Journal of Pure and Applied Algebra 4 (1974) 41–45. © North-Holland Publishing Company

BIPOLYNOMIAL HOPF ALGEBRAS

Douglas C. RAVENEL
Columbia University, New York, N.Y., U.S.A.

and

W. Stephen WILSON
Princeton University, Princeton, N.J., U.S.A.

Communicated by A. Heller
Received 25 May 1973

0. Introduction

Definition. A graded connected bicommutative Hopf algebra is said to be *bipolynomial* if both it and its dual are polynomial algebras.

We will show that being bipolynomial of finite type over $\mathbf{Z}_{(p)}$ or $\mathbf{Z}_p$ determines the Hopf algebra structure. This result has applications in [2].

1. The simple case

Husemoller [1] studies a Hopf algebra $B_{(p)}$ over $\mathbf{Z}_{(p)}$, the integers localized at the prime p (or $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$). As an algebra, $B_{(p)}$ is polynomial on generators a_k of degree $2p^k d$ (for $\mathbf{Z}_2$, degree $2^k d$). As a Hopf algebra it is bipolynomial; in fact, it is isomorphic as Hopf algebras to its own dual. Let $B_{(p)}(n)$ be the sub-Hopf algebra generated by $a_0, ..., a_n$. We will show, by the following series of lemmas, that a bipolynomial Hopf algebra over $\mathbf{Z}_{(p)}$ which is the same size as $B_{(p)}$ is isomorphic as Hopf algebras to $B_{(p)}$.

Let $B_{(p)}(n)^*$ be the Hopf algebra dual to $B_{(p)}(n)$.

Lemma 1.1. *As an algebra, a minimal set of generators for $B_{(p)}(n)^*$ is given by the elements e_k dual to $(a_0)^{p^k}$ with respect to the monomial basis of $B_{(p)}(n)$.*

Proof. For $B_{(p)}$, see [1]. $B(0)^*$ is just the algebra of divided powers, i.e., $\gamma_i \gamma_j = (i,j)\gamma_{i+j}$ where γ_i is dual to $(a_0)^i$ and (i,j) is the binomial coefficient in $\mathbf{Z}_{(p)}$ (or $\mathbf{Z}_p$).

It is well known that the necessary generators here are the $\gamma_p k = e_k$. The inclusions $B_{(p)}(0) \to B_{(p)}(n) \to B_{(p)}$ take monomials to monomials, and so the maps are $\mathbf{Z}_{(p)}$ split. Since the maps are $\mathbf{Z}_{(p)}$ split, the dual maps $B_{(p)}(0)^* \leftarrow B_{(p)}(n)^* \leftarrow B_{(p)}^*$ are all onto. Since monomials go to monomials, the e_k for $B_{(p)}^*$ go to the e_k for $B_{(p)}(n)^*$. Since $B_{(p)}^* \to B_{(p)}(n)^*$ is onto and the e_k generate $B_{(p)}^*$, the e_k must also generate $B_{(p)}(n)^*$. Since $B_{(p)}(n)^* \to B_{(p)}(0)^*$ is onto, and the e_k are all necessary in $B_{(p)}(0)^*$, they must all be necessary in $B_{(p)}(n)^*$. □

Let H^* be a polynomial algebra over $\mathbf{Z}_{(p)}$ (or $\mathbf{Z}_p$) on generators f_k of degree $2p^k d$ (for $\mathbf{Z}_2$, $2^k d$).

Lemma 1.2. *If we have an onto algebra map $H^* \to B_{(p)}(n-1)^*$ in degrees $\leq 2p^n d$ ($2^n d$ for $\mathbf{Z}_2$), then it lifts to an algebra isomorphism $H^* \to B_{(p)}(n)^*$ in degrees $\leq 2p^n d$ ($2^n d$ for $\mathbf{Z}_2$):*

$$(\mathrm{A}^*)\qquad \begin{array}{ccc} B_{(p)}(n-1)^* & \longleftarrow & H^* \\ \uparrow & \swarrow & \\ B_{(p)}(n)^* & & \end{array}$$

Proof. $B_{(p)}(n-1)^* \simeq B_{(p)}(n)^* \simeq H^*$ in degrees $< 2p^n d$ ($2^n d$ for $\mathbf{Z}_2$), so our only concern is how to lift f_n.

Some multiple of e_n, say $cp^k e_n$, may be decomposable, where c is a unit in $\mathbf{Z}_{(p)}$. If this is so, then $p^k e_n$ is decomposable. k is greater than zero by Lemma 1.1. (In fact $k = n$, but all we care is that $\infty \geq k > 0$.) If the image of f_n in $B_{(p)}(n-1)^*$ is decomposable, then e_n is not in the image and $H^* \to B_{(p)}(n-1)^*$ is not onto by 1.1. So f_n goes to $cp^j e_n + g(e_1, e_2, \ldots, e_{n-1})$, where g is a polynomial and c is a unit. Thus we have $p^j e_n$ and $p^k e_n$ in the image, so if e_n is to be in the image we must have $j = 0$. So f_n goes to $ce_n + g(e_1, \ldots, e_{n-1})$ in $B_{(p)}(n-1)^*$. To obtain our lift for f_n, we just send it to $ce_n + g(e_1, \ldots, e_{n-1})$ in $B_{(p)}(n)^*$. This commutes and gives us our algebra map in degrees $\leq 2p^n d$, and since c is a unit, e_n is in the image and we have an isomorphism in those degrees. □

Let H be a bipolynomial Hopf algebra with the algebra structure given by $\mathbf{Z}_{(p)}[g_0, g_1, \ldots]$, where degree $g_k = 2p^k d$ ($2^k d$ for $\mathbf{Z}_2$). By rank considerations, H^* as an algebra must be $\mathbf{Z}_{(p)}[f_0, f_1, \ldots]$ as above.

Proposition 1.3. *$H \simeq B_{(p)}$ as Hopf algebras.*

Proof. We will define an isomorphism $B_{(p)} \to H$ step by step, starting with $B_{(p)}(0)$. Map $B_{(p)}(0) \to H$ by letting a_0 go to g_0.

Claim (n). If we have a Hopf algebra map $B_{(p)}(n-1) \to H$ which takes generators to generators, then we can extend the map to a Hopf algebra map of $B_{(p)}(n)$ in such a way that a_n goes to a generator:

$$\text{(A)}\qquad \begin{array}{ccc} B_{(p)}(n-1) & \dashrightarrow & H \\ \downarrow & \nearrow & \\ B_{(p)}(n) & & \end{array}$$

This will prove the proposition, because by induction we get a Hopf algebra map $B_{(p)} \to H$ which is an algebra isomorphism, and so it is a Hopf algebra isomorphism.

Proof of claim (n). $B_{(p)}(n-1) \to H$ takes generators to generators, so since H is a polynomial algebra, $H_{2p^n d} = \mathbf{Z}_{(p)} \oplus \text{image}\, B_{(p)}(n-1)_{2p^n d}$. We can send a_n to any element which has the proper coproduct and get a map of Hopf algebras $B_{(p)}(n) \to H$ extending $B_{(p)}(n-1) \to H$. Such a map $B_{(p)}(n) \to H$ gives a $\mathbf{Z}_{(p)}$ isomorphism in degree $2p^n d$ iff a_n goes to an algebra generator. Therefore all we must do is extend the (coalgebra) map $B_{(p)}(n-1) \to H$ to a coalgebra map $B_{(p)}(n) \to H$ in dimensions $\leqslant 2p^n d$ in such a way that $B_{(p)}(n)_{2p^n d} \to H_{2p^n d}$ is a $\mathbf{Z}_{(p)}$ isomorphism. Because the a_i go to polynomial generators in H, $B_{(p)}(n-1) \to H$ is $\mathbf{Z}_{(p)}$ split and the dual map is onto. We have now reduced the problem to a purely coalgebra statement, the dual of which we have already proven as Lemma 1.2, so we are done. Similarly for $\mathbf{Z}_p$. □

2. The general case

We adopt the notation of [1]. Our Hopf algebra $B_{(p)}$ will be denoted $B_{(p)}[x, 2d]$ with generators $a_k(x)$ of degree $2p^k d$. We let $B_{(p)}[x, 2d](2n)$ be the sub-Hopf algebra of $B_{(p)}[x, 2d]$ generated by the $a_k(x)$ with degree $a_k(x) = 2p^k d \leqslant 2n$. Note that this is different from our $B_{(p)}(n)$. We need to generalize the lemmas found in Section 1. Note that

$$[\otimes_j B_{(p)}[x_j, 2d_j](2n)]^* \simeq \otimes_j B_{(p)}[x_j, 2d_j](2n)^*$$

(for j over a finite indexing set). Let $e_k(x_j) \in \otimes_j B_{(p)}[x_j, 2d_j]^*$ be dual to $[a_0(x_j)]^{p^k}$ in the monomial basis for $\otimes_j B_{(p)}[x_j, 2d_j](2n)$.

Lemma 2.1. *As an algebra, a minimal set of generators for*

$$\otimes_j B_{(p)}[x_j, 2d_j](2n)^*$$

is given by the $e_k(x_j)$ for $d_j \leqslant n$.

Proof. It is enough to do this for $B_{(p)}[x_j, 2d_j](2n)$, which is done in Lemma 1.1. □

Let H^* be a polynomial algebra of finite type over $\mathbf{Z}_{(p)}$ (or $\mathbf{Z}_p$).

Lemma 2.2. *If we have an onto algebra map* ($d_j < n$)

$$H^* \to \otimes_j B_{(p)}[x_j, 2d_j](2(n-1))^*$$

which is an isomorphism in degrees $\leq 2(n-1)$, *then it lifts to an algebra isomorphism*

$$H^* \to \otimes_j B_{(p)}[x_j, 2d_j](2n)^* \otimes_i B_{(p)}[x_i, 2n](2n)^*$$

in degrees $\leq 2n$.

Proof. We need only worry about what happens in dimension $2n$. The $2n$-dimensional generators of $\otimes_j B_{(p)}[x_j, 2d_j](2(n-1))^*$ are the $e_k(x_j)$, where $n = p^k d_j$. So, since we have this onto map, just as in Lemma 1.2 there must be degree $2n$ generators $e'_k(x_j)$ in H^* such that $e'_k(x_j)$ maps to $e_k(x_j)$ + decomposables, $p^k d_j = n$. As in 1.2, we just map $e'_k(x_j)$ to $e_k(x_j)$ + same decomposables in $\otimes_j B_{(p)}[x_j, 2d_j](2n)^*$. Now look at the kernel of this map in dimension $2n$. Find $\mathbf{Z}_{(p)}$ generators e_k of the kernel. They will all be algebra generators of H^*. Map them to $e_0(x_k)$ in $B_{(p)}[x_k, 2n](2n)^*$. This completes the proof. □

Proposition 2.3. *If H is a bipolynomial Hopf algebra of finite type over* $\mathbf{Z}_{(p)}$ (*or* $\mathbf{Z}_p$), *then*

$$H \simeq \otimes_j B_{(p)}[x_j, 2d_j]$$

as Hopf algebras (for $\mathbf{Z}_2$ *replace* $2d_j$ *by* d_j).

Proof. We define an isomorphism

$$\otimes_j B_{(p)}[x_j, 2d_j] \to H$$

step by step starting with $\otimes_i B_{(p)}[x_i, 2](2) \to H$ which is easily constructed.

Claim (n). If we have a Hopf algebra map ($d_j < n$)

$$\otimes_j B_{(p)}[x_j, 2d_j](2(n-1)) \to H\,,$$

which is an isomorphism for dimensions $\leq 2(n-1)$, then we can extend this to a Hopf algebra map

$$\otimes_j B_{(p)}[x_j, 2d_j](2n) \otimes_i B_{(p)}[x_i, 2n](2n) \to H\,,$$

which is an isomorphism for dimensions $\leq 2n$.

This will prove the proposition by letting $n \to \infty$ and the fact that $B_{(p)}[x_j, 2d_j] \simeq B_{(p)}[x_j, 2d_j](2n)$ in degrees $\leq 2n$.

Proof of claim (n). All of the algebra generators of $\otimes_j B_{(p)}[x_j, 2d_j](2(n-1))$ are of degree $\leqslant 2(n-1)$. The given map is an isomorphism in this range and so generators go to generators. Since both algebras are polynomial algebras, this means that the map is $\mathbf{Z}_{(p)}$ split and the dual is onto. Our only problem is in deciding where to send the new generators, i.e. $a_k(x_j)$ for $p^k d_j = n$, and $a_0(x_i)$. Any coalgebra extension to these elements also gives a Hopf algebra extension. If it is a coalgebra isomorphism through dimension $\leqslant 2n$, then it is a Hopf algebra isomorphism. We have reduced the problem to a pure coalgebra problem which is dual to Lemma 2.2 which we have already proven, so we are done. □

For $\mathbf{Z}_2$, just eliminate the factor of 2 everywhere.

We would like to thank the referee for finding a mistake in the original version of this paper where we thought we could prove the result over the integers. The problem is still open there. (*Added in proof*: Brian Shay has found counterexamples to the integer analogue of Proposition 2.3.)

References

[1] D. Husemoller, The structure of the Hopf algebra $H_*(BU)$ over a $\mathbf{Z}_{(p)}$-algebra, Am. J. Math. 93 (1971) 329–349.

[2] W.S. Wilson, The Ω-spectrum for Brown–Peterson cohomology, Part I, Comment. Math. Helv., to appear; Part II, Am. J. Math., to appear.