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# NOTAS DE MATEMATICAS Y SIMPOSIA

NUMERO 1

# CONFERENCE ON HOMOTOPY THEORY

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CONTENIDO



CONFERENCIAS

# DIEUDONNE MODULES FOR ABELIAN HOPF ALGEBRAS

## Preliminary Report in Honor of SAMUEL EILENBERG

Douglas C. Ravenel(*)

By Abelian Hopf algebra we mean graded connected biassociative strictly bi-commutative Hopf algebra of finite type over a perfect field $k$ of characteristic $p$. Let $\underline{A}$ denote the category of such objects. $\underline{A}$ is known to be abelian ([12]) and our purpose here is to show that it is isomorphic to a certain category of modules. An analogous theorem for the nongraded case was proved long ago by Dieudonné, and the modules that he used have been studied extensively (see [1], Chapter V, and [4]). I am grateful to Bill Singer for first bringing this work to my attention and suggesting the problem of carrying it over to the graded case.

The ring $D$ in question is a noncommutative power series over $W(k)$ (the Witt ring of $k$) in two variables $F$ and $V$ subject to the relations

$$FV = VF = p$$

$$Fw = w^{\sigma}F, \quad Vw^{\sigma} = wV$$

(*) Research partially supported by N.S.F.

for $w \in W(k)$, where $w^\sigma$ denotes the action of the Frobenius automorphism of $k$ lifted to $W(k)$.

In our case we will obtain modules over a commutative graded ring $E = W(k)[[F,V]]/(FV-p)$ where $\dim F = 1$, $\dim V = -1$. $F$ will be seen to correspond to the Frobenius endomorphism of a Hopf algebra $A$ which sends $x \in A$ to $x^p$, while $V$ corresponds to the dual of $F$, commonly known as the Verschiebung.

The relation between abelian Hopf algebras and E-modules will be described in Theorem 3" below, which is our main result.

Our first result is a decomposition theorem.

Definition. Let $n$ be an integer prime to $p$. An Abelian Hopf algebra is of type $n$ if each of its primitives and generators has dimension $np^i$ for some $i$. Let $\underline{T_n A} \subset A$ denote the full subcategory of type $n$ Abelian Hopf algebras.

Theorem 1. There is a canonical categorical splitting $A \cong \prod_{(n,p)=1} \underline{T_n A}$, i.e.

a) Every Abelian Hopf algebra is canonically a direct product of type $n$ Abelian Hopf algebras.

b) There are no nontrivial maps between a type $n$ Hopf algebra and a type $m$ Hopf algebra for $m \neq n$.

c) Moreover, $\underline{T_1 A} \cong \underline{T_n A}$ $\forall n$

Such a decomposition is well-known for the Hopf algebra $H_*(BU;k)$ (see [3] for example) The general decomposition is



established by showing that the endomorphism ring of $H_*(BU;k)$ acts canonically on any abelian Hopf algebra. Part (b) follows from the fact that a Hopf algebra map sends primitives to primitives. Part (c) is trivial.

We now construct a set of projective generators for $\underline{T_1 A}$. Let $B_n \in \underline{A}$ be $k[b_1, b_2, \ldots, b_n]$ with $\dim b_i = i$ and coproduct $\psi b_i = \sum_{s+t=i} b_s \otimes b_t$ where $b_0 = 1$. Let $W_n$ be the type 1 factor of $B_{p^n}$. It is a polynomial algebra $k[w_0, w_1, \ldots, w_n]$ with $\dim w_i = p^i$. The coproduct is obtained lifting to $W(k)$ and defining the Witt polynomials $f_m(w) = \sum_{i=0}^{m} p^i w_i^{p^{m-i}}$, $0 \le m \le n$, to be primitive.

Theorem 2. $W_n$ is a projective object in $\underline{A}$, and its dual $W_n^*$ is therefore injective.

Proof. Let $S_r$ be the simple object $k[x_r]/x_r^p$, $\dim x_r = r$. Any Abelian Hopf algebra can be built up out of these simple objects by multiple extensions, so it suffices to show $\mathrm{Ext}^1_{\underline{A}}(W_n, S_r) = 0$ $\forall r$, which is a simple calculation.

Now let $\underline{W} \subset \underline{T_1 A}$ denote the full subcategory whose objects are the $W_n$. Let $\underline{FW}$ denote the category of contravariant functors from $\underline{W}$ to the category of finite $W(k)$ modules. This category is abelian. We define a functor

$$\underline{D} : \underline{T_1 A} \longrightarrow \underline{FW}$$

by

$$\underline{D}(A)(W_n) = \mathrm{Hom}_{\underline{A}}(W_n, A).$$

Now we can state our main result:

<u>Theorem</u> 3. The functor $\underline{D}$ defined above is an equivalence of abelian categories.

The proof is analogous to that of Theorem V, §1,4.3 of [1].

Theorem 3 can be described in a more useful way by analyzing the structure of $\underline{W}$. Let $V_n : W_{n-1} \hookrightarrow W_n$ be the inclusion and let $F_n : W_{n+1} \to W_n$ be defined by $F_n(w_i) = w_{i-1}^p$. Note that $V_n F_{n-1} = F_n V_{n+1} = p$. Then we have

<u>Lemma</u> 4. The endomorphism ring of $W_n$ is $W(k)/p^{n+1}$ and these endomorphisms along with the $F_n$ and $V_n$ generate all of the morphisms of $\underline{W}$.

Hence Theorem 3 can be paraphrased as

<u>Theorem</u> 3'. A type 1 Abelian Hopf algebra is characterized by a sequence of $W(k)$ modules $W_n(A) = \mathrm{Hom}(W_n, A)$ and maps $F_n : W_n(A) \to W_{n+1}(A)$ and $V_n : W_n(A) \to W_{n-1}(A)$ where $V_n F_{n-1} = F_n V_{n+1} = p$.

If we identify $f \in W_n(A)$ with the element $f(w_n) \in A$, we have $(F_n f)(w_{n+1}) = f(w_n)^p \in A$, i.e. $F_n$ corresponds to the Frobenius endomorphism of $A$, while $V_n$ corresponds similarly



to the dual endomorphism, i.e. the Verschiebung.

To make this more concise let $\underline{E}_O^+$ denote the where $A_O$ is projective and $A_1$ is polynomial. (If $A$ is not finitely generated, one can still construct and $A_O$ and $A_1$ but they need not be of finite type).

This is a consequence of

<u>Theorem</u> 6. $\mathrm{Ext}^2_{\underline{A}}(B,A) = 0$ for all $A$ iff $B$ is polynomial.

We will conclude by identifying some well-known Hopf algebra functors with standard functors from homological algebra. It is convenient at this point to embed $\underline{E}_O^+$ in $\underline{E}$, the full category of graded E-modules and maps of all degrees. Hence for $M, N \in \underline{E}$, $\mathrm{Hom}_{\underline{E}}(M,N)$ is also an E-module. Moreover, if $N$ is nonnegative and $M$ does not have any generators in positive dimensions then $\mathrm{Hom}_{\underline{E}}(M,N)$ will also be nonnegatively graded. Define modules $P = E/VE$, $R = E/FE$.

<u>Theorem</u> 7. Let $A \in \underline{T_1 A}$. Then $\mathrm{Hom}_{\underline{E}}(P, C(A))$ is isomorphic to the abelian restricted Lie algebra of primitives of $A$ (where $F$ corresponds to the restriction), and $\mathrm{Ext}^1_{\underline{E}}(T, C(A))$ is isomorphic to the abelian restrict Lie coalgebra (with $V$ corresponding to the corestriction) of decomposable elements of $A$.

The functors $\mathrm{Ext}^1_{\underline{E}}(P, C(A))$ and $\mathrm{Hom}_{\underline{E}}(R, C(A))$ are the functors $\hat{P}$ and $\hat{Q}$ respectively defined in [6] and also in [5]

§3. Hence an extension in $\underline{T_1 A}$ induces six term exact sequences relating these functors as was shown in [6]. (Note that $Ext^2_{\underline{E}}(P,-) = Ext^2_{\underline{E}}(R,-) = 0$). It is evident that the connecting homomorphisms of these sequences must be E-module maps, i.e. they must preserve the restriction and corestriction respectively. Hence the argument of 4.10 of [6] (which leads to contradictions of Theorems 2 and 4) is incorrect.

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N.B. These results were also obtained by C. Schoeller, "Etude de la Categorie des Algebres de Hopf Commutatives Connexes sur un Corps", Manuscripta Math. 3(1970), 133-155.

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