_J \mid J \text{ is as in (b) with } j_1 < n-1 \text{ and } j_q < nj-1\}.$$

(e) *The coalgebra structure follows from the Hopf ring properties and the structure of* $K(n)_*\mathbf{K}_1$ *given in Theorem* 5.7.

(f) *The Verschiebung map for* $\overline{K(n)}_*\mathbf{K}_q$ *is computed on the* a_I *of* (b) *as follows:*

$$V(a_I) = 0 \qquad \text{if } i_1 = 0;$$

$$V(a_I) = a_{(i_1-1, i_2-1, \ldots, i_q-1)} \qquad \text{if } \bar{i}_1 \ne 0;$$

$$V(a_I) = (-1)^{q-1} a_{(n[(i_1-1)/n]+\bar{i}_2-1, i_3-1, \ldots, i_q-1, nj-1)} \qquad \text{if } \bar{i}_1 = 0,\ i_1 \ne 0. \qquad \square$$

LEMMA 11.2. *In* $\overline{K(n)}_*\mathbf{K}_2$,

(a) $a_{(i)} \circ a_{(k)} = -a_{(k)} \circ a_{(i)}$,
(b) $a_{(i)} \circ a_{(i)} = 0$,
(c) $a_{(i)} \circ a_{(k)} = 0 \qquad \text{if } i < n \text{ and } k < n(j-1),$

$$= a_{(i-n)} \circ a_{(k+n)} \qquad \text{if } n \le i \text{ and } k < n(j-1). \qquad \square$$

Proof. Parts (a) and (b) are just 9.1. For part (c), in $\overline{K(n)}_*\mathbf{K}_2$

$$\begin{aligned} a_{(i)} \circ a_{(k)} &= a_{(i)} \circ F(a_{(k+n-1)}) && \text{by 5.7(d)} \\ &= F(V(a_{(i)}) \circ a_{(k+n-1)}) && \text{by 7.1(c)} \\ &= F(a_{(i-1)} \circ a_{(k+n-1)}) && \text{by 7.1(e)} \\ &= F(a_{(i-1)} \circ V(a_{(k+n)})) && \text{by 7.1(e)} \\ &= F(a_{(i-1)}) \circ a_{(k+n)} && \text{by 7.1(c)} \\ &= a_{(i-n)} \circ a_{(k+n)} && \text{by 5.7(d).} \qquad \square \end{aligned}$$

Remark. From 11.2 it is clear that $a_I = 0$ unless the i_k are distinct and

$$\sum_{k=1}^{q} [i_k/n] \ge (q-1)(j-1).$$

Partial proof of 11.1. It is immediate from 11.2 that every a_I can be written in the form of (b) or is equal to zero. By now, part (c) should be a standard computation using 5.7, 7.1 and 11.2. Part (d) follows tediously from part (c). Part (e) follows from Hopf ring properties, (a) and (c). Part (f) follows from 7.1 and 11.2. Part (a) and some of (b) are all that remain. $\square$

COROLLARY 11.3. *The Hopf ring* $K(n)_*\mathbf{K}_*$ *is the free*

$$K(n)_*[Z/(p^j)] \textit{ Hopf ring}$$

on the Hopf algebra $K(n)_*\mathbf{K}_1$. $\square$

Proof. The proof is similar to the proof of 9.3 but uses 11.1 in place of 9.2. $\square$

To complete our description of $K(n)_*\mathbf{K}_*$ we give the following result which we need in the proof of 11.1.

PROPOSITION 11.4. *Let* α: $\mathbf{K}(Z/(p^j), q) \to \mathbf{K}(Z/p^{i+1}, q)$ *and* β: $\mathbf{K}(Z/(p^{j+1}), q) \to \mathbf{K}(Z/(p^j), q)$ *be the maps induced by the inclusion* $Z/(p^j) \hookrightarrow Z/(p^{j+1})$ *and the standard surjection* $Z/(p^{j+1}) \to Z/(p^j)$ *respectively. Then for p an odd prime,*

$$\alpha_*(a_{(i_1,i_2,\ldots,i_q)}) = a_{(i_1,i_2+n,\ldots,i_q+n)} \in \overline{K(n)}_*\mathbf{K}(Z/(p^{j+1}), q),$$

and

$$\beta_*(a_{(i_1+n,i_2+n,\ldots,i_q+n)}) = a_{(i_1,i_2,\ldots,i_q)} \in \overline{K(n)}_*\mathbf{K}(Z/(p^j), q). \qquad \square$$

Remark. Note that both α_* and β_* preserve the form of 11.1(b).

Proof. The statement about β_* follows from Lemma 5.9 and the fact that $Z/(p^{j+1}) \to Z/(p^j)$ is a ring homomorphism and β_* therefore preserves $\circ$ products.

For α_*, the commutative diagram, with obvious maps,

$$\begin{tikzcd}
Z/(p^j) \otimes Z/(p^{j+1}) \arrow[r, hook] \arrow[d] & Z/(p^{j+1}) \otimes Z/(p^{j+1}) \arrow[dd] \\
Z/(p^j) \otimes Z/(p^j) \arrow[d] & \\
Z/(p^j) \arrow[r, hook] & Z/(p^{j+1}),
\end{tikzcd}$$

gives rise to the following homotopy commutative diagram, where μ is the map inducing $\circ$ product:

$$\begin{tikzcd}
\mathbf{K}(Z/(p^j),1)\times\mathbf{K}(Z/(p^{j+1}),q-1) \arrow[r, "\alpha\times 1"] \arrow[d, "1\times\beta"] & \mathbf{K}(Z/(p^{j+1}),1)\times\mathbf{K}(Z/(p^{j+1}),q-1) \arrow[dd, "\mu"] \\
\mathbf{K}(Z/(p^j),1)\times\mathbf{K}(Z/(p^j),q-1) \arrow[d, "\mu"'] & \\
\mathbf{K}(Z/(p^j),q) \arrow[r, "\alpha"] & \mathbf{K}(Z/(p^{j+1}),q).
\end{tikzcd}$$

From 5.9 we know $\alpha_*(a_{(i_1)}) = a_{(i_1)}$, so applying $\overline{K(n)}_*(-)$ to the above diagram we have

$$\begin{aligned} a_{(i_1, i_2+n, \dots, i_q+n)} &= a_{(i_1)} \circ a_{(i_2+n, \dots, i_q+n)} \\ &= \mu_*(a_{(i_1)} \otimes a_{(i_2+n, \dots, i_q+n)}) \\ &= \mu_*(\alpha_* \otimes 1_*)(a_{(i_1)} \otimes a_{(i_2+n, \dots, i_q+n)}) \\ &= \alpha_*\mu_*(1_* \otimes \beta_*)(a_{(i_1)} \otimes a_{(i_2+n, \dots, i_q+n)}) \\ &= \alpha_*\mu_*(a_{(i_1)} \otimes a_{(i_2, \dots, i_q)}) \\ &= \alpha_*(a_{(i_1, i_2, \dots, i_q)}) \end{aligned} \qquad \square$$

The proof of 11.1 is by induction. It uses the complete description of the bar spectral sequence.

THEOREM 11.5. *Let p be an odd prime with $\mathbf{K}_* = \mathbf{K}(Z/(p^j), *)$, $j > 0$. In the bar spectral sequence*

$$E^r_{**}\overline{K(n)}_*\mathbf{K}_{q-1} \Rightarrow \overline{K(n)}_*\mathbf{K}_q,$$

we have

(a) $$\begin{aligned} E^2_{**}\overline{K(n)}_*\mathbf{K}_{q-1} &\simeq H_{**}\overline{K(n)}_*\mathbf{K}_{q-1} \\ &\simeq E(\sigma([1]-[0])) \otimes \Gamma(\phi([1]-[0])^{*p^{j-1}}), \quad q = 1, \\ &\simeq \otimes_I E(\sigma a_I) \otimes_I \Gamma(\phi a_{(\bar{i}_1 - 1, i_2 - 1, \dots, i_{q-1}-1)}), \quad 1 < q \le n, \\ &\qquad n(j-1) < i_1 < i_2 < \cdots < i_{q-1} < nj, \\ &\simeq \overline{K(n)}_*, \quad n < q; \end{aligned}$$

(b) *the differentials are given by*

$$d^{2p^r-1}\gamma_{p^r}\phi a_J = r_I \sigma a_{(i_1+1, i_2+1, \dots, i_{q-1}+1)}$$

where

$$\begin{aligned} r_I &\neq 0, & i_q &= nj - 1 \\ j - 1 &= mq + c - 1 & 0 &< c \le q \\ r &= mm + \bar{i}_c + 1 \end{aligned}$$

$$
\begin{aligned}
n(j-1) \le i_1 < i_2 < \cdots < i_{q-1} < nj - 1 \\
j_1 &= \bar{i}_{c+1} - \bar{i}_c - 1 \\
j_k &= i_{c+k} - \bar{i}_c - 1, && 1 < k \le q - c \\
j_{q-c+1} &= nj + i_1 - i_c - 1 \\
j_{q-c+k} &= i_k - \bar{i}_c + n - 1 && 1 < k < c;
\end{aligned}
$$

(c) *modulo decomposables in* $E^{\infty}_{*,*}$, a_I *is represented by*

$$(-1)^{(c-1)(q-c+1)}\gamma_{p^{mn+\bar{i}_c}}\phi a_J$$

where $[i_1/n] = mq + c - 1$, $0 < c \le q$, J *is as in* (b) *and* I *is as in* 11.1(b). □

Proof of 11.1. The differentials of 11.5(b) fit the description of 6.10(b). This computes the $E^{\infty}_{*,*}$ term of the spectral sequence. Part (c) of 11.5 shows that the a_I of 11.1(b) are indeed all non-zero and distinct. The generators of E^{∞} are in 1-1 correspondence with the a_I of 11.1(b). The extension problems are all solved by 11.1(c) which follows from the Hopf ring structure. Part (a) of 11.1 follows and the proof of 11.1 is complete. □

Remark 11.6. Let a_I be as in 11.1(b) with $0 < q < n$. Define $\hat{I} = (\hat{i}_1, \ldots, \hat{i}_{n-q})$ as in 11.1(b) with $\{\bar{i}_k\} \cup \{\bar{\hat{i}}_k\} = \{0, 1, \ldots, n-1\}$ and $[i_1/n] + [\hat{i}_1/n] = j - 1$. Let x_0 be the $k = 0$, $q = n$ case of a_I in 11.1(a). It is clear that $a_I \circ a_{\hat{I}} = \pm x_0$ in $\overline{K(n)}_*\mathbf{K}_n$. Furthermore, it is clear that $\hat{I}$ is uniquely determined by this property.

LEMMA 11.7. *In* $\overline{K(n)}_*\mathbf{K}_*$, *with* I *and* J *as in* 11.1(b),

$$V^n a_I = \pm a_J \quad \textit{if and only if } a_{\hat{I}} = \pm F^n a_{\hat{J}}. \qquad \square$$

Proof.

$$
\begin{aligned}
x_0 &= \pm F^n(x_0) = \pm F^n(a_J \circ a_{\hat{J}}) \\
&= \pm F^n((V^n a_I) \circ a_{\hat{J}}) = \pm a_I \circ F^n(a_{\hat{J}}).
\end{aligned}
$$

By the uniqueness of $\hat{I}$ this concludes the proof. □

This lemma demonstrates the close relationship that exists between $\overline{K(n)}_* \mathbf{K}_q$ and $\overline{K(n)}_* \mathbf{K}_{n-q}$. Lemma 11.7 shows that the algebra structure of $\overline{K(n)}_* \mathbf{K}_q$ is determined by the coalgebra structure of $\overline{K(n)}_* \mathbf{K}_{n-q}$, (and vice-versa). This coalgebra structure is determined by 11.1(f). In the proof of 11.5, we compute the r such that $V^r a_I$ is a primitive and what that primitive is for each I. This is all we need for our proofs; and this allows us to read off the $\rho(I)$ of 11.1(a). For more detail we state the following lemma without proof.

LEMMA 11.8. *Suppose* $V^r a_I = \pm a_J$, *for* I *and* J *as in* 11.1(b). *Then* I *and* J *are related as follows:*

(a) $$\bar{i}_k = \begin{cases} \bar{r} + \bar{j}_{q-t+k} - n & \text{for } 1 \le k \le t \\ \bar{r} + \bar{j}_{k-t} & \text{for } t+1 \le k \le q, \end{cases}$$

(b) $$\bar{j}_k = \begin{cases} \bar{i}_{k+t} - \bar{r} & \text{for } 1 \le k \le q-t \\ \bar{i}_{k+t-q} + n - \bar{r} & \text{for } q-t+1 \le k \le q, \end{cases}$$

(c) $[i_1/n] = [j_1/n] + q\,[r/n] + t$

where

$$\begin{aligned} t &= \max\{\{0\} \cup \{s \mid \bar{j}_{q+1-s} + \bar{r} \ge n\}\} \\ &= \max\{\{0\} \cup \{s \mid \bar{i}_s < \bar{r}\}\}. \end{aligned}$$ □

This can be used not only to read off the iterated Verschiebung, but by the duality of 11.7, the iterated Frobenius.

Proof of Theorem 11.5. The proof is by induction on q and j. We know the result (9.4) for $j = 1$, all q. We know $q = 0$ and 1 which begins our induction. We assume 11.5 for everything less than q and j. We now compute the spectral sequence

$$E^r_{*,*}\overline{K(n)}_* \mathbf{K}_{q-1} \Rightarrow \overline{K(n)}_* \mathbf{K}_q$$

Part (a) follows from the $q - 1$ version of 11.1 together with 6.1, 6.6, 6.7 and 8.1.

We proceed to the proof of part (c). The α of 11.4 induces

$$\alpha_*: H_*{}_*\overline{K(n)}_*\mathbf{K}(Z/(p^{j-1}), q-1) \to H_*{}_*\overline{K(n)}_*\mathbf{K}(Z/(p^j), q-1).$$

For I and J as in part (c), $i_1 < n(j-1)$, we can use induction to see that $a_{(i_1, i_2-n, \ldots, i_q-n)}$ has been identified in $H_*{}_*\overline{K(n)}_*\mathbf{K}(Z/(p^{j-1}), q-1)$ as $(-1)^{(c-1)(q-c+1)}\gamma_{p^{mn+i_c}}\phi a_{(j_1, j_2-n, \ldots, j_{q-1}-n)}$. Applying α_* we see that part (c) follows if $i_1 < n(j-1)$. Furthermore, these elements are also permanent cycles by naturality. To make the identification for $i_1 \geq n(j-1)$ we will apply the iterated Verschiebung just as in the $q = 2$ case of 9.4. This technique works for all $i_1 > 0$. Actually the technique of 9.4 could be used to make the identifications once $q > j$. The differentials can be computed as in 9.4 once $q > j+1$. This explains the problem with $q = 2$ in 9.4. To prove part (c) for $i_1 \geq n(j-1)$, we just apply $V^{mn+\bar{i}_c}$ to a_I. Using 11.1(f) we see that this is

$$(-1)^{(c-1)(q-c+1)}a_{(0)} \circ a_{(j_1+1+n(j-1), j_2+1, \ldots, j_{q-1}+1)}$$

which has already been shown to be represented by $(-1)^{(c-1)(q-c+1)}\phi a_J$. This is equal to $V^{mn+\bar{i}_c}$ of $(-1)^{(c-1)(q-c+1)}\gamma_{p^{mn+\bar{i}_c}}\phi a_J$. Thus, since the iterated Verschiebungs are identified, part (c) holds modulo decomposables. The only elements in that filtration killed by V are the decomposables. We must now show that all of these elements are infinite cycles. They must survive because the a_I which they would represent reduce non-trivially to $\overline{K(n)}_*\mathbf{K}(Z/(p^{j-1}), q)$ by 11.4 and 11.1 for $j-1$.

To compute the differentials of part (b) we must first show that $\gamma_{p^r}\phi a_J$ cannot survive. The argument is the same as for $q = 2$ in 9.4. The map

$$p^j: \mathbf{K}_q \to \mathbf{K}_q$$

is null homotopic. If $\gamma_{p^r}\phi a_J$ survived to represent an element z, then $p^j{}_*z$ would be zero. Well, $p^j{}_*z = p^{j-1}{}_*(V(z))^{*p}$. From the spectral sequence we see that $V(z)$ is represented by an a_I with $i_q = nj-1$ and $i_1 \geq n(j-1)$. Then

$$\begin{aligned} p^j{}_*z &= p^{j-1}{}_*(a_I)^{*p} \\ &= p^{j-1}{}_*(-1)^{q-1}a_{(n(j-1), i_1+1, \ldots, i_{q-1}+1)} \\ &= (-1)^{q-1}a_{(0, i_1+1, \ldots, i_{q-1}+1)} \end{aligned}$$

recalling that $p_* = [p] \circ = F(V-)$. This last element is known to be non-zero because we have previously verified that it is represented by a permanent cycle in the spectral sequence. One can now check that 6.10 applies to our situation. This will determine r and the target for degree reasons. $\square$

The preceding proof is a bit sketchy because the reader is assumed to have a good grasp of this type of computation by this stage of the paper.

12. The integral Eilenberg-MacLane spaces. In this section we will compute the Hopf algebra structure of $\overline{K(n)}_* \mathbf{K}(Z, q)$ for p odd and all q. We also describe $\overline{K(n)}^* \mathbf{K}(Z, q)$ as a formal group of dimension $\binom{n-1}{q-2}$ for $q \geq 2$.

We recall definition 6.4 of the algebra

$$S(x_0, x_1, \ldots) \simeq \mathbf{F}_p[x_0, x_1, \ldots]/(x_0{}^p, x_1{}^p - x_0, \ldots, x_i{}^p - x_{i-1}, \ldots).$$

In practice we can abbreviate this to $S(x_0)$ without ambiguity.

We define

$$b_J \in \overline{K(n)}_* \mathbf{K}(Z, q + 1)$$

for $J = (j_1, j_2, \ldots, j_q)$ with $j_1 \geq 0$, $0 \leq \bar{j}_1 < j_2 < \cdots < j_q < n$, as the image

$$\delta_* a_I = b_J$$

where δ: $\mathbf{K}(Z/(p^j), q) \to \mathbf{K}(Z, q + 1)$ is the standard map, $j > [j_1/n]$, and $i_1 = j_1$, $i_k = n(j - 1) + j_k$, $k > 1$. The elements b_J are well defined by 11.4.

THEOREM 12.1. *Let p be an odd prime.*

(a) *As algebras,*

$\overline{K(n)}_* \mathbf{K}(Z, q + 1)$

$\simeq \overline{K(n)}_*[Z]$, *the group ring of Z over $K(n)_* \simeq \mathbf{F}_p$,* $\qquad q = -1;$

$$
\begin{array}{ll}
E(e_1), \text{ the exterior algebra on a} & \\
\text{one dimensional element,} & q = 0; \\
\otimes_J S(b_J), & \text{if } 0 < q < n; \\
0 \le j_1 < j_2 < \cdots < j_q < n-1 & \\
\otimes_J \mathbf{F}_p[b_J]/(b_J{}^{*p} - (-1)^{q-1} b_J), & \text{if } q = n; \\
j_1 = nk,\ k \ge 0,\ j_m = m-1,\ m > 1, & \\
\overline{K(n)}_* \simeq \mathbf{F}_p, & \text{if } q > n.
\end{array}
$$

(b) *Given $i > 0$ and b_J with $0 \le j_1 < \cdots < j_q < n-1$ there is some b_I such that $b_I{}^{*p^i} = \pm b_J$.*

(c) $b_J{}^{*p} = 0$ if $j_1 \le j_q < n-1$;

$$= b_{(j_1-n+1, j_2+1, \ldots, j_q+1)} \quad \text{if } n-1 \le j_1,\ j_q < n-1$$

$$= (-1)^{q-1} b_{(n[j_1/n], \bar{j}_1+1, j_2+1, \ldots, j_{q-1}+1)} \quad \text{if } j_q = n-1.$$

(d) *The coalgebra structure follows from the coalgebra structure of $\overline{K(n)}_* \mathbf{K}(Z/(p^j), q)$ and the fact that each b_J is in the image of δ_* for some j.*

(e) $V(b_J) = 0$ if $j_1 = 0$;

$$= b_{(j_1-1, \ldots, j_q-1)} \quad \text{if } \bar{j}_1 \ne 0;$$

$$= (-1)^{q-1} b_{(n[(j_1-1)/n]+j_2-1, j_3-1, \ldots, j_q-1, n-1)} \quad \text{if } \bar{j}_1 = 0,\ j_1 > 0. \qquad \square$$

COROLLARY 12.2. *For $q > 0$,*

$$\varinjlim_{j} \overline{K(n)}_* \mathbf{K}(Z/(p^j), q) \simeq \overline{K(n)}_* \mathbf{K}(Z, q+1),$$

where the limit is made with α_* of 11.4. $\square$

Proof. This is immediate from 12.1 and the definition of b_J. $\square$

The proof is really very simple. We will show that the direct limit over j of the bar spectral sequences

$$E^r_{*,*}\overline{K(n)}_*\mathbf{K}(Z/(p^j), q-1) \Rightarrow \overline{K(n)}_*\mathbf{K}(Z/(p^j), q)$$

gives the bar spectral sequence

$$E^r_{*,*}\overline{K(n)}_*\mathbf{K}(Z, q) \Rightarrow \overline{K(n)}_*\mathbf{K}(Z, q+1)$$

for $q > 1$.

THEOREM 12.3. *Let* p *be an odd prime with* $\mathbf{K}_* = \mathbf{K}(Z, *)$. *In the bar spectral sequence*

$$E^r_{*,*}\overline{K(n)}_*\mathbf{K}_q \Rightarrow \overline{K(n)}_*\mathbf{K}_{q+1},$$

we have

(a) $$\begin{aligned} H_{*\,*}\overline{K(n)}_*\mathbf{K}_q &\simeq E^2_{**}\overline{K(n)}_*\mathbf{K}_q \simeq E^\infty_{*,*}\overline{K(n)}_*\mathbf{K}_q \\ &\simeq \otimes_J \Gamma(\phi b_J), \qquad 1 < q \le n, \\ &\qquad 0 \le j_1 < j_2 < \cdots < j_{q-1} < n-1, \\ &\simeq \overline{K(n)}_*, \qquad n < q; \end{aligned}$$

(b) *modulo decomposables in* E^∞_{**}, b_I, $I = (i_1, \ldots, i_q)$ $q > 1$, *is represented by*

$$(-1)^{(c-1)(q-c+1)}\gamma_{p^{mn+\bar{i}_c}}\phi b_J$$

where

$$[i_1/n] = mq + c - 1, \qquad 0 < c \le q, \qquad i_1 \ge 0,$$
$$0 \le \bar{i}_1 < i_2 < \cdots < i_q < n,$$

and

$$j_k = \bar{i}_{c+k} - \bar{i}_c - 1, \qquad 0 < k \le q - c$$
$$j_{q-c+k} = \bar{i}_k - \bar{i}_c + n - 1 \qquad 0 < k < c. \qquad \square$$

Proof of 12.1. Part (c) follows from 11.1(c) and the definition of b_J. Part (d) is automatic from part (a) and (b). Part (b) follows from (c). Part (e) follows from the definition of the b_J and 11.1(f). All that remains is part (a). The case $\mathbf{K}(Z, 0)$ is trivial as is $\mathbf{K}(Z, 1) \simeq S^1$. Since $\mathbf{K}(Z, 2) \simeq \mathbf{C}P^\infty$, this is 5.6. By Theorem 12.3(a) the bar spectral sequence collapses. Part (b) of 12.3 identifies all of the appropriate b_J and then part (c) of 12.1 allows us to solve all extension problems to get the desired result. □

Proof of 12.3. We begin our induction by knowing $\overline{K(n)}_*\mathbf{K}(Z, 2)$ by 5.6 since $\mathbf{K}(Z, 2) \simeq CP^\infty$. We also know $\delta_*: \overline{K(n)}_*\mathbf{K}(Z/(p^j), 1) \to \overline{K(n)}_*\mathbf{K}(Z, 2)$ by 5.7. Assume we know $\overline{K(n)}_*\mathbf{K}(Z, q)$ from 12.1 for $q \geq 2$. Part (a) follows from 12.1(a) and 6.8, for $1 < q \leq n$, and 6.7 for $q = n + 1$. It is trivial for $q > n + 1$. The spectral sequence collapses because it is an even degree spectral sequence. Part (b) follows by induction, assuming $\overline{K(n)}_*\mathbf{K}(Z, q)$ from 12.1; naturality of the spectral sequence, using the map

$$\delta: \mathbf{K}(Z/(p^j), q - 1) \to \mathbf{K}(Z, q);$$

part (c) of 11.5; and the definition of b_J. □

We now describe $\overline{K(n)}^*\mathbf{K}(Z, q)$ for $q > 2$. The $q = 2$ case is trivial.

THEOREM 12.4. *Let p be an odd prime and $\mathbf{K}_q = \mathbf{K}(Z, q)$. For $q > 2$:*

(a) $\overline{K(n)}^*\mathbf{K}_q = \mathbf{F}_p[[x_S]]$ *where*

$$S = (s_1, s_2, \ldots, s_{q-2}) \text{ with } 0 < s_1 < s_2 < \cdots < s_{q-2} < n$$

and

$$\dim x_S = 2(1 + p^{s_1} + p^{s_2} + \cdots p^{s_{q-2}}).$$

and the topology (Section 7) is the m-adic one, m being the maximal ideal.

(b) *The coalgebra structure of $\overline{K(n)}^*\mathbf{K}_q$ is determined by*

$$V(x_S) = (-1)^q x_T{}^{p^a} \text{ where } a = s_1 - 1,\ t_k = s_{k+1} - s_1,$$

$$1 < k \leq q - 3 \text{ and } t_{q-2} = n - s_1.$$ □

Proof. The proof follows in a straightforward manner from 12.1 and 12.3 by duality. □

13. Exact sequences, $K(n)^*$, and the Johnson question. In this section we will sketch some related results.

COROLLARY 13.1. *Let p be an odd prime, then we have the following short exact sequences in the category of Hopf algebras over $K(n)_*$*

$$K(n)_* \longrightarrow \mathbf{K}(n)_*\mathbf{K}(Z/(p^j), q) \xrightarrow{\alpha_*{}^i} K(n)_*\mathbf{K}(Z/(p^{j+i}), q)$$

$$\xrightarrow{\beta_*{}^j} K(n)_*\mathbf{K}(Z/(p^i), q) \longrightarrow K(n)_*$$

$$K(n)_* \longrightarrow K(n)_*\mathbf{K}(Z/(p^j), q) \xrightarrow{\delta_*} K(n)_*\mathbf{K}(Z, q+1)$$

$$\xrightarrow{p_*{}^j} K(n)_*\mathbf{K}(Z, q+1) \longrightarrow K(n)_* \qquad \square$$

Proof. This follows from our description of these Hopf algebras and maps given in 11.1, 11.4 and 12.1. □

Since $K(n)^*X \simeq \hom_{K(n)_*}(K(n)_*X, K(n)_*)$ we can compute $K(n)^*\mathbf{K}_q$ directly from 11.1 and 12.1 by duality. For $\mathbf{K}_q = \mathbf{K}(Z/(p^j), q)$, $0 < q < n$, $K(n)^*\mathbf{K}_q$ is just a tensor product of truncated polynomial algebras whose height can be determined from the computation of the iterated Verschiebung in the proof of 11.5. The entire Hopf algebra structure follows by dualizing 11.1.

David C. Johnson has asked the following exciting question. Is it possible that given any space X, then $v_n{}^k x_n \neq 0$ for all k and all $0 \neq x_n \in \mathrm{BP}_n X$. Rephrased the question becomes: Is $\mathrm{BP}_*X \to v_n{}^{-1}\mathrm{BP}_*X$ injective for degrees less than or equal to n.

This is a very strong statement about the unstable BP_* module structure of spaces, and if true would be of great interest. The main evidence for this conjecture is the following result.

THEOREM 13.3. *For p an odd prime, if $0 \neq x_n \in BP_nX$ reduces nontrivially to $H_n(X; \pi)$, for $\pi = Z$ or $Z/(p^j)$ some $j > 0$, then $v_n{}^k x_n \neq 0$ for $k > 0$.* □

The proof is straightforward once the next result is known.

THEOREM 13.4. *Let p be an odd prime. Let $\iota_n \in BP_n\mathbf{K}(Z/(p^j), n)$ be the fundamental class. Then $v_n{}^k p^{j-1}\iota_n \neq 0$ for all $k \geq 0$.* □

Sketch of Proof. We can compute $v_n{}^{-1}\mathrm{BP}_*\mathbf{K}(Z/(p^j), n)$ in the same fashion as 10.5. The injection

$$K(n)_*\mathbf{K}(Z/(p), n) \xrightarrow{\alpha_*{}^{j-1}} K(n)_*\mathbf{K}(Z/(p^j), n)$$

forces an injection on all of the group associated with the computation of $v_n{}^{-1}\mathrm{BP}_*(-)$ of these Eilenberg-MacLane spaces. In particular

$$\alpha_*{}^{j-1}\colon v_n{}^{-1}\mathrm{BP}_*\mathbf{K}(Z/(p), n) \to v_n{}^{-1}\mathrm{BP}_*\mathbf{K}(Z/(p^j), n)$$

is an injection and

$$\alpha_*{}^{j-1}(\iota_n) = p^{j-1}\iota_n. \qquad \square$$

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REFERENCES

[1] J. F. Adams, *Stable Homotopy and Generalised Homology,* University of Chicago Press, Chicago, 1974.

[2] D. W. Anderson and L. Hodgkin, "The K-theory of Eilenberg-MacLane complexes," *Topology* **7** (1968), pp. 317-329.

[3] N. A. Baas, "On bordism theory of manifolds with singularities," *Math. Scand.* **33** (1973), pp. 279-302.

[4] E. H. Brown and F. P. Peterson, "A spectrum whose Z_p-cohomology is the algebra of reduced p-th powers," *Topology* **5** (1966), pp. 149-154.

[5] H. Cartan, "Operations dans les constructions acycliques," Seminaire H. Cartan, 1954/55, Expose 6.

[6] P. E. Conner and E. E. Floyd, *Differential Periodic Maps,* Springer, Berlin, 1964.

[7] T. tom Dieck, "Actions of finite abelian p-groups without stationary points," *Topology* **9** (1970), pp. 359-366.

[8] E. E. Floyd, "Actions of $(Z_p)^k$ without stationary points," *Topology* **10** (1971), pp. 327-336.

[9] L. Hodgkin, "On the K-theory of Lie groups," *Topology* **6** (1967), pp. 1-36.

[10] D. C. Johnson and W. S. Wilson, "Projective dimension and Brown-Peterson homology," *Topology* **12** (1973), pp. 327-353.

[11] D. C. Johnson and W. S. Wilson, "BP operations and Morava's extraordinary K-theories," *Math. Zeit.* **144** (1975), pp. 55-75.

[12] D. C. Johnson and W. S. Wilson, "The projective dimension of the complex bordism of Eilenberg-MacLane spaces," *Osaka Jour. Math.* **14** (1977), pp. 533-536.

[13] P. S. Landweber, "New applications of commutative algebra to Brown-Peterson homology," Proc. of 1978 Waterloo Topology Conference, Lecture Notes in Mathematics, No. 741. Berlin-Heidelberg-New York, Springer, 1979.

[14] H. R. Margolis, "Periodic spectra," preprint.

[15] H. R. Miller and D. C. Ravenel, "Morava stabilizer algebras and the localization of Novikov's E_*-term," *Duke J. Math.* **44** (1977), pp. 433-447.

[16] H. R. Miller, D. C. Ravenel, and W. S. Wilson, "Periodic phenomena in the Adams-Novikov spectral sequence," *Ann. of Math.* **106** (1977), pp. 469-516.

[17] H. R. Miller and W. S. Wilson, "On Novikov's Ext^1 modulo an invariant prime ideal," *Topology* **15** (1976), pp. 131-141.

[18] R. J. Milgram, "The bar construction and abelian H-spaces," *Ill. Jour. of Math.* **11** (1967), pp. 242-250.

[19] J. W. Milnor, "Construction of universal bundles II," *Ann. of Math.* **63** (1956), pp. 430-436.

[20] J. C. Moore, in Seminaire H. Cartan, 1954/55 and 1959/60.

[21] J. Morava, "A product for odd-primary bordism of manifolds with singularities," *Topology,* **18** (1979), pp. 177-186.

[22] D. G. Quillen, "On the formal group laws of unoriented and complex cobordism theory," *Bull. A.M.S.* **75** (1969), pp. 1293-1298.

[23] D. C. Ravenel, "The structure of Morava stabilizer algebras," *Inventiones Math.* **37** (1976), pp. 109-120.

[24] ______, "The cohomology of the Morava stabilizer algebras," *Math. Z.* **152** (1977), pp. 287-297.

[25] D. C. Ravenel and W. S. Wilson, "The Hopf ring for complex cobordism," *Jour. of Pure and Applied Alg.* **9** (1977), pp. 241-280.

[26] M. Rothenberg and N. E. Steenrod, "The cohomology of classifying spaces of H-spaces," *Bull. Amer. Math. Soc.* **71** (1965), pp. 872-875, and mimeographed notes.

[27] N. Shimada and N. Yagita, "Multiplications in complex bordism theory with singularities," *Pub. RIMS,* Kyoto Univ. **12** (1976), pp. 259-293.

[28] E. H. Spanier, *Algebraic Topology,* McGraw-Hill, New York, 1966.

[29] J. D. Stasheff, "Associated fibre spaces," *Mich. Math. Jour.* **15** (1968), pp. 457-470.

[30] D. Sullivan, "Singularities in spaces," Proceedings of the Liverpool Singularities Symposium II. *Lecture Notes in Mathematics, No. 209.* Berlin-Heidelberg-New York, Springer, 1971.

[31] R. W. Thomason and W. S. Wilson, "Hopf rings in the bar spectral sequence," Quarterly Jour. Math., to appear.

[32] J. W. Vick, "Some applications of the Rothenberg-Steenrod spectral sequence," *Osaka J. Math.* **11** (1974), pp. 87-103.

[33] U. Würgler, "On products in a family of cohomology theories associated to the invariant prime ideals of $\pi_*(\mathrm{BP})$, *Comment. Math. Helv.* **52** (1977), pp. 457-481.

[34] ______, On the relation of Morava K-theories to Brown-Peterson homology. *Topology and algebra* (*Proc. Colloq., Eidgenöss. Tech. Hochsch., Zurich,* 1977), pp. 269-280. *Monographies Enseignement Math.,* **26**. *Enseignement Math., Univ. Genève, Geneva,* 1978. **55** (14).

[35] N. Yagita, "A topological note on the Adams spectral sequence based on Morava's K-theory," *Proc. Amer. Math. Soc.* **72** (1978), pp. 613-617.