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The non-existence of odd primary Arf invariant elements in stable homotopy

BY DOUGLAS C. RAVENEL†
University of Washington, Seattle

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0. *Introduction.* In this paper we will show that certain elements of order p (p an odd prime) on the 2-line of the Adams–Novikov spectral sequence support non-trivial differentials and therefore do not detect elements in the stable homotopy groups of spheres. These elements are analogous to the so-called Arf invariant elements of order 2, hence the title. However, it is evident that the methods presented here do not extend to the prime 2.

From our point of view the methods used here are possibly more interesting than the result itself. They rely heavily on the deeper algebraic structure of complex cobordism theory originally perceived by Jack Morava (21) and exploited in our earlier papers ((18), (20), (24) and (25)). Our main application of these ideas is Theorem 4, which states that a large number of elements in the Novikov E_2-term are non-zero. These elements can be detected by a certain homomorphism from the Novikov E_2-term to the ordinary mod p cohomology of the group of order p. The existence and basic properties of this homomorphism, which we are unable to construct explicitly, follow from our previous results (Theorems 11, 12 and 13) and an additional fact from algebraic number theory (Theorem 14). The idea is to map the Novikov E_2-term, $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$ to $\mathrm{Ext}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)$.

Morava's work, as interpreted by the above theorems, implies that this Ext is essentially the mod p continuous cohomology of a certain pro-p group S_n, which in turn is the group of proper units in a division algebra over the p-adic numbers with Hasse invariant $1/n$. (For details see (24).) Theorem 14 asserts that for $n = p-1$ this division algebra has a primitive pth root of unity and it follows that the group S_{p-1} has a subgroup of order p. It is the cohomology of this subgroup that detects the elements mentioned in Theorem 4.

The proofs of Theorems 11 and 13 do not reveal the insight that led to their formulation, which I will try to indicate now. Quillen (23) observed that the complex cobordism ring is isomorphic to the Lazard ring, over which the universal one-dimensional formal group law is defined. He then used this connexion between formal groups and complex cobordism to determine the algebra of cohomology operations for BP cohomology, the dual of which is BP_*BP. Haynes Miller has observed (17) that BP_*BP is a cogroupoid object in the category of commutative $\mathbb{Z}_{(p)}$-algebras; he calls such an object a Hopf algebroid. This means that for any commutative $\mathbb{Z}_{(p)}$-algebra R, the set of algebra

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homomorphisms from BP_*BP to R is a groupoid, i.e. a small category in which every morphism is an equivalence. Peter Landweber (10) has observed that in this case the category is that of p-typical formal group laws over R and isomorphisms between them. The structure of this small category is reflected in the structure of BP_*BP.

Very loosely speaking, the Novikov E_2-term may be thought of as the cohomology of this groupoid with coefficients in BP_*. The homomorphism from this cohomology to that of S_n can be thought of as a restriction map obtained by restricting to a smaller class of formal group laws, namely those of height n (to be defined). It is known ((7), pp. 72–86) that the automorphism group of any height n one-dimensional formal group defined over a field containing $\mathbb{F}_{p^n}$ is $S_n \oplus \mathbb{F}_{p^n}^\times$. This accounts for the connexion between $\mathrm{Ext}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)$ and H^*S_n implied by Theorems 11 and 13. To define the height of a formal group law F we first define power series $[m]_F(x)$ for $m > 0$ by induction on m. Let $[1]_F(x) = x$ and $[m]_F(x) = F(x, [m-1]_F(x))$. Of particular interest is $[p]_F(x)$. If the ground ring has characteristic p, it can be shown (7) that the leading term of $[p]_F(x)$ is a non-zero multiple of x^{p^n} for some n, and this is defined to be the height of F.

It seems very unlikely that these methods can be carried over to the classical Adams spectral sequence.

The plan of the paper is as follows. In Section 1 we state Theorem 4 and use it to prove our main result, Corollary 5, which states that certain elements

$$b_i \in \mathrm{Ext}_{BP_*BP}^{2,(2p-2)p^{i+1}}(BP_*, BP_*)$$

support non-trivial differentials (d_{2p-1}) in the Novikov spectral sequence, for $i > 0$. For $i = 1$, this result is due to Toda (Theorem 1). Our argument is an induction based on certain relations among these elements (Theorem 3). We use Theorem 4 to prove that the images of the computed differentials are non-zero. The computation of Theorem 3 is equally valid in the Adams spectral sequence, but to draw the desired conclusion from it would require showing that the images of the differentials are non-zero in the Adams E_{2p-1}-term. This would be very difficult if not impossible. In the first non-trivial case for $p = 3$, the desired element is zero in the Adams E_2-term (Proposition 2). In the Novikov spectral sequence the E_2- and E_{2p-1}-terms are equal for dimensional reasons, so Theorem 4 suffices.

As a consequence of our main result, we prove Theorem 6 which says that certain BP_* modules cannot be realized as the BP-homology of any connective spectrum.

In Section 2 we prove Theorem 4 in the manner described above.

In Section 3 we examine the elements b_i in the classical Adams (as opposed to the Novikov) spectral sequence. Although there is a map from the Novikov to the Adams spectral sequence, the fact that our elements support non-trivial differentials in the former does not imply that they do so in the latter. Indeed it can be shown that, for $p = 3$, the element b_2, which has dimension 106, is a permanent cycle (see the discussion following the statement of Theorem 17). However, we are able to show (Theorem 17) that this never happens for $p \geqslant 5$. In the process of proving this we use a new technique (the proof of Lemma 18) for computing the mod I_n reduction of certain elements in the

Novikov E_2-term. We mention this method now because it may be useful in other contexts.

It is a pleasure to thank the referee for his diligence in suggesting many improvements in this paper. We also thank S. Oka for correcting an earlier proof of Theorem 6.

1. *The main result.* To define the elements in question recall there is a spectral sequence due to Adams (3)

$$E_2^{**} = \mathrm{Ext}_{A_p}^{**}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_{*(p)}^s,$$

where A_p is the mod p Steenrod algebra and $\pi_{*(p)}^s$ denotes the p-component of the stable homotopy of the sphere spectrum. For $p > 2$ there are elements

$$h_i \in \mathrm{Ext}_{A_p}^{1, qp^i}(\mathbb{F}_p, \mathbb{F}_p),$$

where $q = 2(p-1)$ (and for $p = 2$, $h_i \in \mathrm{Ext}_{A_2}^{1, 2^i}(\mathbb{F}_2, \mathbb{F}_2)$) and $i \geqslant 0$. These elements correspond to the indecomposable elements $\mathscr{P}^{p^i} \in A_p (Sq^{2^i} \in A_2)$ and are commonly known as the Hopf invariant elements. For $p = 2$ Adams showed (1) that these elements support non-trivial differentials for $i \geqslant 4$. Using similar methods, Liulevicius (11), and Shimada and Yamanoshita (29) obtained the analogous result for $p > 2$ and $i \geqslant 1$.

For $p = 2$, the h_i^2 are the celebrated Arf invariant elements, so named because Browder (5) showed that for $i > 0$, h_i^2 is a permanent cycle in the Adams spectral sequence if and only if there exists a framed $(2^{i+1}-2)$-manifold with non-trivial Arf invariant. At the present time such manifolds are known to exist for $i \leqslant 5$ ((4), (16)).

For $p > 2$ one can define analogous elements $b_i \in \mathrm{Ext}_{A_p}^{2, qp^{i+1}}(\mathbb{F}_p, \mathbb{F}_p)$ as p-fold Massey products (see (14))

$$b_i = -\langle h_i, h_i \ldots h_i \rangle.$$

Equivalently b_i corresponds to the Adem relation for $\mathscr{P}^{(p-1)p^i}\mathscr{P}^{p^i}$ and is represented in the cobar construction (see (11) or (20)) by the element

$$\sum_{0<j<p} \frac{1}{p}\binom{p}{j}[\xi_1^{jp^i} | \xi_1^{(p-j)p^i}].$$

The element b_0 is a permanent cycle and detects the homotopy element $\beta_1 \in \pi_{pq-2}^s$ given by the p-fold Toda bracket (see (31))

$$\beta_1 = -\langle \alpha_1, \alpha_1 \ldots \alpha_1 \rangle,$$

where $\alpha_1 \in \pi_{q-1}^s$ is the element detected by h_0. α_1 is the first non-trivial positive dimensional element in the p-component of stable homotopy while β_1 is the first non-trivial element in the cokernel of the J-homomorphism. (Most of these facts can be found in (31).)

Theorem 17 states that the elements b_i for $i > 0$ do not detect stable homotopy elements for $p > 3$. The first result in this direction was obtained in 1967 by Toda (32), (33)).

THEOREM 1. *In the Adams spectral sequence for $p > 2$ there are non-trivial differentials $d_{2p-1} b_1 = h_0 b_0^p$, up to multiplication by a non-zero element of $\mathbb{F}_p$, i.e. b_1 does not detect a homotopy element and $\alpha_1 \beta_1^p = 0$.*

Toda's proof involves a subtle geometric argument related to the non-associativity of the mod p Moore spectrum.

Theorem 1 made it natural (to the author at least) to conjecture that

$$d_{2p-1}b_{i+1} = h_0 b_i^p$$

for all i, and that one would prove this by some generalization of Toda's extended power construction (32). This program was discouraged by the following result, which was first proved by J. P. May in (15), part II.

PROPOSITION 2. *For* $p = 3$, $h_0 b_1^3 = 0$ *in* $\mathrm{Ext}_{A_3}(\mathbb{F}_3, \mathbb{F}_3)$, i.e. b_2 *cannot support the expected non-trivial differential.*

Proof. We use a certain Massey product identity ((14), cor. 3·2) and very simple facts about $\mathrm{Ext}_{A_3}(\mathbb{F}_3, \mathbb{F}_3)$ to show $h_0 b_1^2 = 0$. We have

$$\begin{aligned} h_0 b_1^2 &= -h_0\langle h_1, h_1, h_1\rangle b_1 \\ &= -\langle h_0, h_1, h_1\rangle h_1 b_1 \end{aligned}$$

by (14). We will show in the proof of Theorem 3 that $h_1 b_1 = h_2 b_0$, so

$$\begin{aligned} h_0 b_1^2 &= -\langle h_0, h_1, h_1\rangle h_2 b_0 \\ &= -\langle h_1, h_0, h_1\rangle h_2 b_0 \\ &= -h_1\langle h_0, h_1, h_2\rangle b_0. \end{aligned}$$

The element $\langle h_0, h_1, h_2\rangle$ is represented in the cobar construction by $[\xi_1^9|\xi_2] + [\xi_2^3|\xi_1]$ which is the coboundary of $[\xi_3]$ so $h_0 b_1^2 = 0$. ∎

We will see below that this problem disappears if instead of using the classical Adams spectral sequence, we use the Adams–Novikov spectral sequence (see (22), (34) or (20)), i.e. the analogue of the Adams spectral sequence based on complex cobordism theory or equivalently, if we localize at a prime, Brown–Peterson homology theory (6). Recall there is a multiplicative homology theory BP_* with

$$BP_*(pt.) = \mathbb{Z}_{(p)}[v_1, v_2 \ldots] \quad (\text{where} \dim v_i = 2(p^i - 1)),$$

which will hereafter be denoted simply by BP_*. The analogue in this theory of the dual of the Steenrod algebra is $BP_* BP$ which as an algebra is

$$BP_*[t_1, t_2 \ldots] \quad \text{with} \quad \dim t_i = 2(p^i - 1),$$

(see (23) or (2)). There is a spectral sequence due to Novikov and Adams

$$\mathrm{Ext}^{**}_{BP_*BP}(BP_*, BP) \Rightarrow \pi^s_{*(p)}.$$

(For more details we refer the reader to (20).)

One can show by other methods that the analogue of Proposition 2 for the Adams–Novikov, E_2-term is $h_0 b_1^2 = \pm h_0 b_0 \beta_4 = \pm \alpha_1 \beta_1 \beta_4$ (see (20) or (19) for the definition of β_4). The latter term corresponds to an element of filtration 7 (whereas $h_0 b_1^2$ has filtration 5) in the Adams E_2-term. This is an example of one of the advantages of the Adams–Novikov spectral sequence: elements tend to have lower filtration and the E_2-term therefore gives more information.

There is a natural map from the Adams–Novikov spectral sequence to the Adams spectral sequence (see (20), section 9) which enables us to pull b_i back to

$$\mathrm{Ext}^{2,qp^{i+1}}_{BP_*BP}(BP_*, BP_*),$$

i.e. we define b_i to be the element represented in the cobar construction by

$$-\sum_{0<j<p^{i+1}} \frac{1}{p}\binom{p^{i+1}}{j}[t_1^j|t_1^{p^{i+1}-j}].$$

We also define $$h_0 = -[t_1] \in \mathrm{Ext}^{1,q}_{BP_*BP}(BP_*, BP_*).$$

Now we can state our main results. As in Theorem 1, the equations are valid up to a non-zero factor which we ignore.

THEOREM 3. *In the Adams–Novikov spectral sequence for $p > 2$*

$$d_{2p-1}b_{i+1} \equiv h_0 b_i^p \text{ modulo } \ker b_0^{a_i},$$

where $a_i = p(p^i-1)/(p-1)$.

THEOREM 4. *The elements $b_0^{i_0}b_1^{i_1}\ldots b_k^{i_k}$ and $h_0 b_0^{i_0}\ldots b_k^{i_k} \in \mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$ are non-zero for all exponents $i_0, i_1 \ldots \geqslant 0$.*

COROLLARY 5. *$d_{2p-1}b_{i+1} \neq 0$ for all $i \geqslant 0$.*

Proof. The Adams–Novikov spectral sequence has the convenient sparseness property $E_2^{s,t} = 0$ if $t \not\equiv 0 \bmod q$. Hence $d_r = 0$ if $r \not\equiv 1 \bmod q$ so $E_2 = E_{2p-1}$ so $b_0^{a_i}h_0 b_i^p \neq 0$ in E_{2p-1} since it is non-trivial in E_2. ∎

THEOREM 6. *There is no connective spectrum X such that*

$$BP_* X = BP_*/(p, v_1^{p^i}, v_2^{p^i})$$

for $i > 0$ and $p > 2$.

Proof. Using methods developed by Smith (30) one can show that such an X must be an 8-cell complex and that there must be cofibrations

(i) $\Sigma^{2p^i(p^2-1)}Y \xrightarrow{f} Y' \to X$,

(ii) $\Sigma^{2p^i(p-1)}V(0) \xrightarrow{g} V(0) \to Y$,

(iii) $\Sigma^{2p^i(p-1)}V(0) \xrightarrow{g'} V(0) \to Y'$,

where $V(0)$ is the mod p Moore spectrum, g and g' induce multiplication by $v_1^{p^i}$ in $BP_*V(0) = BP_*/(p)$, and f induces multiplication by $v_2^{p^i}$ in

$$BP_*Y = BP_*Y' = BP_*/(p, v_1^{p^i}).$$

$V(0)$ and the maps g, g' certainly exist, e.g. Smith showed that there is a map

$$\alpha: \Sigma^{2(p-1)}V(0) \to V(0)$$

which includes multiplication by v_1. Hence α^{p^i} induces multiplication by $v_1^{p^i}$, but it may not be the only map that does so.

Hence we have to show that the existence of f leads to a contradiction. Consider the composite

$$S^{2p^i(p^2-1)} \xrightarrow{j} S^{2p^i(p^2-1)}Y \xrightarrow{f} Y' \xrightarrow{k} S^{2+2p^i(p-1)},$$

where j is the inclusion of the bottom cell and k is the collapse onto the top cell. We will show that the resulting element in $\pi^s_{2p^{i+1}(p-1)-2}$ would be detected in the Novikov spectral sequence by b_i, thus contradicting Corollary 5. The cofibrations (ii) and (iii) induce the following short exact sequence of BP_* modules

$$0 \to \Sigma^{2p^i(p-1)} BP_*/(p) \xrightarrow{v_1^{p^i}} BP_*/(p) \to BP_*/(p, v_1^{p^i}) \to 0,$$

and the cofibration $$S^0 \xrightarrow{p} S^0 \to V(0)$$

induces $$0 \to BP_* \xrightarrow{p} BP_* \to BP_*/(p) \to 0.$$

Hence we get connecting homomorphisms

$$\delta_1 : \operatorname{Ext}^0_{BP_*BP}(BP_*, BP_*/(p, v_1^{p^i})) \to \operatorname{Ext}^1_{BP_*BP}(BP_*, BP_*/(p))$$

and $$\delta_0 : \operatorname{Ext}^1_{BP_*BP}(BP_*, BP_*/(p)) \to \operatorname{Ext}^2_{BP_*BP}(BP_*, BP_*).$$

The element $fj \in \pi_{2p^i(p^2-1)} Y'$ is detected by $v_2^{p^i} \in \operatorname{Ext}^0_{BP_*BP}(BP_*, BP_*/(p, v_1^{p^i})$. The main result (theorem 1·7) of (9) implies that

$$\delta_0 \, \delta_1(v_2^{p^i}) \in \operatorname{Ext}^2_{BP_*BP}(BP_*, BP_*)$$

detects the element $kfj \in \pi_{2p^{i+1}(p-1)-2} S^0$.

In the proof of Lemma 18 below it is shown that this element is b_{i+1}, so we have the desired contradiction. ∎

The proof of Theorem 3 is, modulo Theorem 1 (which is also valid in the Adams–Novikov spectral sequence), an algebraic induction argument which is also valid in the Adams spectral sequence.

Theorem 4 is obviously not valid in the Adams spectral sequence, by Proposition 2. Its proof is the main ingredient of this paper; it relies on the theory of Morava stabilizer algebras developed in (18) and (24) along with an additional result from algebraic number theory (Theorem 14). Note that it does not assert that the indicated elements are linearly independent, which indeed they are not, but merely that they are all non-zero. Equation (10) below gives some linear relations among them, which are essential to our argument.

Proof of Theorem 3. We begin with a computation in $\operatorname{Ext}_{BP_*BP}(BP_*, BP/(p))$. We use the symbol b_i to denote the mod p reduction of the b_i defined above in

$$\operatorname{Ext}_{BP_*BP}(BP_*, BP_*).$$

We also let h_i denote the element $-[t_1^{p^i}]$. In the cobar construction we have

$$d[t_2] = -[t_1|t_1^p] + v_1 \sum_{0<j<p} \frac{1}{p}\binom{p}{j} [t_1^j|t_1^{p-j}]$$

so

(7) $$v_1 b_0 = -h_0 h_1.$$

In (13) May developed a general theory of Steenrod operations which is applicable to this Ext group. His operations are similar to the classical ones in ordinary cohomology, except for the fact that $\mathscr{P}^0 \neq 1$. Rather we have $\mathscr{P}^0 h_i = h_{i+1}$ and $\mathscr{P}^0 b_i = b_{i+1}$. We also have $\beta\mathscr{P}^0 h_i = b_i$, $\beta\mathscr{P}^0 b_i = 0$, $\beta\mathscr{P}^0 v_1 = 0$, $\mathscr{P}^1 b_i = b_i^p$ and the Cartan formula implies that $\mathscr{P}^{j} b_i^{p^j} = b_i^{p^{j+1}}$. Applying $\beta\mathscr{P}^0$ to (7) gives

$$0 = b_0 h_2 - h_1 b_1. \tag{8}$$

(The analogous equation in $\mathrm{Ext}_{A_3}(\mathbb{F}_3, \mathbb{F}_3)$ is used in the proof of Proposition 2.) If we apply the operation $\mathscr{P}^{p^{i-1}}\mathscr{P}^{p^{i-2}}\ldots\mathscr{P}^1$ to (8) we get

$$h_{1+i} b_1^{p^i} = h_{2+i} b_0^{p^i}. \tag{9}$$

Now associated with the short exact sequence

$$0 \to BP_* \xrightarrow{p} BP_* \to BP_*/(p) \to 0$$

there is a connecting homomorphism

$$\delta\colon \mathrm{Ext}^{s,*}_{BP_*BP}(BP_*, BP_*/(p)) \to \mathrm{Ext}^{s+1,*}_{BP_*BP}(BP_*, BP_*)$$

with $\delta(h_{i+1}) = b_i$. Applying δ to (9) gives

$$b_i b_1^{p^i} = b_{i+1} b_0^{p^i} \in \mathrm{Ext}_{BP_*BP}(BP_*, BP_*). \tag{10}$$

We can now prove the theorem by induction on i, using Theorem 1 to start the induction. We have for $i > 0$

$$\begin{aligned} d_{2p-1}(b_{i+1})\, b_0^{p^i} &= d_{2p-1}(b_{i+1} b_0^{p^i}) \\ &= d_{2p-1}(b_i b_1^{p^i}) \\ &= d_{2p-1}(b_i)\, b_1^{p^i} \\ &\equiv h_0 b_{i-1}^p b_1^{p^i} \bmod \ker b_0^{a_{i-1}} \\ &\equiv h_0 (b_{i-1} b_1^{p^{i-1}})^p \\ &\equiv h_0 (b_i b_0^{p^{i-1}})^p \\ &\equiv h_0 b_i^p b_0^{p^i} \end{aligned}$$

so

$$d_{2p-1} b_{i+1} \equiv h_0 b_i^p \bmod \ker b_0^{a_i}. \blacksquare$$

2. *The proof of Theorem* 4. We will prove Theorem 4 by showing that the indicated elements map non-trivially to $\mathrm{Ext}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)$, where

$$I_n = (p, v_1 \ldots v_{n-1}) \subset BP_* \quad \text{and} \quad n = p-1,$$

and the map is induced by the obvious map $BP_* \to v_n^{-1}BP_*/I_n$. We first need to recall some of the results from (18) and (24) concerning this Ext group. Let

$$K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}] \quad \text{for} \quad n > 0$$

and make it a BP_* module by sending v_i to zero for $i \neq n$. Then let

$$K(n)_* K(n) = K(n)_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} K(n)_*.$$

Then we have

THEOREM 11 (18). *For all* $n > 0$

$$\mathrm{Ext}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n) = \mathrm{Ext}_{K(n)_*K(n)}(K(n)_*, K(n)_*). \blacksquare$$

THEOREM 12 (24). *$K(n)_*K(n)$ is a commutative (non-cocommutative) Hopf algebra over $K(n)_*$. Its algebra structure is given by*

$$K(n)_*K(n) = K(n)_*[t_1, t_2 \ldots]/(v_n t_i^{p^n} - v_n^{p^i} t_i).$$

Its coproduct Δ is given by $\displaystyle\sum_{i\geqslant 0}^{F} \Delta(t_i) = \sum_{i,j\geqslant 0}^{F} t_i \otimes t_j^{p^i},$

where $t_0 = 1$ and Σ^F denotes the sum with respect to the formal group law over $K(n)_*$ determined by the homomorphism $BP_* \to K(n)_*$. $\blacksquare$

Let $\mathbb{F}_{p^n}$ denote the field with p^n elements and make it a non-graded $K(n)_*$-module by sending v_n to 1. In (17) we showed that the non-graded Hopf algebra

$$K(n)_*K(n) \otimes_{K(n)_*} \mathbb{F}_{p^n}$$

is isomorphic to the dual of the group ring over $\mathbb{F}_{p^n}$ of a certain p-adic Lie group S_n which we now proceed to define. Let $W(\mathbb{F}_{p^n})$ denote the Witt ring of $\mathbb{F}_{p^n}$, i.e. the (degree n) extension of the p-adic integers $\mathbb{Z}_p$ obtained by adjoining (p^n-1)th roots of unity.

$W(\mathbb{F}_p) = \mathbb{Z}_p$, the p-adic integers, and $W(\mathbb{F}_{p^n})$ has an automorphism over $\mathbb{Z}_p$ which is a lifting of the Frobenius automorphism (which sends x to x^p) on $\mathbb{F}_{p^n}$. The image of an element $w \in W(\mathbb{F}_{p^n})$ under this automorphism will be denoted by w^σ. Let

$$E_n = W(\mathbb{F}_{p^n})\langle\langle T\rangle\rangle/(T^n - p),$$

where T is a non-commuting power series variable with $Tw = w^\sigma T$ for $w \in W(\mathbb{F}_{p^n})$. In particular if $\omega \in W(\mathbb{F}_{p^n})$ is a root of unity, then $T\omega = \omega^p T$. Then E_n is a $\mathbb{Z}_p$-algebra of rank n^2 generated by T and the roots of unity in $W(\mathbb{F}_{p^n})$. It is a complete local ring with maximal ideal (T) and residue field $\mathbb{F}_{p^n}$. The group S_n is defined to be the group of units of E_n which are congruent to 1 modulo (T). This group S_n is a pro-p group and there is a short exact sequence

$$1 \to S_n \to E_n^\times \to \mathbb{F}_{p^n}^\times \to 1.$$

The relevance of this group to the Novikov spectral sequence is due to Theorem 11 and

THEOREM 13 (24). *As Hopf algebras*

$$K(n)_*K(n) \otimes_{K(n)_*} \mathbb{F}_{p^n} = \mathrm{Hom}_c(\mathbb{F}_{p^n}[S_n], \mathbb{F}_{p^n}). \blacksquare$$

To explain this notation, note that the group algebra $\mathbb{F}_{p^n}[S_n]$ has a natural topology induced by that of S_n. $\mathrm{Hom}_c(\mathbb{F}_{p^n}[S_n], \mathbb{F}_{p^n})$ is the Hopf algebra of continuous linear maps from this group algebra to $\mathbb{F}_{p^n}$. To describe the isomorphism of the Theorem more explicitly, note that continuous linear maps on $\mathbb{F}_{p^n}[S_n]$ are in one-to-one correspondence with continuous $\mathbb{F}_{p^n}$-valued functions on S_n itself. Now each element of S_n can be written uniquely as

$$1+\sum_{i>0} e_i T^i,$$

where each e_i satisfies $e_i^{p^n} = e_i$, i.e. each is either zero or a root of unity in $W(\mathbb{F}_{p^n})$. We can thus define continuous $\mathbb{F}_{p^n}$-valued functions t_i on S_n by $t_i(1+\sum_{j>0} e_i T^j) = \bar{e}_i$, where $\bar{e}_i$ denotes the mod p reduction of e_i. In this way we get the isomorphism of Theorem 13 above.

All of the above assertions are proved in (24). We now need another result from algebraic number theory which will guarantee that S_{p-1} has a subgroup of order p. The cohomology of this subgroup will be used below to detect the elements mentioned in Theorem 4.

Let $\mathbb{Q}_p$ denote the p-adic numbers (the fraction field of the p-adic integers) and let $D_n = E_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. This latter object is a division algebra over $\mathbb{Q}_p$. Such division algebras are classified by an invariant in $\mathbb{Q}/\mathbb{Z}$, the Hasse invariant, which in the case of D_n is $1/n$ (see (28)).

THEOREM 14. (i) *Every degree n extension of the field* $\mathbb{Q}_p$ *is a subfield of the division algebra* D_n. (ii) S_{p-1} *contains a subgroup of order* p.

Proof. For (i) see (28), p. 138, or (27), p. 202.

For (ii), let K be the field obtained from $\mathbb{Q}_p$ by adjoining pth roots of unity. The degree of this extension is $p-1$, so (i) implies that K can be embedded in D_{p-1}. The roots of unity in K are integers, so they map to elements of E_{p-1}, which is the ring of integers of D_{p-1}. They are congruent to 1 modulo the maximal ideal in E_{p-1}, and hence are in S_{p-1}, because in the residue field the only pth root of unity is one. Hence the subgroup of order p in $K^\times$ gives a subgroup of order p in S_{p-1}. ∎

We now have all the ingredients necessary to prove Theorem 4. Let A denote the dual of $\mathbb{F}_{p^n}[\mathbb{Z}/(p)]$. Then Theorems 13 and 14 (ii) imply that there is an epimorphism of Hopf algebras over $\mathbb{F}_{p^n}$ (where $n = p-1$)

$$f\colon K(n)_* K(n) \otimes_{K(n)_*} \mathbb{F}_{p^n} \to A.$$

Although the embedding of $\mathbb{Z}/(p)$ in S_{p-1} and hence the map f are not unique, and although we cannot construct any such f explicitly, we can get enough partial information about it to prove Theorem 4. We first determine the structure of A as a Hopf algebra.

LEMMA 15. *Let A be the linear dual of* $\mathbb{F}_{p^n}[\mathbb{Z}/(p)]$. *As a Hopf algebra*

$$A = \mathbb{F}_{p^n}[t]/(t^p - t) \quad \textit{with} \quad \Delta t = t \otimes 1 + 1 \otimes t.$$

Proof. As a Hopf algebra we have $\mathbb{F}_{p^n}[Z/(p)] = \mathbb{F}_{p^n}[u]/(u^p - 1)$ with $\Delta u = u \otimes u$, where u corresponds to a generator of the group $Z/(p)$. We define an element $t \in A$ by its

Kronecker pairing $\langle u^i, t\rangle = i$. Since the product in A is dual to the coproduct in the group algebra, we have

$$\langle u^i, t^k\rangle = \langle \Delta(u^i), t\otimes t^{k-1}\rangle$$
$$= \langle u^i, t\rangle \langle u^i, t^{k-1}\rangle$$

so by induction on k

$$\langle u^i, t^k\rangle = i^k. \tag{16}$$

We also have $\langle u^i, 1\rangle = 1$.

We show that $\{1, t, t^2 \dots t^{p-1}\}$ is a basis for A by relating it to the dual basis of the group algebra. Define $x_j \in A$ by

$$x_j = \sum_{0<k<p} (jt)^k$$

for $0 < j < p$ and $x_0 = 1 + \sum_{0<j<p} x_j$. Then

$$\langle u^i, x_j\rangle = \langle u^i, \sum_{0<k<p} (jt)^k\rangle$$
$$= \sum_{0<k<p} j^k i^k$$
$$= \sum_{0<k<p} (ij)^k$$
$$= \begin{cases} -1 & \text{if } ij = 1 \bmod p \\ 0 & \text{otherwise} \end{cases}$$

and

$$\langle u^i, x_0\rangle = \langle u^i, 1 + \sum_{0<j<p} x_j\rangle$$
$$= \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

so $\{x_0, -x_1, -x_2 \dots -x_{p-1}\}$ is the dual basis up to permutation.

Moreover (16) implies that $t^p = t$ so A has the desired algebra structure.

For the coalgebra structure we use the fact that the coproduct in A is dual the product in the group algebra. We have

$$\langle u^i \otimes u^j, t\otimes 1 + 1\otimes t\rangle = i + j$$

and

$$\langle u^i \otimes u^j, \Delta(t)\rangle = \langle u^{i+j}, t\rangle = i + j$$

so $\Delta t = t\otimes 1 + 1 \otimes t$. ∎

To proceed with the proof of Theorem 4, we now show that under the epimorphism

$$f\colon K(n)_* K(n) \otimes_{K(n)_*} \mathbb{F}_{p^n} \to A \quad (\text{where} \quad n = p-1), \quad f(t_1) \neq 0.$$

From the remarks following Theorem 13, t_1 can be regarded as a continuous function from S_n to $\mathbb{F}_{p^n}$. It follows then that the non-triviality of $f(t_1)$ is equivalent to the non-vanishing of the function t_1 on the non-trivial element of order p in S_n. Suppose $x \in S_{p-1}$ is such an element. We can write

$$x = 1 + \sum_{i>0} e_i T^i$$

with $e_i \in W(\mathbb{F}_{p^n})$ and $e_i^{p^n} = e_i$. Recalling that $T^{p-1} = p$, we compute

$$1 = x^p \equiv 1 + pe_1 T + (e_1 T)^p \bmod (T)^{1+p}$$

and $$(e_1 T)^p \equiv e_1^{(p^p-1)/(p-1)} T^p \bmod (T)^{1+p}$$

so it follows that $$e_1 + e_1^{(p^p-1)/(p-1)} \equiv 0 \bmod p.$$

(Remember that $t_1(x)$ is the mod p reduction of e_1.) Clearly one solution to this equation is $e_1 \equiv 0 \bmod p$ and hence $e_1 = 0$. We exclude this possibility by showing that it implies that $x = 1$. Suppose inductively that $e_i = 0$ for $i < k$. Then $x \equiv 1 + e_k T^k \bmod (T^{k+1})$ and $x^p \equiv 1 + pe_k T^k \bmod (T^{k+p})$ so $e_k \equiv 0 \bmod p$. Since $e_k^{p^n} - e_k = 0$, this implies $e_k = 0$.

Hence, if $e_1 \equiv 0 \bmod p$, $x = 1$, so t_1 is non-zero on non-trivial elements of order p, so $f(t_1) \neq 0$.

Since f is a map of Hopf algebras, $f(t_1)$ is primitive, so $f(t_1) = ct$ where $c \in \mathbb{F}_{p^n}$ is non-zero. Now recall that

$$\mathrm{Ext}_A(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}) = H^*(\mathbb{Z}/(p); \mathbb{F}_{p^n}) = E(h) \otimes P(b),$$

where $E(\ )$ and $P(\ )$ denote exterior and polynomial algebras over $\mathbb{F}_{p^n}$ respectively, and $h = [t] \in H^1$ and

$$b = \sum_{0<j<p} \frac{1}{p}\binom{p}{j} [t^j | t^{p-j}] \in H^2.$$

Let f^* denote the composition

$$\begin{aligned}\mathrm{Ext}_{BP_*BP}(BP_*, BP_*) &\to \mathrm{Ext}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n) \\ &\xrightarrow{\cong} \mathrm{Ext}_{K(n)_*K(n)}(K(n)_*, K(n)) \to \mathrm{Ext}_A(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}) \xrightarrow{\cong} H^*(\mathbb{Z}/(p); \mathbb{F}_{p^n}).\end{aligned}$$

Then it follows that $f^*(h_0) = -ch$ and $f^*(b_i) = -c^{p^{i+1}} b$ and Theorem 4 is proved. ∎

Note that the scalar c must satisfy $1 + c^{(p^p - p)/(p-1)} = 0$. Since $c^{p^{p-1}-1} = 1$, the equation is equivalent to $1 + c^{(p^{p-1}-1)/(p-1)} = 0$. It follows that $c = w^{(p-1)/2}$ for some generator w of $\mathbb{F}_{p^{p-1}}^\times$, so c is not contained in any proper subfield of $\mathbb{F}_{p^{p-1}}$. Hence tensoring with this field is essential to the construction of the detecting map f.

3. *The classical Adams spectral sequence*. We now turn to the classical Adams spectral sequence. Our result is

THEOREM 17. *For $p \geq 5$, the elements b_{i+1} $(i \geq 0)$ in the classical Adams spectral sequence all support non-trivial differentials, i.e. none of them detect homotopy elements.*

Before proving this, we describe the situation for $p = 3$. In a subsequent paper we will compute the Novikov spectral sequence through a range of dimensions beyond 106, the dimension of b_2. There we will see that $d_5 \beta_7 = \pm d_5 b_2$ and $b_2 \pm \beta_7$ is a permanent cycle. It reduces in the classical Adams spectral sequence to b_2, which is therefore a permanent cycle. (The element there corresponding to β_7 has filtration ≥ 7 and supports a non-trivial differential.) It is quite possible that b_{i+1} for $i \geq 2$ is a permanent cycle in the Adams spectral sequence for similar reasons.

To prove Theorem 17 we need the following information about Novikov E_2-term.

LEMMA 18. *For $p \geqslant 3$*

(i) $\mathrm{Ext}^{2,qp^{i+2}}_{BP_*BP}(BP_*, BP_*)$ *is generated by the* $[(i+3)/2]$ *elements* $\beta_{a_{i,j}/p^{i+3-2j}}$, *where* $j = 1, 2, \ldots [(i+3)/2]$, $a_{i,j} = (p^{i+2}+p^{i+3-2j})/(p+1)$, *and* $[(i+3)/2]$ *is the largest integer* $\leqslant (i+3)/2$. *Each of these elements has order* p.

(ii) *Each of these elements except* $\beta_{p^{i+1}/p^{i+1}}$ *reduces to zero in*

$$\mathrm{Ext}^{2,qp^{i+2}}_{BP_*BP}(BP_*, BP_*/I_3). \blacksquare$$

LEMMA 19. *For $p \geqslant 5$, any element of* $\mathrm{Ext}^{2,qp^{i+2}}_{BP_*BP}(BP_*, BP_*)$ *(for $i \geqslant 0$) which maps to* b_{i+1} *in the Adams E_2-term supports a non-trivial differential* d_{2p-1}. ∎

We will prove these Lemmas below. Note that the preceding discussion shows that the Lemma 19 is false for $p = 3$.

We now prove Theorem 17 modulo Lemmas 18 and 19. The natural map from the Adams–Novikov to the classical Adams spectral sequence (see (20), section 9) comes from a map of Adams resolutions. It follows that if x is a permanent cycle in the Adams E_2-term, there is a permanent cycle $\tilde{x}$ in the Novikov E_2-term which detects the same element in homotopy as x and has filtration less than or equal to that of x, equality holding if and only if $\tilde{x}$ maps to x. Lemma 19 indicates that b_{i+1} is not the image of any permanent cycle in filtration 2. Hence if b_{i+1} is a permanent cycle, there must be permanent cycles $\tilde{b}_{i+1}$ in the Novikov E_2-term having filtration 0 or 1. But the sparseness property mentioned in the proof of Corollary 5 guarantees that no such elements exist. ∎

Before proving the Lemmas we remark that in (20), corollary 9·6, we showed that for $p > 2$, the only elements of $\mathrm{Ext}^{2,*}_{A_p}(\mathbb{F}_p, \mathbb{F}_p)$ which could possibly detect homotopy elements are the b_i ($i \geqslant 0$), $h_0 h_i$ ($i \geqslant 2$) and 3 or 4 others (depending on p). Theorem 17 shows that the b_i cannot be permanent cycles for $i \geqslant 1$ and $p \geqslant 5$. Whether the elements $h_0 h_i$ survive is still an open question, as is that of the survival of b_i^p for $i \geqslant 1$. Mahowald has recently shown (12) that for $p = 2$ the elements $h_1 h_{i+1}$ (the analogues of $h_0 h_i$) do survive and there is hope that his construction may generalize to odd primes.

Proof of Lemma 18. (i) Can be read off from the description of $\mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$ given in (19) or (20).

To prove (ii) we recall the definition of the elements in question. We have short exact sequences of BP_*BP-comodules

$$0 \to BP_* \xrightarrow{p} BP_* \xrightarrow{\rho_1} BP_*/(p) \to 0, \tag{20}$$

$$0 \to BP_*/(p) \xrightarrow{v_1^{p^{i+3-2j}}} BP_*/(p) \xrightarrow{\rho_2} BP_*/(p, v_1^{p^{i+3-2j}}) \to 0. \tag{21}$$

Let δ_0 and δ_1 denote the respective connecting homomorphisms. Then we have $v_2^{a_{i,j}} \in \mathrm{Ext}^0_{BP_*BP}(BP_*, BP_*/(p, v_1^{p^{i+3-2j}}))$ and $\beta_{a_{i,j}/p^{i+3-2j}} = \delta_0 \delta_1(v_2^{a_{i,j}})$. The element $\beta_{p^{i+1}/p^{i+1}}$ (i.e. the above element for $j = 1$) can be shown to be b_{i+1} as follows. Since (26)

$$\eta_R(v_2) \equiv v_2 + v_1 t_1^p - v_1^p t_1 \bmod p, \tag{22}$$

$$\delta_1(v_2^{p^{i+1}}) = t_1^{p^{i+2}} - v_1^{p^{i+2}-p^{i+1}} t_1^{p^{i+1}}$$

and $\delta_0(t_1^{p^{i+2}}) = b_{i+1}$. Moreover (22) implies that in $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*/(p))$,

$$v_1^{p^j} t_1^{p^{j+1}} \simeq v_1^{p^{j+1}} t_1^{p^j}, \quad \text{so} \quad v_1^{p^{i+2}-p^{i+1}} t_1^{p^{i+1}} \simeq v_1^{p^{i+2}-1} t_1.$$

This element is the mod p reduction of $p^{-i-2}\delta_0(v_1^{p^{i+2}})$ and is therefore in ker δ_0. Hence $\delta_0\,\delta_1(v_2^{p^{i+1}}) = \delta_0(t_1^{p^{i+2}}) = b_{i+1}$.

This definition of $\beta_{p^{i+1}/p^{i+1}}$ differs from that of (19) and (20), where for $i > 0$ it is defined to be $\delta_0\,\delta_1(v_2^{p^2} - v_1^{p^2-1}v_2^{p^2-p+1})^{p^{i-1}}$.

In principle one can compute this element explicitly in the cobar construction (see (20), section 1) and reduce mod I_3, but that would be very messy. A much easier method can be devised using Yoneda's interpretation of elements in Ext groups as equivalence classes of exact sequences (see for example (8), ch. IV). Consider the following diagram.

$$(23)\qquad \begin{tikzcd} 0 \arrow[r] & BP_* \arrow[r, "p"] \arrow[d, "p_1"] & BP_* \arrow[rr, "v_1^{p^{i+3-2j}}\rho_1"] \arrow[dr, "p_2"] & & BP_*/(p) \arrow[r, "\rho_2"] \arrow[d, "p_3"] & BP_*/(p, v_1^{p^{i+3-2j}}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & BP_*/(p, v_1, v_2) \arrow[rr] & & M_1 \arrow[r] & N_2 \arrow[r] & BP_*/(p, v_1^{p^{i+3-2j}}) \arrow[r] & 0. \end{tikzcd}$$

The top row is obtained by splicing (20) and (21) and it corresponds to an element in $\mathrm{Ext}^2_{BP_*BP}(BP_*/(p, v_1^{p^{i+3-2j}}), BP_*)$. Composing this element with

$$v_2^{a_{i,j}} \in \mathrm{Ext}^0_{BP_*BP}(BP_*, BP_*/(p, v_1^{p^{i+3-2j}}))$$

gives
$$\beta_{a_{i,j}/p^{i+3-2j}}.$$

We let p_1 be the standard surjection. It follows from Yoneda's result that if we choose BP_*BP-comodules M_1 and M_2, and comodule maps p_2 and p_3 such that the diagram commutes and the bottom row is exact, then the latter will determine the element of
$$\mathrm{Ext}^2_{BP_*BP}(BP_*/(p, v_1^{p^{i+3-2j}}), BP_*/(p, v_1, v_2))$$
which, when composed with $v_2^{a_{i,j}}$, will give the mod I_3 reduction of $\beta_{a_{i,j}/p^{i+3-2j}}$. We choose $M_1 = BP_*/(p^2, pv_1, v_1^2, pv_2)$ and $M_2 = BP_*/(p, v_1^{2+p^{i+3-2j}})$ and let p_2 and p_3 be the standard surjections. It is easy to check that M_1 and M_2 are comodules over BP_*BP, i.e. that the corresponding ideals in BP_* are invariant. (The ideal used to define M_1 is simply $I_2^2 + I_1 I_3$.) Moreover, the resulting diagram has the desired properties.

The resulting bottom row of (23) is the splice of the 2 following short exact sequences.

$$(24)\qquad 0 \to BP_*/(p, v_1, v_2) \xrightarrow{p} BP_*/(p^2, pv_1, pv_2, v_1^2) \to BP_*/(p, v_1^2) \to 0,$$

$$(25)\qquad 0 \to BP_*/(p, v_1^2) \xrightarrow{v_1^{p^{i+3-2j}}} BP_*/(p, v_1^{2+p^{i+3-2j}}) \to BP_*/(p, v_1^{p^{i+3-2j}}) \to 0.$$

Let δ_0', δ_1' denote the corresponding connecting homomorphisms. The element we are interested in then is $\delta_0'\,\delta_1'(v_2^{a_{i,j}})$. Again, we refer the reader to (20), section 1, for a description of the cobar construction used to make this computation.

To compute $\delta_1'(v_2^{a_{i,j}})$ we use the formula $d(v_2^n) = (v_2 + v_1 t_1^p - v_1^p t_1)^n - v_2^n$, implied by (22), in the cobar construction for $BP_*/(p, v_1^{2+p^{i+3-2j}})$. Recall that

$$a_{i,j} = (p^{i+2} + p^{i+3-2j})/(p+1) \quad \text{for} \quad 1 \leqslant j \leqslant [(i+3)/2].$$

Hence $a_{i,j} = p^{i+3-2j} \bmod p^{i+4-2j}$ and $d(v_2^{a_{i,j}}) = v_2^{b_{i,j}} v_1^{p^{i+3-2j}}[t_1^{p^{i+4-2j}}]$, so
$$\delta_1'(v_2^{a_{i,j}}) = v_2^{b_{i,j}}[t_1^{p^{i+4-2j}}],$$
where $b_{i,j} = a_{i,j} - p^{i+3-2j} = (p^{i+2} - p^{i+4-2j})/(p+1)$.

For $j = 1$, $b_{i,1} = 0$ and

$$\delta_0' \delta_1' v_2^{a_{i,1}} = -\sum_{0<k<p} \frac{1}{p}\binom{p}{k}[t_1^{kp^i}|t_1^{(p-k)p^i}] = -b_{i+1}.$$

For $j > 1$, $b_{i,j}$ is divisible by p and $dv_2^{b_{i,j}} \equiv 0 \bmod (p^2, pv_1, v_1^2)$ and

$$v_2^{b_{i,j}} d[t_1^{p^{i+4-2j}}] \equiv 0 \bmod pv_2,$$

so $\delta_1' v_2^{a_{i,j}} \in \mathrm{Ext}^1_{BP_*BP}(BP_*, BP_*/(p, v_1^2))$ pulls back in (24) to an element of

$$\mathrm{Ext}^1_{BP_*BP}(BP_*, BP_*/(p^2, pv_1, pv_2, v_1^2)) \quad \text{and} \quad \delta_0' \delta_1'(v_2^{a_{i,j}}) = 0,$$

completing the proof. ∎

Proof of Lemma 19. Any element of $\mathrm{Ext}^{2,qp^{i+2}}_{BP_*BP}(BP_*, BP_*)$ can be written uniquely as $cb_{i+1} + x$ where x is in the subgroup generated by the elements $\beta_{a_{i,j}/p^{i+3-2j}}$ for $j > 1$. In (20), theorem 9·4, we showed that x maps to zero in the classical Adams E_2-term. Hence it suffices to show that no such x can have the property

$$d_{2p-1}(x) = d_{2p-1}(b_{i+1}).$$

In (30) Smith showed that for $p \geqslant 5$ there is an 8-cell spectrum $V(2)$ with $BP_* V(2) = BP_*/(p, v_1, v_2)$, and a map $f: S^0 \to V(2)$ inducing a surjection in BP homology. f also induces the standard map

$$f_*: \mathrm{Ext}_{BP_*BP}(BP_*, BP_*) \to \mathrm{Ext}_{BP_*BP}(BP_*, BP_*/I_3).$$

Lemma 16 asserts that $f_*(\beta_{a_{i,j}/p^{i+3-2j}}) = 0$ for $j > 1$, so $f_*(d_{2p-1}(x)) = 0$ where x is as above. However, Theorem 3 and the proof of Theorem 4 show that

$$g_*(d_{2p-1}(b_{i+1})) \neq 0$$

where g_* is induced by the obvious map

$$g: BP_* \to v_{p-1}^{-1} BP_*/I_{p-1}.$$

Since g factors through BP_*/I_3, this shows that $f_*(d_{2p-1}(b_{i+1})) \neq 0$, completing the proof. ∎

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