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On β-elements in the Adams-Novikov spectral sequence

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Dedicated to Professor Takao Matumoto on his sixtieth birthday

Abstract

In this paper we detect invariants in the comodule consisting of β-elements over the Hopf algebroid $(A(m+1), G(m+1))$ defined in[**Rav02**], and we show that some related Ext groups vanish below a certain dimension. The result obtained here will be extensively used in [**NR**] to extend the range of our knowledge for $\pi_*(T(m))$ obtained in[**Rav02**].

Contents

1. *Introduction*

In this paper we describe some tools needed in the method of infinite descent, which is an approach to finding the E_2-term of the Adams-Novikov spectral sequence converging to the stable homotopy groups of spheres. It is the subject of [**Rav86**, Chapter 7], [**Rav04**, Chapter 7] and [**Rav02**].

We begin by reviewing some notation. Fix a prime p. Recall the Brown-Peterson spectrum BP. Its homotopy groups and those of $BP \wedge BP$ are known to be polynomial algebras

$$\pi_*(BP) = \mathbf{Z}_{(p)}[v_1, v_2 \ldots] \quad \text{and} \quad BP_*(BP) = BP_*[t_1, t_2 \ldots].$$

In [**Rav86**, Chapter 6] the second author constructed intermediate spectra

$$S^0_{(p)} = T(0) \longrightarrow T(1) \longrightarrow T(2) \longrightarrow T(3) \longrightarrow \cdots \longrightarrow BP$$

with $T(m)$ is equivalent to BP below the dimension of v_{m+1}. This range of dimensions grows exponentially with m. $T(m)$ is a summand of p-localization of the Thom spectrum of the stable vector bundle induced by the map $\Omega SU(p^m) \to \omega SU = BU$. In [**Rav02**] we constructed truncated versions $T(m)_{(j)}$ for $j \geq 0$ with

$$T(m) = T(m)_{(0)} \longrightarrow T(m)_{(1)} \longrightarrow T(m)_{(2)} \longrightarrow \cdots \longrightarrow T(m+1)$$

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These spectra satisfy

$$\begin{aligned} BP_*(T(m)) &= \pi_*(BP)[t_1, \ldots, t_m] \\ \text{and} \quad BP_*(T(m)_{(j)}) &= BP_*(T(m)) \left\{ t_{m+1}^{\ell} : 0 \le \ell < p^j \right\} \end{aligned}$$

Thus $T(m)_{(j)}$ has p^j 'cells,' each of which is a copy of $T(m)$.

For each $m \ge 0$ we define a Hopf algebroid

$$\begin{aligned} \Gamma(m+1) &= (BP_*, BP_*(BP)/(t_1, t_2, \ldots, t_m)) \\ &= BP_*[t_{m+1}, t_{m+2}, \ldots] \end{aligned}$$

with structure maps inherited from $BP_*(BP)$, which is $\Gamma(1)$ by definition. Let

$$\begin{aligned} A &= BP_*, \\ A(m) &= \mathbf{Z}_{(p)}[v_1, \ldots, v_m] \\ \text{and} \qquad G(m+1) &= A(m+1)[t_{m+1}] \end{aligned}$$

with t_{m+1} primitive. Then there is a Hopf algebroid extension

$$(A(m+1), G(m+1)) \to (A, \Gamma(m+1)) \to (A, \Gamma(m+2)). \tag{1.1}$$

In order to avoid excessive subscripts, we will use the notation

$$\widehat{v}_i = v_{m+i}, \quad \text{and} \quad \widehat{t}_i = t_{m+i}.$$

We will use the usual notation without hats when $m = 0$. We will use the notation

$$\widehat{v}_i = v_{m+i}, \quad \widehat{t}_i = t_{m+i}, \quad \widehat{\beta}_{i/e_1, e_0} = \frac{\widehat{v}_2^i}{p^{e_0} v_1^{e_1}} \quad \text{and} \quad \widehat{\beta}'_{i/e_1} = \frac{\widehat{v}_2^i}{piv_1^{e_1}}.$$

We will also use the notations $\widehat{\beta}_{i/e_1} = \widehat{\beta}_{i/e_1,1}$ and $\widehat{\beta}'_{i/e_1} = \widehat{\beta}'_{i/e_1,1}$ for short. We will use the usual notation without hats when $m = 0$.

Given a Hopf algebroid (B, Γ) and a Γ-comodule M, we will abbreviate $\operatorname{Ext}_\Gamma(B, M)$ by $\operatorname{Ext}_\Gamma(M)$ and $\operatorname{Ext}_\Gamma(B)$ by $\operatorname{Ext}_\Gamma$. With this in mind, there are change-of-rings isomorphisms

$$\begin{aligned} \operatorname{Ext}_{BP_*(BP)}(BP_*(T(m))) &= \operatorname{Ext}_{\Gamma(m+1)} \\ \text{and} \qquad \operatorname{Ext}_{BP_*(BP)}(BP_*(T(m)_{(j)})) &= \operatorname{Ext}_{\Gamma(m+1)}\left(T_m^{(j)}\right) \\ \text{where} \qquad T_m^{(j)} &= A\left\{\widehat{t}_1^{\ell} : 0 \le \ell < p^j\right\}. \end{aligned}$$

Very briefly, *the method of infinite descent involves determining the groups*

$$\operatorname{Ext}_{\Gamma(m+1)}\left(T_m^{(j)}\right) \qquad \text{and} \qquad \pi_*\left(T(m)_{(j)}\right)$$

by downward induction on m and j.

To begin with, we know that

$$\operatorname{Ext}^0_{\Gamma(m+1)}\left(A\left\{t_{m+1}^{\ell} : 0 \le \ell < p^j\right\}\right) = A(m)\left\{\widehat{v}_1^{\ell} : 0 \le \ell < p^j\right\}.$$

To proceed further, we make use of a short exact sequence of $\Gamma(m+1)$-comodules

$$0 \longrightarrow BP_* \xrightarrow{\iota_0} D^0_{m+1} \xrightarrow{\rho_0} E^1_{m+1} \longrightarrow 0, \tag{1.2}$$

where D^0_{m+1} is weak injective (meaning that its higher Ext groups vanish) with ι_0 inducing an isomorphism in Ext^0. It has the form

$$D^0_{m+1} = A(m)[\widehat{\lambda}_1, \widehat{\lambda}_2, \ldots] \subset \mathbf{Q} \otimes BP_*$$

with

$$\widehat{\lambda}_i = p^{-1}\widehat{v}_i + \dots .$$

Thus we have an explicit description of E^1_{m+1}, which is a certain subcomodule of the chromatic module $N^1 = BP_*/(p^\infty)$.

It follows that the connecting homomorphism δ_0 associated with (1.2) is an isomorphism

$$\mathrm{Ext}^s_{\Gamma(m+1)}(E^1_{m+1}) \xrightarrow{\cong} \mathrm{Ext}^{s+1}_{\Gamma(m+1)}$$

and more generally

$$\mathrm{Ext}^s_{\Gamma(m+1)}(E^1_{m+1} \otimes T^{(j)}_m) \xrightarrow{\cong} \mathrm{Ext}^{s+1}_{\Gamma(m+1)}(T^{(j)}_m)$$

for each $s \geq 0$. The determination of this group for $s = 0$ will be the subject of [**Nak**]. In this paper we will limit our attention to the case $s > 0$.

Unfortunately there is no way to embed E^1_{m+1} in a weak injective comodule in a way that induces an isomorphism in Ext^0 as in (1.2). (This is explained in [**NR**, Remark7.4].) Instead we will study the Cartan-Eilenberg spectral sequence for $\mathrm{Ext}_{\Gamma(m+1)}(E^1_{m+1} \otimes T^{(j)}_m)$ associated with the extension (1.1). Its E_2-term is

$$\begin{aligned} \tilde{E}^{s,t}_2(T^{(j)}_m) &= \mathrm{Ext}^s_{G(m+1)}(\mathrm{Ext}^t_{\Gamma(m+2)}(T^{(j)}_m \otimes E^1_{m+1})) \\ &= \mathrm{Ext}^s_{G(m+1)}(\overline{T}^{(j)}_m \otimes \mathrm{Ext}^t_{\Gamma(m+2)}(E^1_{m+1})) \\ &\qquad \text{where } \overline{T}^{(j)}_m = A(m+1)\left\{\widehat{t}^{\,\ell}_1 : 0 \leq \ell < p^j\right\} \end{aligned} \tag{1.3}$$

and differentials $\tilde{d}_r : \tilde{E}^{s,t}_2 \to \tilde{E}^{s+r,t-r+1}_2$. Note that $T^{(j)}_m = A \otimes_{A(m+1)} \overline{T}^{(j)}_m$. We use the tilde to distinguish this spectral sequence from the resolution spectral sequence. We did not use this notation in [**Rav02**].

The short exact sequence of $\Gamma(m+1)$-comodules (1.2) is also a one of $\Gamma(m+2)$-comodules, and D^0_{m+1} is also weak injective over $\Gamma(m+2)$ (this was proved in [**Rav02**, Lemma 2.2]), but this time the map ι_0 does not induce an isomorphism in Ext^0. However, the connecting homomorphism

$$\delta_0 : \mathrm{Ext}^t_{\Gamma(m+2)}(E^1_{m+1} \otimes T^{(j)}_m) \to \mathrm{Ext}^{t+1}_{\Gamma(m+2)}(T^{(j)}_m)$$

is an isomorphim of $G(m+1)$-comdules for $t > 0$. Note that over $\Gamma(m+2)$, $T^{(j)}_m$ is a direct sum of p^j suspended copies of A, so the isomorphism above is the tensor product with $\overline{T}^{(j)}_m$ with

$$\delta_0 : \mathrm{Ext}^t_{\Gamma(m+2)}(E^1_{m+1}) \to \mathrm{Ext}^{t+1}_{\Gamma(m+2)}.$$

We will abbreviate the group on the right by U^{t+1}_{m+1}. Its structure up to dimension $(p^2+p)|\widehat{v}_2|$ was determined in [**NR**, Theorem 7.10]. It is p-torsion for all $t \geq 0$ and v_1-torsion for $t > 0$. Moreover, it is shown that each U^t_{m+1} for $t \geq 2$ is a certain suspension of U^2_{m+1} below dimension $p|\widehat{v}_3|$.

Let $\overline{E}^1_{m+1} = \mathrm{Ext}^0_{\Gamma(m+2)}(E^1_{m+1})$. For $j = 0$, the Cartan-Eilenberg E_2-term of (1.3) is

$$\tilde{E}^{s,t}_2(T^{(0)}_m) = \begin{cases} \mathrm{Ext}^s_{G(m+1)}(\overline{E}^1_{m+1}) & \text{for } t = 0 \\ \mathrm{Ext}^s_{G(m+1)}(U^{t+1}_{m+1}) & \text{for } t \geq 1. \end{cases}$$

While it is impossible to embed the $\Gamma(m+1)$-comodule E^1_{m+1} into a weak injective by a map inducing an isomorphism in Ext^0, it is possible to do this for the $G(m+1)$-comodule $\overline{E}^1_{m+1}$.

In Theorem 2.4 below we will show that there is a pullback diagram of $G(m+1)$-comodules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{E}^1_{m+1} & \xrightarrow{\iota_1} & W_{m+1} & \xrightarrow{\rho_1} & B_{m+1} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{E}^1_{m+1} & \longrightarrow & v_1^{-1}\overline{E}^1_{m+1} & \longrightarrow & \overline{E}^1_{m+1}/(v_1^\infty) & \longrightarrow & 0 \end{array} \tag{1.4}$$

where W_{m+1} is weak injective, ι_1 induces an isomorphism in Ext^0, and B_{m+1} is the $A(m+1)$-submodule of $\overline{E}^1_{m+1}/(v_1^\infty)$ generated by

$$\left\{ \frac{\widehat{v}_2^i}{ipv_1^i} : i > 0 \right\}.$$

The object of this paper is to study B_{m+1} and related Ext groups. Since the ith element above is $\widehat{\beta}'_{i/i}$, the elements of B_{m+1} are the beta elements of the title.

In [**NR**] we construct a variant of the Cartan-Eilenberg spectral sequence converging to $\mathrm{Ext}_{\Gamma(m+1)}(T_m^{(j)})$. Its $\tilde{E}_1$-term has the following chart:

	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$t=2$	0	$\mathrm{Ext}^0(U^3)$	$\mathrm{Ext}^1(U^3)$	$\mathrm{Ext}^2(U^3)$	$\cdots$
$t=1$	0	$\mathrm{Ext}^0(U^2)$	$\mathrm{Ext}^1(U^2)$	$\mathrm{Ext}^2(U^2)$	$\cdots$
$t=0$	$\mathrm{Ext}^0(\overline{D})$	$\mathrm{Ext}^0(W)$	$\mathrm{Ext}^0(B)$	$\mathrm{Ext}^1(B)$	$\cdots$
	$s=0$	$s=1$	$s=2$	$s=3$	

where all Ext groups are over $G(m+1)$ and the subscripts (equal to $m+1$) on U^{t+1}, $\overline{D}^0$, W and B have been omitted to save space.

Tensoring (1.4) with $\overline{T}_m^{(j)}$, we get the following chart:

(1.5)

	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$t=2$	0	$\mathrm{Ext}^0(\overline{T}_m^{(j)}U^3)$	$\mathrm{Ext}^1(\overline{T}_m^{(j)}U^3)$	$\mathrm{Ext}^2(\overline{T}_m^{(j)}U^3)$	$\cdots$
$t=1$	0	$\mathrm{Ext}^0(\overline{T}_m^{(j)}U^2)$	$\mathrm{Ext}^1(\overline{T}_m^{(j)}U^2)$	$\mathrm{Ext}^2(\overline{T}_m^{(j)}U^2)$	$\cdots$
$t=0$	$\mathrm{Ext}^0(\overline{D})$	$\mathrm{Ext}^0(\overline{T}_m^{(j)}W)$	$\mathrm{Ext}^0(\overline{T}_m^{(j)}B)$	$\mathrm{Ext}^1(\overline{T}_m^{(j)}B)$	$\cdots$
	$s=0$	$s=1$	$s=2$	$s=3$	

where the tensor product signs have been omitted to save space.

The construction of B_{m+1} will be given in §2. After introducing our basic methodology in §3, we determine the groups

$$\mathrm{Ext}^0(\overline{T}_m^{(j)} \otimes B_{m+1})$$

for the cases $j = 0$, $j = 1$ and $j > 1$ in the next three sections. Here

$$\overline{T}_m^{(j)} = A(m+1)\left\{t_{m+1}^{\ell} : 0 \le \ell < p^j\right\}.$$

In §7 we determine the higher Ext groups for $j = 1$ in a range of dimensions. Our calculations require some results about binomial coefficients and Quillen operations that are collected in Appendices A and B respectively.

2. *The construction of B_{m+1}*

Proposition 2.1. **A 4-term exact sequence of $G(m+1)$-comodules.** *The short exact sequence (1.2) gives a 4-term exact sequence*

$$\begin{array}{c} A(m+1) \\ \| \\ 0 \longrightarrow U_{m+1}^0 \xrightarrow{\iota_0} A(m)[p^{-1}\widehat{v}_1] \xrightarrow{\rho_0} \overline{E}_{m+1}^1 \xrightarrow{\delta_0} U_{m+1}^1 \longrightarrow 0. \end{array}$$

Let

$$\begin{aligned} V_{m+1} &= A(m)[p^{-1}\widehat{v}_1]/A(m+1) \\ &= A(m+1)\left\{\frac{\widehat{v}_1^i}{p^i} : i > 0\right\} \subset BP_*/(p^\infty). \end{aligned}$$

There is a short exact sequence of $G(m+1)$-comodules

$$0 \longrightarrow V_{m+1} \longrightarrow \overline{E}_{m+1}^1 \longrightarrow U_{m+1}^1 \longrightarrow 0$$

which is not split.

Proof. The comodule D_{m+1}^0 was described explicitly in [**Rav02**, Theorem 3.9]. It has the form

$$D_{m+1}^0 = A(m)[\widehat{\lambda}_1, \ldots] \subset p^{-1}BP_*$$

with

$$\widehat{\lambda}_i = \begin{cases} \dfrac{\widehat{v}_1}{p} & \text{for } i = 1 \\ \dfrac{\widehat{v}_2}{p} + \dfrac{\widehat{v}_1 v_1^{p\omega}}{p^2} + \dfrac{(p^{p-1}-1)v_1\widehat{v}_1^p}{p^{p+1}} & \text{for } i = 2 \\ \dfrac{v_i}{p} + \ldots & \text{for } i > 2 \end{cases}$$

and

$$\eta_R(\widehat{\lambda}_i) = \begin{cases} \widehat{\lambda}_1 + \widehat{t}_1 & \text{for } i = 1 \\ \widehat{\lambda}_2 + \widehat{t}_2 + (p^{p-1}-1)v_1 \displaystyle\sum_{0<j<p} p^{-1}\binom{p}{j}\widehat{\lambda}_1^{p-j}\widehat{t}_1^j & \text{for } i = 2 \\ \widehat{\lambda}_i + \widehat{t}_i + \ldots & \text{for } i > 2 \end{cases}$$

It follows that $\mathrm{Ext}^0_{\Gamma(m+2)}(D_{m+1}^0) = A(m)[\widehat{\lambda}_1]$ as claimed.

In order to understand the relation between $\overline{E}^1_{m+1}$ and U^1_{m+1}, consider the following diagram of $\Gamma(m+2)$-comodules with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & BP_* & \longrightarrow & D^0_{m+1} & \longrightarrow & E^1_{m+1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & BP_* & \longrightarrow & p^{-1}BP_* & \longrightarrow & BP_*/(p^\infty) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & BP_* & \longrightarrow & D^0_{m+2} & \longrightarrow & E^1_{m+2} & \longrightarrow & 0
\end{array}$$

The vertical maps are monomorphisms, and there is no obvious map either way between D^0_{m+1} and D^0_{m+2}. The description of the $U^1_{m+1} = \operatorname{Ext}^1_{\Gamma(m+2)}$ above is in terms of the connecting homomorphism for the bottom row. The element

$$\frac{\widehat{v}_2^i}{pi} \in E^1_{m+2}$$

is invariant and maps to the similarly named element in U^1_{m+1}. To describe its image in terms of the cobar complex, we pull it back to $\widehat{v}_2^i/pi \in D^0_{m+2}$ and compute its coboundary, which is

$$d\left(\widehat{v}_2^i/pi\right) = \left((\widehat{v}_2 + p\widehat{t}_2)^i - \widehat{v}_2^i\right)/pi = \widehat{v}_2^{i-1}\widehat{t}_2 + \ldots$$

However, the element $\widehat{v}_2^i/pi$ is *not present* in E^1_{m+1}. To see this, consider the case $i = 1$. In $p^{-1}BP_*$ we have

$$\begin{aligned}
\frac{\widehat{v}_2}{p} &= \widehat{\lambda}_2 - \frac{\widehat{v}_1 v_1^{p\omega}}{p^2} + \frac{(1-p^{p-1})v_1\widehat{v}_1^p}{p^{p+1}} \\
&= \widehat{\lambda}_2 - \frac{\widehat{\lambda}_1 v_1^{p\omega}}{p} + \frac{(1-p^{p-1})v_1\widehat{\lambda}_1^p}{p} \\
&\notin D^0_{m+1} = A(m)[\widehat{\lambda}_1, \widehat{\lambda}_2, \ldots].
\end{aligned}$$

Instead of $\widehat{v}_2/p$, consider the element $\widehat{\lambda}_2$ itself. Its image in E^1_{m+1} is invariant, so it defines a nontrivial element in $\overline{E}^1_{m+1}$. The computation of the image of $(p\widehat{\lambda}_2)^i/pi$ under the connecting homomorphism gives the same answer as before.

The right unit formula above implies that the short exact sequence does not split. □

DEFINITION 2.2. *Let M be a graded torsion $G(m+1)$-comodule of finite type, and let M_i have order p^{a_i}. Then the* **Poincaré series** *for M is defined by*

$$g(M) = \sum a_i t^i. \tag{2.3}$$

Given two such power series $f_1(t)$ and $f_2(t)$, the inequality $f_1(t) \le f_2(t)$ means that each coefficient of $f_1(t)$ is dominated by the corresponding one in $f_2(t)$.

THEOREM 2.4. **Construction of B_{m+1}.** *Let $B_{m+1} \subset \overline{E}^1_{m+1}/(v_1^\infty)$ be the sub-$A(m+1)$-module generated by the elements*

$$\widehat{\beta}'_{i/i} = \frac{\widehat{v}_2^i}{ipv_1^i}$$

for all $i > 0$. It is a $G(m+1)$-subcomodule whose Poincaré series is

$$g(B_{m+1}) = g_{m+1}(t) \sum_{k \geq 0} \frac{x^{p^{k+1}}(1 - y^{p^k})}{(1 - x^{p^{k+1}})(1 - x_2^{p^k})},$$

where

$$\begin{aligned} y &= t^{|v_1|}, \\ x &= t^{|\widehat{v}_1|}, \\ x_i &= t^{|\widehat{v}_i|} \qquad \text{for } i > 1 \\ \text{and} \qquad g_{m+1}(t) &= \prod_{1 \leq i \leq m+1} \frac{1}{1 - t^{|v_i|}}. \end{aligned}$$

Let W_{m+1} be the pullback in the diagram (1.4). Then W_{m+1} is a weak injective with $\mathrm{Ext}^0_{G(m+1)}(W_{m+1}) = \mathrm{Ext}^0_{G(m+1)}(\overline{E}^1_{m+1})$, i.e., the map $\overline{E}^1_{m+1} \to W_{m+1}$ induces an isomorphism in Ext^0.

Proof. To show that B_{m+1} is a $G(m+1)$-subcomodule, note that

$$\begin{aligned} \eta_R(\widehat{v}_2) &\equiv \widehat{v}_2 + v_1 \widehat{t}_1^p - v_1^{p\omega} \widehat{t}_1 \qquad \bmod p \\ \text{so} \qquad \eta_R(\widehat{v}_2)^i) &= \left(\widehat{v}_2 + v_1 \widehat{t}_1^p - v_1^{p\omega} \widehat{t}_1\right)^p \qquad \bmod pi \\ \text{and} \qquad \eta_R(\widehat{\beta}'_{i/i}) &\in B_{m+1} \otimes G(m+1). \end{aligned}$$

so B_{m+1} is a $G(m+1)$-comodule.

For the Poincaré series, let $F_k B_{m+1} \subset B_{m+1}$ denote the submodule of exponent p^k with $F_0 B_{m+1} = \phi$. Then the Poincaré series of

$$F_k B_{m+1} / F_{k-1} B_{m+1} = A(m+1)/I_1 \left\{ \widehat{\beta}_{ip^{k-1}/ip^{k-1}, p^k} \colon i > 0 \right\}$$

is

$$\begin{aligned} g\left(F_k B_{m+1} / F_{k-1} B_{m+1}\right) &= g(A(m+1)/I_2) \sum_{i > 0} x^{ip^k} \frac{1 - y^{ip^{k-1}}}{1 - y} \\ &= g_{m+1}(t) \sum_{i > 0} \left(x^{ip^k} - (x^p y)^{ip^{k-1}} \right) \\ &= g_{m+1}(t) \sum_{i > 0} \left(x^{ip^k} - x_2^{ip^{k-1}} \right) \\ &= g_{m+1}(t) \left(\frac{x^{p^k}}{1 - x^{p^k}} - \frac{x_2^{p^{k-1}}}{1 - x_2^{p^{k-1}}} \right). \end{aligned}$$

Summing these for all positive k gives the desired formula.

To show $\mathrm{Ext}^0_{G(m+1)}(W_{m+1})$ is as claimed it is enough to show that the connecting homomorphism

$$\mathrm{Ext}^0_{G(m+1)}(B_{m+1}) \longrightarrow \mathrm{Ext}^1_{G(m+1)}(\overline{E}^1_{m+1})$$

is monomorphic. Since the target group is in the Cartan-Eilenberg $\tilde{E}_2$-term converging to $\mathrm{Ext}^1_{\Gamma(m+1)}(E^1_{m+1})$, we have the composition

$$\eta : \mathrm{Ext}^0_{G(m+1)}(B_{m+1}) \longrightarrow \mathrm{Ext}^1_{\Gamma(m+1)}(E^1_{m+1}) \xrightarrow{\ \delta_0\ } \mathrm{Ext}^2_{\Gamma(m+1)}.$$

So it is sufficient to show that η is monomorphic. Since B_{m+1} is in $\mathrm{Ext}^0_{\Gamma(m+2)}(N^2)$, we have the following diagram

$$\begin{array}{ccccc}
\mathrm{Ext}^0_{\Gamma(m+1)}(M^1) & \longrightarrow & \mathrm{Ext}^0_{\Gamma(m+1)}(N^2) & \longrightarrow & \mathrm{Ext}^1_{\Gamma(m+1)}(N^1) \\
\| & & \uparrow & & \| \\
v_1^{-1}\mathrm{Ext}^1_{\Gamma(m+1)} & & \mathrm{Ext}^0_{G(m+1)}(B_{m+1}) & \xrightarrow{\eta} & \mathrm{Ext}^2_{\Gamma(m+1)}
\end{array}$$

The right equality holds because $\mathrm{Ext}^1_{\Gamma(m+1)}(M^0) = 0$, and the top row is exact. Since $\mathrm{Ext}^0_{\Gamma(m+1)}(M^1)$ is the $v_1^{-1}A(m)$-module generated by $\widehat{v}_1^i/ip$ the map η is monomorphic as desired.

The Poincaré series of W_{m+1} is given by

$$\begin{aligned}
g(W_{m+1}) &= g(\overline{E}^1_{m+1}) + g(B_{m+1}) = g(V_{m+1}) + g(U^1_{m+1}) + g(B_{m+1}) \\
&= g_{m+1}(t)\left(\frac{x}{1-x} + \sum_{j\geq 0}\frac{x_2^{p^j}}{1-x_2^{p^j}} + \sum_{j\geq 0}\frac{x^{p^{j+1}}(1-y^{p^j})}{(1-x^{p^{j+1}})(1-x_2^{p^j})}\right) \\
&= g_{m+1}(t)\left(\frac{x}{1-x} + \sum_{j\geq 0}\frac{x^{p^{j+1}}}{1-x^{p^{j+1}}}\right) = g_{m+1}(t)\sum_{j\geq 0}\frac{x^{p^j}}{1-x^{p^j}} \\
&= \frac{g(\mathrm{Ext}^1_{\Gamma(m+1)})}{1-x} \qquad \text{by [\textbf{Rav02}, Theorem 3.17]} \\
&= \frac{g\left(\mathrm{Ext}^0_{G(m+1)}(W_{m+1})\right)}{1-x}.
\end{aligned}$$

This means that W_{m+1} is weak injective by [**Rav02**, Theorem 2.6]. □

3. *Basic methods for finding comodule primitives*

From now on, all Ext groups are understood to be over $G(m+1)$.

Definition 3.1. *[**Rav04**, Definition 7.1.8] A $G(m+1)$-comodule M is called j-**free** if the comodule tensor product $\overline{T}^{(j)}_m \otimes_{A(m+1)} M$ is weak injective, i.e.,*

$$\mathrm{Ext}^n(A(m+1), \overline{T}^{(j)}_m \otimes_{A(m+1)} M) = 0$$

*for $n > 0$. The elements of Ext^0 are called j-**primitives**.*

We will often abbreviate $\mathrm{Ext}(A(m+1), N)$ by $\mathrm{Ext}(N)$ for short. We will see in Proposition 3.3 that it is enough to consider a certain subgroup $L_j(M)$ of M to detect elements of $\mathrm{Ext}^0(\overline{T}^{(j)}_m \otimes M)$. Given a right $G(m+1)$-comodule M and the structure map $\psi_M : M \to G(m+1) \otimes M$, define the *Quillen operation* $\widehat{r}_i : M \to M$ $(i \geq 0)$ on $z \in M$ by $\psi_M(z) = \sum_i \widehat{r}_i(z) \otimes \widehat{t}_1^i$. *In this paper all comodules are right comodules. In most cases the structure map is determined by the right unit formula.*

Definition 3.2. **The group $L_j(M)$.** *Denote the subgroup $\bigcap_{n\geq p^j} \ker \widehat{r}_n$ of M by $L_j(M)$. By definition, we have a sequence of inclusions*

$$L_0(M) \subset L_1(M) \subset \ldots\ldots \subset L_j(M) \subset \ldots\ldots$$

and $L_0(M) = \mathrm{Ext}^0(M)$.

The following result allows us to identify j-primitives with $L_j(M)$.

PROPOSITION 3.3. *[***Rav02***, Lemma 1.12]* **Identification of the j-primitives with $L_j(m)$.** *For a $G(m+1)$-comodule M, the map*

$$(c \otimes 1)\psi_M : L_j(M) \longrightarrow \mathrm{Ext}^0(\overline{T}_m^{(j)} \otimes M)$$

is an isomorphism between $A(m+1)$-modules, where c is the conjugation map.

When we detect elements of $L_j(M)$, it is enough to consider elements killed by $\widehat{r}_{p^j}$ $(j \geq 0)$, as one sees by the following proposition.

PROPOSITION 3.4. **A property of Quillen operations.** *If the Quillen operation $\widehat{r}_{p^j}$ on a $G(m+1)$-comodule M is trivial, then all operations $\widehat{r}_n$ for $p^j \leq n < p^{j+1}$ are trivial.*

Proof. Since $\widehat{r}_i\widehat{r}_j = \begin{pmatrix} i+j \\ i \end{pmatrix}\widehat{r}_{i+j}$ [**Nak**, Lemma 3.1] we have a relation $\widehat{r}_{n-p^j}\widehat{r}_{p^j} = \begin{pmatrix} n \\ p^j \end{pmatrix}\widehat{r}_n$. Observing that the congruence $\begin{pmatrix} n \\ p^j \end{pmatrix} \equiv s \mod (p)$ for $sp^j \leq n < (s+1)p^j$, $\begin{pmatrix} n \\ p^j \end{pmatrix}$ is invertible in $\mathbf{Z}_{(p)}$ whenever $p^j \leq n < p^{j+1}$, and the result follows. □

In the following sections we will determine the structure of $L_0(B_{m+1})$ in Proposition 4.2 and 4.4 and $L_1(B_{m+1})$ in Proposition 5.1 and 5.4 in all dimensions, and $L_j(B_{m+1})$ $(j > 1)$ in Theorem 6.1 below dimension $|\widehat{v}_2^{p^j+1}/v_1^{p^j}|$. Then we need a method for checking whether all j-primitives $(j > 1)$ are listed or not.

The following lemma gives an explicit criterion the j-freeness of a comodule M.

LEMMA 3.5. **A Poincaré series characterization of j-free comodules.** *For a graded torsion connective $G(m+1)$-comodule M of finite type, we have an inequality*

$$g(M)(1 - x^{p^j}) \;\leq\; g(L_j(M)) \qquad \textit{where } x = t^{|\widehat{v}_1|} \tag{3.6}$$

with equality holding iff M is j-free.

Proof. Let $I \subset A(m+1)$ be the maximal ideal. We have the inequality

$$g(\overline{T}_m^{(j)} \otimes M) \leq g(\mathrm{Ext}^0(\overline{T}_m^{(j)} \otimes M)) \cdot g(G(m+1)/I)$$

by [**Rav04**] Theorem 7.1.34, where the equality holds iff M is a weak injective. Observe that

$$g(\overline{T}_m^{(j)} \otimes M) = g(M)\frac{1 - x^{p^j}}{1 - x},$$
$$g(G(m+1)/I) = \frac{1}{1 - x}$$

$$\text{and} \qquad g(\mathrm{Ext}^0(\overline{T}_m^{(j)} \otimes M)) = g(L_j(M)).$$

□

LEMMA 3.7. **A Poincaré series formula for the first Ext^1 group.** *For a graded torsion connective $G(m+1)$-comodule M of finite type, suppose*

$$\frac{g(L_j(M))}{1-x^{p^j}} - g(M) \equiv ct^d \qquad \bmod t^{d+1}$$

Then the first nontrivial element in $\mathrm{Ext}^1(\overline{T}_m^{(j)} \otimes M)$ occurs in dimension d, and the order of the group $G = \mathrm{Ext}^{1,d}(\overline{T}_m^{(j)} \otimes M)$ is p^c.

Proof. Since the inequality of (3.6) is an equality below dimension d, M is j-free in that range, so $\mathrm{Ext}^1(\overline{T}_m^{(j)} \otimes M)$ vanishes below dimension d. Each element $x \in G$ is represented by a short exact sequence of the form

$$0 \longrightarrow \overline{T}_m^{(j)} \otimes M \longrightarrow M' \longrightarrow \Sigma^d A(m+1) \longrightarrow 0.$$

If x has order p^i, then we get a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{T}_m^{(j)} \otimes M & \longrightarrow & M' & \longrightarrow & \Sigma^d A(m+1) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{T}_m^{(j)} \otimes M & \longrightarrow & M'' & \longrightarrow & \Sigma^d A(m+1)/(p^i) & \longrightarrow & 0 \end{array}$$

Since G is a finite abelian p-group, it is a direct sum of cyclic groups. We can do the above for each of its generators and assemble them into an extension

$$0 \longrightarrow \overline{T}_m^{(j)} \otimes M \longrightarrow M''' \longrightarrow \Sigma^d G \otimes_{\mathbf{Z}_{(p)}} A(m+1) \longrightarrow 0$$

with $\mathrm{Ext}^0_{G(m+1)}(M''') = L_j(M)$ through dimension d and $\mathrm{Ext}^{1,d}_{G(m+1)}(M''') = 0$, so M''' is weak injective through dimension d.

If $|G| = p^b$, then we have

$$\begin{aligned} g(M''') &= g(\overline{T}_m^{(j)} \otimes M) + g(\Sigma^d G \otimes_{\mathbf{Z}_{(p)}} A(m+1)) \\ &= g(M)\left(\frac{1-x^{p^j}}{1-x}\right) + bt^d g_{m+1}(t) \end{aligned}$$

Since M''' is weak injective through dimension d, we have

$$\begin{aligned} g(M''') &\equiv \frac{g\left(\mathrm{Ext}^0_{G(m+1)}(M''')\right)}{1-x} \qquad \bmod t^{d+1} \\ &\equiv \frac{g\left(L_j(M)\right)}{1-x} \\ &\equiv g(M)\left(\frac{1-x^{p^j}}{1-x}\right) + ct^d \end{aligned}$$

so $b = c$. □

4. 0-*primitives in* B_{m+1}

In this section we determine the structure of $\mathrm{Ext}^0(B_{m+1})$, i.e., the primitives in B_{m+1} in the usual sense. We treat the cases $m > 0$ and $m = 0$ separately. The latter is more complicated because v_1 is not invariant over $\Gamma(1)$. Recall that the $G(m+1)$-comodule structure of B_{m+1} is given by the right unit map η_R.

LEMMA 4.1. **An approximation of the right unit.** *The right unit map* $\eta_R : A(m+2)_* \to G(m+2)$ *on the Hazewinkel generators are expressed by*

$$\eta_R(\widehat{v}_1) = \widehat{v}_1 + p\widehat{t}_1,$$
$$\eta_R(\widehat{v}_2) \equiv \widehat{v}_2 + v_1\widehat{t}_1^p - v_1^{p\omega}\widehat{t}_1 \mod (p)$$

where $\omega = p^m$.

Proof. These directly follow from [**MRW**] (1.1) and (1.3). □

For a given integer n, denote the exponent of a prime p in the factorization of n by $\nu_p(n)$ as usual. In particular, $\nu_p(0) = \infty$. When the integer is a binomial coefficient $\binom{n}{k}$, we will write $\nu_p\binom{n}{k}$ instead of $\nu_p\left(\binom{n}{k}\right)$.

Let $\widehat{h}_j$ be the 1-dimensional cohomology class of $\widehat{t}_1^{p^j}$.

PROPOSITION 4.2. **Structure of** $\mathrm{Ext}^0(B_{m+1})$ **for** $m > 0$**.** *For* $m > 0$, $\mathrm{Ext}^0(B_{m+1})$ *is the* $A(m)$*-module generated by*

$$\left\{ p^k\widehat{v}_1^s\widehat{\beta}'_{ip^k/t}\colon\ i > 0,\ s \geq 0,\ k \geq 0,\ 0 < t \leq p^k \text{ and } \nu_p(i) \leq \nu_p(s) \right\}.$$

The first nontrivial element in $\mathrm{Ext}^1(B_{m+1})$ *is*

$$\widehat{h}_0\widehat{\beta}_1 \in \mathrm{Ext}^{1,2(p+1)(p\omega-1)}(B_{m+1}).$$

Proof. We may put $s = ap^\ell$ and $i = bp^\ell$ with $p \nmid b$ and $a \geq 0$. Observe that

$$\begin{aligned}\psi\left(\frac{\widehat{v}_1^{ap^\ell}\widehat{v}_2^{bp^{\ell+k}}}{bp^{\ell+1}v_1^t}\right) &= \frac{\widehat{v}_1^{ap^\ell}(\widehat{v}_2^{p^k} + v_1^{p^k}\widehat{t}_1^{p^{k+1}} - v_1^{p^{k+1}\omega}\widehat{t}_1^{k})^{bp^\ell}}{bp^{\ell+1}v_1^t} \quad \text{since } p \nmid b \\ &= \frac{\widehat{v}_1^{ap^\ell}\widehat{v}_2^{bp^{\ell+k}}}{bp^{\ell+1}v_1^t} \quad \text{since } t \leq p^k\end{aligned}$$

and so the exhibited elements are invariant. On the other hand, we have nontrivial Quillen operations

$$\begin{aligned}\widehat{r}_1(p^k\widehat{v}_1^s\widehat{\beta}'_{ip^k/t}) &= -\frac{\widehat{v}_1^s\widehat{v}_2^{ip^k-1}}{p^{1-k}v_1^{t-p\omega}} + \frac{s}{i}\cdot\frac{\widehat{v}_1^{s-1}\widehat{v}_2^{ip^k}}{v_1^t} & \text{if } \nu_p(s) < \nu_p(i) \\ \text{and} \quad \widehat{r}_{p^{k+1}}(p^k\widehat{v}_1^s\widehat{\beta}'_{ip^k/t}) &= \frac{\widehat{v}_1^s\widehat{v}_2^{p^k(i-1)}}{pv_1^{t-p^k}} + \dots & \text{if } t > p^k,\end{aligned}$$

where the missing terms in the second expression involve lower powers of $\widehat{v}_1$ in the numerator or smaller powers of v_1 in the denominator.

This means each element $p^k\widehat{v}_1^s\widehat{\beta}'_{ip^k/t}$ with $\nu_p(s) < \nu_p(i)$ supports a nontrivial $\widehat{r}_1$, the targets of which are linearly independent. Similarly, each such monomial with $t > p^k$ supports a nontrivial $\widehat{r}_{p^{k+1}}$. It follows that no linear combination of such elements is invariant, so Ext^0 is as stated.

For the second statement, note that $\widehat{h}_0$ and $\widehat{\beta}_1$ are the first nontrivial elements in Ext^1 and $\mathrm{Ext}^0(B_{m+1})$ respectively, so if their product is nontrivial, the claim follows. It is nontrivial because there is no $x \in B_{m+1}$ with $\widehat{r}_1(x) = \widehat{\beta}_1$. □

We now turn to the case $m = 0$.

Lemma 4.3. **Right unit in $G(1)$.** *The right unit $\eta_R : A(1) \to G(1)$ on the chromatic fraction $\frac{1}{ipv_1^t}$ is*

$$\eta_R\left(\frac{1}{ipv_1^t}\right) = \sum_{k\geq 0}\binom{t+k-1}{k}\frac{(-t_1)^k}{ip^{1-k}v_1^{t+k}}.$$

Note that this sum is finite because a chromatic fraction is nontrivial only when its denominator is divisible by p.

Proof. Recall the expansion

$$\frac{1}{(x+y)^t} = (x+y)^{-t} = x^{-t}(1+y/x)^{-t} = x^{-t}\sum_{k\geq 0}\binom{-t}{k}\frac{y^k}{x^k}$$
$$= \sum_{k\geq 0}\binom{t+k-1}{k}\frac{(-y)^k}{x^{k+t}}$$

and the formula $\eta_R(v_1^t) = (v_1 + pt_1)^t$ by Lemma 4.1. □

Proposition 4.4. **Structure of $\mathrm{Ext}^0(B_1)$.** *For $m = 0$, $\mathrm{Ext}^0(B_1)$ is the $\mathbf{Z}_{(p)}$-module generated by*

$$\left\{p^k\beta'_{ip^k/t}\colon\ i > 0,\ k \geq 0,\ 0 < t \leq p^k \text{ and } \nu_p(i) \leq \nu_p(t)\ \right\}.$$

The first nontrivial element in $\mathrm{Ext}^1(B_1)$ is

$$h_0\beta_1 \in \mathrm{Ext}^{1,2(p^2-1)}(B_{m+1})$$

Proof. When i and t are as stated, we may set $t = ap^\ell$ and $i = bp^\ell$ with $p \nmid b$ and $a > 0$. Observe that

$$\eta_R\left(\frac{v_2^{bp^{\ell+k}}}{bp^{\ell+1}v_1^{ap^\ell}}\right) = \left(v_2^{p^k} + v_1^{p^k}t_1^{p^{k+1}} - v_1^{p^{k+1}}t_1^{p^k}\right)^{bp^\ell}$$
$$\sum_{n\geq 0}\binom{ap^\ell+n-1}{n}\frac{(-t_1)^n}{bp^{\ell+1-n}v_1^{ap^\ell+n}}.$$

For $n > 0$, the binomial coefficient is divisible by $p^{\ell+1-n}$ by Lemma A.3 below, so the expression simplifies to

$$\eta_R\left(\frac{v_2^{bp^{\ell+k}}}{bp^{\ell+1}v_1^{ap^\ell}}\right) = \frac{(v_2^{p^k} + v_1^{p^k}t_1^{p^{k+1}} - v_1^{p^{k+1}}t_1^{p^k})^{bp^\ell}}{bp^{\ell+1}v_1^{ap^\ell}}$$

and $p^k\beta'_{ip^k/t}$ is invariant by an argument similar to that of Lemma 4.2. On the other hand if either of the conditions on i and t fails, we have nontrivial Quillen operations

$$\begin{aligned} r_1\left(p^k\beta'_{ip^k/t}\right) &= -\frac{v_2^{ip^k-1}}{p^{1-k}v_1^{t-p}} - \frac{t}{i}\cdot\frac{v_2^{ip^k}}{v_1^{t+1}} \quad \text{if } \nu_p(i) > \nu_p(t) \\ \text{or} \quad r_{p^{k+1}}\left(p^k\beta'_{ip^k/t}\right) &= \frac{v_2^{(i-1)p^k}}{pv_1^{t-p^k}} \quad \text{if } t > p^k\ . \end{aligned}$$

The rest of the argument, inclduing the identifation of the first nontrivial element in $\mathrm{Ext}^1(B_1)$, is the same as in the case $m > 0$. □

5. 1-*primitives in* B_{m+1}

In this section we determine the structure of $L_1(B_{m+1})$, which includes all elements of $\mathrm{Ext}^0(B_{m+1})$ determined in the previous section. By observing that $\widehat{r}_1(\widehat{v}_1\widehat{\beta}'_p) = \widehat{\beta}_p$ and $\widehat{r}_{p^j}(\widehat{v}_1\widehat{\beta}'_p) = 0$ for $j \geq 1$, the first element of the quotient $L_1(B_{m+1})/L_0(B_{m+1})$ is $\widehat{v}_1\widehat{\beta}'_p$ for $m > 0$. In general, we have

Proposition 5.1. **Structure of $L_1(B_{m+1})$ for $m > 0$.** *For $m > 0$, $L_1(B_{m+1})$ is isomorphic to the $A(m)$-module generated by $p^k\widehat{v}_1^s\widehat{\beta}'_{ip^k/t}$, where $i > 0$, $s \geq 0$, $k \geq 0$ and $0 < t \leq p^k$, and the integers i and s satisfy the following condition: there is a non-negative integer n such that $s \equiv 0, 1, \dots p-1$ mod (p^{n+1}) and $\nu_p(i) < n + p$.*

Note that the description of $L_1(B_{m+1})$ differs from that of $L_0(B_{m+1})$ given in Proposition 4.2 only in the restriction on i and s. In that case it was $\nu_p(i) \leq \nu_p(s)$. If $\nu_p(s) = n+1$ (i.e., $s \equiv 0 \bmod (p^{n+1})$), then an integer i satisfying $\nu_p(i) \leq n+1$ also satisfies $\nu_p(i) < n+p$. Hence we have $L_0(B_{m+1}) \subset L_1(B_{m+1})$ as desired.

Proof. In Proposition 4.2 we have already seen that $p^k\widehat{\beta}'_{ip^k/t}$ is invariant iff $0 < t \leq p^k$. If follows that

$$\widehat{r}_{p^\ell}(p^k\widehat{v}_1^s\widehat{\beta}'_{ip^k/p^k}) = \widehat{r}_{p^\ell}(\widehat{v}_1^s)\cdot p^k\widehat{\beta}'_{ip^k/p^k} = p^{p^\ell}\begin{pmatrix} s \\ p^\ell \end{pmatrix}\widehat{v}_1^{s-p^\ell}\cdot\frac{\widehat{v}_2^{ip^k}}{ipv_1^{p^k}}.$$

Since we are dealing with 1-primitives, we can ignore the case $\ell = 0$. For $\ell = 1$, this is clearly trivial if $s < p$. When $s \geq p$, choose an integer n such that $p^n \mid \begin{pmatrix} s \\ p \end{pmatrix}$. By Lemma A.4 this means $n = 0$ unless s is p-adically close to an integer ranging from 0 to $p-1$. Then $\widehat{r}_p$ is trivial if $\nu_p(i) < n + p$. We can show that all Quillen operations $\widehat{r}_{p^\ell}$ for $\ell > 1$ are trivial under the same condition since

$$\nu_p\left(p^p\begin{pmatrix} s \\ p \end{pmatrix}\right) \leq \nu_p\left(p^{p^\ell}\begin{pmatrix} s \\ p^\ell \end{pmatrix}\right)$$

which follows from

$$q\nu_p\left(p^{p^\ell}\begin{pmatrix} s \\ p^\ell \end{pmatrix}\right) = p^\ell + 1 + \alpha(s-p^\ell) - \alpha(s) \quad \text{by Lemma A.2}$$

$$\text{and} \quad q\left[\nu_p\left(p^{p^\ell}\begin{pmatrix} s \\ p^\ell \end{pmatrix}\right) - \nu_p\left(p^p\begin{pmatrix} s \\ p \end{pmatrix}\right)\right] = p^\ell - p + \alpha(s-p^\ell) - \alpha(s-p)$$

$$\begin{aligned}&\geq \alpha(p^\ell - p) + \alpha(s - p^\ell) - \alpha(s-p)\\ &\geq 0.\end{aligned}$$

□

Note also that the condition on i and s in Proposition 5.1 is automatically satisfied whenever $i < p^p$, which means that we may set $n = 0$. Since

$$\widehat{r}_p(\widehat{v}_1^s) = p^p \binom{s}{p} \widehat{v}_1^{s-p}$$

and p^p kills all of B_{m+1} below the dimension of $\widehat{\beta}_{p^p/p^p}$, $\widehat{v}_1$ is effectively invariant in this range, making B_{m+1} an $A(m+1)$-module.

COROLLARY 5.2. **Poincaré series for $L_1(B_{m+1})$.** *For $m > 0$, the Poincaré series for $L_1(B_{m+1})$ below dimension $p^p|\widehat{v}_2|$ is*

$$g_{m+1}(t) \sum_{k\geq 0} \frac{x^{p^{k+1}} - x_2^{p^k}}{1 - x_2^{p^k}}, \tag{5.3}$$

and in the same range we have

$$L_1(B_{m+1}) = A(m+1)\left\{p^k \widehat{\beta}'_{ip^k/t}\colon\ i > 0,\ k \geq 0 \text{ and } 0 < t \leq p^k\ \right\}.$$

Proof. As is explained in the above, we may consider $L_1(B_{m+1})$ as an $A(m+1)$-module in that range. To determine the Poincaré series $g(L_1(B_{m+1}))$, decompose $L_1(B_{m+1})$ into the following two direct summands:

(i) $S_0 = A(m+1)/I_2\left\{\widehat{\beta}'_i\colon i > 0\right\}$

(ii) $S_k = A(m+1)/I_2\left\{p^k \widehat{\beta}'_{ip^k/t}\colon\ i > 0 \text{ and } p^{k-1} < t \leq p^k\ \right\}$ for $k > 0$

The Poincaré series for these sets are given by

$$\begin{aligned} g(S_0) &= g_{m+1}(t)\cdot(1-y)\sum_{n\geq 0} y^{-1}\frac{x_2^{p^n}}{1-x_2^{p^n}}\\ \text{and}\quad g(S_k) &= g_{m+1}(t)\cdot(1-y)\sum_{n>0}\frac{y^{-p^k}(1-y^{p^k-p^{k-1}})}{1-y}\cdot\frac{x_2^{p^{n+k-1}}}{1-x_2^{p^{n+k-1}}}\\ &= g_{m+1}(t)\sum_{n\geq 0}(y^{-p^k}-y^{-p^{k-1}})\frac{x_2^{p^{n+k}}}{1-x_2^{p^{n+k}}}\end{aligned}$$

which gives

$$\begin{aligned}\frac{g(L_1(B_{m+1}))}{g_{m+1}(t)} &= \sum_{n\geq 0}(y^{-1}-1)\frac{x_2^{p^n}}{1-x_2^{p^n}} + \sum_{0<k\leq n}(y^{-p^k}-y^{-p^{k-1}})\frac{x_2^{p^n}}{1-x_2^{p^n}}\\ &= \sum_{n\geq 0}(y^{-1}-1)\frac{x_2^{p^n}}{1-x_2^{p^n}} + \sum_{n>0}(y^{-p^n}-y^{-1})\frac{x_2^{p^n}}{1-x_2^{p^n}}\\ &= (y^{-1}-1)\frac{x_2}{1-x_2} + \sum_{n>0}(y^{-p^n}-1)\frac{x_2^{p^n}}{1-x_2^{p^n}}\end{aligned}$$

$$= \sum_{n\geq 0} \frac{x_2^{p^n}(y^{-p^n}-1)}{1-x_2^{p^n}}$$

which is equal to (5.3). □

Now we turn to the case $m=0$, for which we make use of Lemma 4.3 again. Observing that $\widehat{r}_1(\beta'_p) = -\beta_{p/2}$ and $\widehat{r}_{p^j}(\beta'_p) = 0$ for $j \geq 1$, the first element of the quotient $L_1(B_{m+1})/L_0(B_{m+1})$ is β'_p. In general, we have

PROPOSITION 5.4. **Structure of $L_1(B_1)$.** *For $m=0$, $L_1(B_1)$ is isomorphic to the $\mathbf{Z}_{(p)}$-module generated by $p^k\beta'_{ip^k/t}$, where $k \geq 0$, $i > 0$ and $0 < t \leq p^k$ satisfying the following condition: there is a non-negative integer n such that $-t = 0, 1, \dots, p-1 \bmod (p^{n+1})$ and $p^{p+n} \not| \, i$.*

Proof. We have

$$\psi\left(\frac{v_2^{ip^k}}{ipv_1^t}\right) = (v_2^{p^k} + v_1^{p^k}t_1^{p^{k+1}} - v_1^{p^{k+1}}t_1^{p^k})^i \sum_{r\geq 0} \binom{t+r-1}{r} \frac{(-pt_1)^r}{ipv_1^{t+r}}$$

in which there are terms

$$\frac{v_2^{(i-1)p^k}t_1^{p^{k+1}}}{pv_1^{t-p^k}}, \quad -\frac{v_2^{(i-1)p^k}t_1^{p^k}}{pv_1^{t-p^{k+1}}} \quad \text{and} \quad (-p)^{p^\ell}\binom{t+p^\ell-1}{p^\ell}\frac{v_2^{ip^k}t_1^{p^\ell}}{ipv_1^{t+p^\ell}} \quad \text{for } \ell \geq 0.$$

Since $t \leq p^k$, the first and the second are trivial, which gives

$$\widehat{r}_{p^\ell}\left(p^k\beta_{ip^k/t}\right) = (-p)^{p^\ell}\binom{t+p^\ell-1}{p^\ell}\frac{v_2^{ip^k}}{ipv_1^{t+p^\ell}}.$$

Choose an integer n such that $p^n \mid \binom{t+p-1}{p}$, which occurs iff $-t = 0, 1, \dots, p-1 \bmod (p^{n+1})$ by Lemma A.4. Then $\widehat{r}_p$ is trivial if $p^{p+n} \not| \, i$. We can also observe that all the higher Quillen operations $\widehat{r}_\ell$ ($\ell \geq 1$) are trivial since

$$\nu_p\left(p^p\binom{t+p-1}{p}\right) \leq \nu_p\left(p^{p^\ell}\binom{t+p^\ell-1}{p^\ell}\right)$$

(see the proof of Proposition 5.1). □

COROLLARY 5.5. **$L_1(B_1)$ as an $A(1)$-module.** *For $m=0$, we have*

$$L_1(B_1) = A(1)\left\{p^k\beta'_{ip^k/t}\colon\ i>0,\ k\geq 0 \text{ and } 0<t\leq p^k\right\}$$

below dimension $p^p|v_2|$. The Poincaré series for $L_1(B_1)$ in this range is the same as (5.3).

Applying Lemma 3.5 and 3.7 to the Poincaré series (5.3), we have the following result.

COROLLARY 5.6. **1-free range for B_{m+1}.** *For $m \geq 0$, B_{m+1} is 1-free below dimension $p(p+1)|\widehat{v}_1|$, and the first element in $\mathrm{Ext}^1(\overline{T}_m^{(1)} \otimes B_{m+1})$ is $\widehat{\beta}_{p/p}\widehat{h}_1$.*

Here we use the notation $\widehat{\beta}_{p/p}$ for its image under the map $(c \otimes 1)\psi_{B_{m+1}}$ (cf. (3.3)).

Proof. By comparing $g(B_{m+1})$ and $g(L_1(B_{m+1}))$ and using Lemma 3.7, we see that the first nontrivial element of $\mathrm{Ext}^1(\widehat{\overline{T}}_m^{(1)} \otimes B_{m+1})$ occurs in the indicated dimension, where the group has order p. The fact that $\widehat{\beta}_{p/p}\widehat{h}_1$ is nontrivial in Ext^1 follows by direct calculation. □

6. *j-primitives in* B_{m+1} *for* $j > 1$

In this section we determine the structure of $L_j(B_{m+1})$ for $j \geq 2$ and $m > 0$ (See [**Rav04**] Lemma 7.3.1 for the $m = 0$ case). The first element of the quotient $L_j(B_{m+1})/L_{j-1}(B_{m+1})$ is $\widehat{\beta}_{p^{j-2}+1/p^{j-2}+1}$, which has nontrivial Quillen operation

$$\widehat{r}_{p^{j-1}}\left(\widehat{\beta}_{p^{j-2}+1/p^{j-2}+1}\right) = \widehat{\beta}_1.$$

In general, we have

THEOREM 6.1. **Structure of $L_j(B_{m+1})$ in low dimensions for $j > 1$.**

(i) *Below dimension $p^{j+1}|\widehat{v}_2|$, $L_j(B_{m+1})$ is the $A(m+1)$-module generated by*

$$\left\{\widehat{\beta}'_{i/t}\colon 0 < t \leq \min(i, p^{j-1})\right\} \cup \left\{\widehat{\beta}_{ap^j+b/t}\colon p^{j-1} < t \leq p^j,\ a > 0 \text{ and } 0 \leq b < p^{j-1}\right\}.$$

(ii) *B_{m+1} is j-free below dimension $|\widehat{v}_1^{p^{j+1}}\widehat{v}_2|$.*

(iii) *The first element in Ext^1 is the p-fold Massey product*

$$\langle\widehat{\beta}_{1+p^{j-1}/p^{j-1}}, \underbrace{\widehat{h}_{1,j}, \ldots, \widehat{h}_{1,j}}_{p-1}\rangle.$$

For the basic properties of Massey products, we refer the reader to [**Rav86**, A1.4] or [**Rav04**, A1.4]

Proof. (i) The listed elements are the only j-primitives below dimensions $p^{j+1}|\widehat{v}_2|$ by Proposition B.3, and the first statement follows.

(ii) To show that B_{m+1} is j-free below the indicated dimension, we need to compute some Poincaré series. This will be a lengthy calculation.

Decompose $L_j(B_{m+1})$ into the following three direct summands:

$$\begin{aligned} S_{0,1} &= A(m+1)\left\{\widehat{\beta}'_{i/t}\colon 0 < t \leq i < p^{j-1}\right\}, \\ S_{0,2} &= A(m+1)\left\{\widehat{\beta}'_{i/t}\colon 0 < t \leq p^{j-1} \leq i\right\}, \\ S_j &= A(m+1)\left\{\widehat{\beta}_{ap^j+b/t}\colon p^{j-1} < t \leq p^j,\ a > 0 \text{ and } 0 \leq b < p^{j-1}\right\}. \end{aligned}$$

We will always work below the dimension of $\widehat{\beta}_{2p^j/p^j}$, which is $|\widehat{v}_1^{p^{j+1}}\widehat{v}_2^{p^j}|$. This means that in the description of S_j above, the only relevant value of a is 1.

Observe that

$$S_{0,1} = \bigcup_{0<k<j} A(m+1)/I_2\left\{\frac{\widehat{v}_2^{ip^{k-1}}}{p^k v_1^{ip^{k-1}-\ell}}\colon 0 \leq \ell < ip^{k-1}, 0 < i < p^{j-k}\right\},$$

so

$$g(S_{0,1}) = g(A(m+1)/I_2) \sum_{0<k<j} \sum_{0<i<p^{j-k}} \frac{(1-y^{ip^{k-1}})(x^{p^k})^i}{1-y}$$

$$\begin{aligned}
&= g_{m+1}(t) \sum_{0<k<j} \sum_{0<i<p^{j-k}} (x^{ip^k} - x_2^{ip^{k-1}}) \\
\frac{g(S_{0,1})}{g_{m+1}(t)} &= \sum_{0<k<j} \left(\frac{x^{p^k}(1-(x^{p^k})^{p^{j-k}-1})}{1-x^{p^k}} - \frac{x_2^{p^{k-1}}(1-(x_2^{p^{k-1}})^{p^{j-k}-1})}{1-x_2^{p^{k-1}}} \right) \\
&= \sum_{0<k<j} \left(\frac{x^{p^k} - x^{p^j}}{1-x^{p^k}} - \frac{x_2^{p^{k-1}} - x_2^{p^{j-1}}}{1-x_2^{p^{k-1}}} \right)
\end{aligned}$$

For $S_{0,2}$, we have

$$S_{0,2} = A(m+1) \left\{ \frac{\widehat{v}_2^i}{ipv_1^{p^{j-1}-\ell}} : 0 \le \ell < p^{j-1}, i \ge p^{j-1} \right\},$$

which is the quotient of

$$\bigcup_{k>0} A(m+1)/I_2 \left\{ \frac{\widehat{v}_2^{ip^{k-1}}}{p^k v_1^{p^{j-1}-\ell}} : 0 \le \ell < p^{j-1}, i > 0 \right\}$$

$$\text{by } \bigcup_{0<k<j} A(m+1)/I_2 \left\{ \frac{\widehat{v}_2^{ip^{k-1}}}{p^k v_1^{p^{j-1}-\ell}} : 0 \le \ell < p^{j-1}, 0 < i < p^{j-k} \right\}.$$

Hence the Poincaré series of $S_{0,2}$ is

$$\begin{aligned}
g(S_{0,2}) &= g(A(m+1)/I_2) \cdot \frac{(1-y^{p^{j-1}})y^{-p^{j-1}}}{1-y} \\
&\qquad \left(\sum_{k>0} \sum_{i>0} (x_2^{p^{k-1}})^i - \sum_{0<k<j} \sum_{0<i<p^{j-k}} (x_2^{p^{k-1}})^i \right) \\
\frac{g(S_{0,2})}{g_{m+1}(t)} &= (y^{-p^{j-1}} - 1) \\
&\qquad \left(\sum_{k>0} \frac{x_2^{p^{k-1}}}{1-x_2^{p^{k-1}}} - \sum_{0<k<j} \frac{x_2^{p^{k-1}}(1-(x_2^{p^{k-1}})^{p^{j-k}-1})}{1-x_2^{p^{k-1}}} \right) \\
&= (y^{-p^{j-1}} - 1) \left(\sum_{k>0} \frac{x_2^{p^{k-1}}}{1-x_2^{p^{k-1}}} - \sum_{0<k<j} \frac{x_2^{p^{k-1}} - x_2^{p^{j-1}}}{1-x_2^{p^{k-1}}} \right) \\
&= (y^{-p^{j-1}} - 1) \left(\sum_{k>j} \frac{x_2^{p^{k-1}}}{1-x_2^{p^{k-1}}} + \sum_{0<k\le j} \frac{x_2^{p^{j-1}}}{1-x_2^{p^{k-1}}} \right) \\
&\equiv (y^{-p^{j-1}} - 1)x_2^{p^j} + \sum_{0<k\le j} \frac{x^{p^j} - x_2^{p^{j-1}}}{1-x_2^{p^{k-1}}}
\end{aligned}$$

in our range of dimensions.

Adding these two gives

$$\begin{aligned}
\frac{g(S_{0,1} \cup S_{0,2})}{g_{m+1}(t)} &= \frac{g(S_{0,1}) + g(S_{0,2})}{g_{m+1}(t)} \\
&= \sum_{0<k<j} \left(\frac{x^{p^k} - x^{p^j}}{1-x^{p^k}} - \frac{x_2^{p^{k-1}} - x_2^{p^{j-1}}}{1-x_2^{p^{k-1}}} \right)
\end{aligned}$$

$$
\begin{aligned}
&+(y^{-p^{j-1}}-1)x_2^{p^j}+\sum_{0<k\le j}\frac{x^{p^j}-x_2^{p^{j-1}}}{1-x_2^{p^{k-1}}}\\
=&\sum_{0<k<j}\left(\frac{x^{p^k}-x^{p^j}}{1-x^{p^k}}+\frac{x^{p^j}-x_2^{p^{k-1}}}{1-x_2^{p^{k-1}}}\right)+\frac{x^{p^j}-x_2^{p^{j-1}}}{1-x_2^{p^{j-1}}}\\
&+(y^{-p^{j-1}}-1)x_2^{p^j}\\
=&\sum_{0<k<j}\frac{(1-x^{p^j})(x^{p^k}-x_2^{p^{k-1}})}{(1-x^{p^k})(1-x_2^{p^{k-1}})}+\frac{x^{p^j}-x_2^{p^{j-1}}}{1-x_2^{p^{j-1}}}\\
&+x^{p^{j+1}}(y^{qp^{j-1}}-y^{p^j}).
\end{aligned}
$$

We also observe that

$$
\begin{aligned}
g(S_j)&=g(A(m+1)/I_2)\frac{x^{p^{j+1}}(1-y^{qp^{j-1}})}{1-y}\cdot\frac{1-x_2^{p^{j-1}}}{1-x_2}\\
&=g_{m+1}(t)\cdot\frac{x^{p^{j+1}}(1-y^{qp^{j-1}})(1-x_2^{p^{j-1}})}{1-x_2}.
\end{aligned}
$$

Summing these three Poincaré series, we obtain

$$
\begin{aligned}
&\frac{g(S_{0,1}\cup S_{0,2}\cup S_j)}{g_{m+1}(t)}\\
&=\frac{g(S_{0,1})+g(S_{0,2})+g(S_j)}{g_{m+1}(t)}\\
&=\sum_{0<k<j}\frac{(1-x^{p^j})(x^{p^k}-x_2^{p^{k-1}})}{(1-x^{p^k})(1-x_2^{p^{k-1}})}+\frac{x^{p^j}-x_2^{p^{j-1}}}{1-x_2^{p^{j-1}}}\\
&\qquad+x^{p^{j+1}}(y^{qp^{j-1}}-y^{p^j})+\frac{x^{p^{j+1}}(1-y^{qp^{j-1}})(1-x_2^{p^{j-1}})}{1-x_2}\\
&=\sum_{0<k<j}\frac{(1-x^{p^j})(x^{p^k}-x_2^{p^{k-1}})}{(1-x^{p^k})(1-x_2^{p^{k-1}})}+\frac{x^{p^j}-x_2^{p^{j-1}}}{1-x_2^{p^{j-1}}}\\
&\qquad+\frac{x^{p^{j+1}}((1-y^{qp^{j-1}})(1-x_2^{p^{j-1}})+(y^{qp^{j-1}}-y^{p^j})(1-x_2))}{1-x_2}\\
&=\sum_{0<k<j}\frac{(1-x^{p^j})(x^{p^k}-x_2^{p^{k-1}})}{(1-x^{p^k})(1-x_2^{p^{k-1}})}+\frac{x^{p^j}-x_2^{p^{j-1}}}{1-x_2^{p^{j-1}}}\\
&\qquad+\frac{x^{p^{j+1}}(1-x_2^{p^{j-1}}+y^{qp^{j-1}}x_2^{p^{j-1}}-y^{p^j}-x_2y^{qp^{j-1}}+x_2y^{p^j})}{1-x_2}\\
&=\sum_{0<k<j}\frac{(1-x^{p^j})(x^{p^k}-x_2^{p^{k-1}})}{(1-x^{p^k})(1-x_2^{p^{k-1}})}+\frac{x^{p^j}-x_2^{p^{j-1}}}{1-x_2^{p^{j-1}}}\\
&\qquad+\frac{x^{p^{j+1}}(1-x_2^{p^{j-1}}-y^{qp^{j-1}}(x_2-x_2^{p^{j-1}})-y^{p^j}(1-x_2))}{1-x_2}.
\end{aligned}
$$

On the other hand, Theorem 2.4 gives

$$
\frac{g(B_{m+1})}{g_{m+1}(t)}\equiv\sum_{0<k\le j+1}\frac{x^{p^k}-x_2^{p^{k-1}}}{(1-x^{p^k})(1-x_2^{p^{k-1}})}
$$

$$\equiv \sum_{0<k<j} \frac{x^{p^k} - x_2^{p^{k-1}}}{(1-x^{p^k})(1-x_2^{p^{k-1}})} + \frac{x^{p^j} - x_2^{p^{j-1}}}{(1-x^{p^j})(1-x_2^{p^{j-1}})} + \frac{x^{p^{j+1}} - x_2^{p^j}}{1-x^{p^{j+1}}}$$

below dimension $|x^{p^{j+1}} x_2^{p^j}|$, so

$$\frac{g(B_{m+1})(1-x^{p^j})}{g_{m+1}(t)} = \sum_{0<k<j} \frac{(x^{p^k} - x_2^{p^{k-1}})(1-x^{p^j})}{(1-x^{p^k})(1-x_2^{p^{k-1}})} + \frac{x^{p^j} - x_2^{p^{j-1}}}{1-x_2^{p^{j-1}}} + \frac{x^{p^{j+1}}(1-y^{p^j})(1-x^{p^j})}{1-x^{p^{j+1}}}.$$

This means

$$\begin{aligned}
&\frac{g(S_{0,1} \cup S_{0,2} \cup S_j) - g(B_{m+1})(1-x^{p^j})}{g_{m+1}(t)} \\
&= \frac{x^{p^{j+1}}(1 - x_2^{p^{j-1}} - y^{qp^{j-1}}(x_2 - x_2^{p^{j-1}}) - y^{p^j}(1-x_2))}{1-x_2} - \frac{x^{p^{j+1}}(1-y^{p^j})(1-x^{p^j})}{1-x^{p^{j+1}}} \\
&\equiv \frac{x^{p^{j+1}}(1 - y^{qp^{j-1}} x_2 - y^{p^j}(1-x_2))}{1-x_2} - \frac{x^{p^{j+1}}(1-y^{p^j} - x_2 + x_2 y^{p^j}}{1-x_2)} \\
&\qquad \text{below dimension } |\widehat{v}_1^{p^j(p+1)}| \\
&= \frac{x^{p^{j+1}} x_2 (1-y^{qp^{j-1}})}{1-x_2}.
\end{aligned}$$

By Lemma 3.5, this means that B_{m+1} is j-free in the range claimed and that the first nontrivial Ext^1 has order p.

(iii) To show that the generator of is Ext^1 the element specified, we first show that the indicated Massey product is defined.

For $j > 1$ and $1 < k < p$ we claim

$$d(\widehat{\beta}_{1+kp^{j-1}/kp^{j-1}}) = \langle \widehat{\beta}_{1+p^{j-1}/p^{j-1}}, \underbrace{\widehat{h}_{1,j}, \ldots, \widehat{h}_{1,j}}_{k-1} \rangle.$$

This can be shown by induction on k and direct calculation as follows. Let

$$s = \widehat{t}_1^p - v_1^{p\omega-1}\widehat{t}_1 \in \overline{T}_m^{(j)} \subset G(m+1).$$

It follows that $w = \widehat{v}_2 - v_1 s$ is invariant. Note that its p^{j-1}th power does not lie in $\overline{T}_m^{(j)}$. Then we have

$$\begin{aligned}
\eta_R\left(\widehat{\beta}_{1+kp^{j-1}/kp^{j-1}}\right) &= \eta_R\left(\frac{\widehat{v}_2^{kp^{j-1}} w}{p v_1^{kp^{j-1}}}\right) \\
&= \sum_{0<\ell\le k} \binom{kp^{j-1}}{\ell p^{j-1}} \frac{\widehat{v}_2^{\ell p^{j-1}} w}{p v_1^{\ell p^{j-1}}} \otimes s^{(k-\ell)p^{j-1}} \\
&= \sum_{0<\ell\le k} \binom{k}{\ell} \frac{\widehat{v}_2^{\ell p^{j-1}} w}{p v_1^{\ell p^{j-1}}} \otimes s^{(k-\ell)p^{j-1}} \\
&= \sum_{0<\ell\le k} \binom{k}{\ell} \widehat{\beta}_{1+\ell p^{j-1}/\ell p^{j-1}} \otimes s^{(k-\ell)p^{j-1}}
\end{aligned}$$

$$= \langle \widehat{\beta}_{1+p^{j-1}/p^{j-1}}, \underbrace{\widehat{h}_{1,j}, \ldots, \widehat{h}_{1,j}}_{k-1} \rangle.$$

This means that our p-fold Massey product is defined.

We claim the first element in Ext^1 is represented by

$$\begin{aligned}
&\sum_{0<\ell<p} \frac{1}{p} \left(\begin{array}{c} p \\ \ell \end{array} \right) \widehat{\beta}_{1+\ell p^{j-1}/\ell p^{j-1}} \otimes s^{(p-\ell)p^{j-1}} \\
&\qquad = \sum_{0<\ell<p} \frac{1}{p} \left(\begin{array}{c} p \\ \ell \end{array} \right) \widehat{\beta}_{1+\ell p^{j-1}/\ell p^{j-1}} \otimes \left(\widehat{t}_1^{p^j} - v_1^{p^{j-1}(p\omega-1)} \widehat{t}_1^{p^{j-1}} \right)^{p-\ell} \\
&\qquad = \sum_{0<\ell<p} \frac{1}{p} \left(\begin{array}{c} p \\ \ell \end{array} \right) \widehat{\beta}_{1+\ell p^{j-1}/\ell p^{j-1}} \otimes \widehat{t}_1^{p^j(p-\ell)} \\
&\qquad = \widehat{\beta}_{1+qp^{j-1}/qp^{j-1}} \otimes \widehat{t}_1^{p^j} + \ldots
\end{aligned}$$

The only element in B_{m+1} in this dimension is $\widehat{\beta}_{1+p^j/p^j}$, which is primitive, so this element in Ext^1 is notrivial. □

7. *Higher Ext groups for $j = 1$*

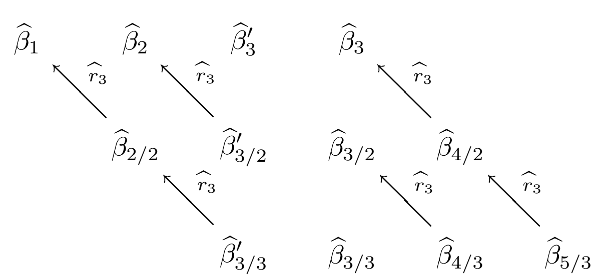
In this section we exhibit some calculations of $\mathrm{Ext}^s(\overline{T}_m^{(j)} \otimes B_{m+1})$ for $s > 0$. Recall the *small descent spectral sequence*, constructed in [**Rav02**, Theorem 1.17], which converges to $\mathrm{Ext}(\overline{T}_m^{(j)} \otimes B_{m+1})$ with

$$E_1^{*,s} = E(\widehat{h}_j) \otimes P(\widehat{b}_j) \otimes \mathrm{Ext}(\overline{T}_m^{(j+1)} \otimes B_{m+1})$$

with $\widehat{h}_j \in E_1^{1,0}$ and $\widehat{b}_j \in E_1^{2,0}$, and $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$. In particular, d_1 is induced by the action of $\widehat{r}_{p^j}$ on B_{m+1} for s even and $\widehat{r}_{qp^j}$ for s odd. The case $m = 0$ has already been treated in [**Rav04**, Chapter 7], so we may assume that $m > 0$. We examine the simplest case, $j = 1$. Recall that B_{m+1} is 2-free below dimension $|\widehat{v}_2^{p^2+1}/v_1^{p^2}|$ and $\mathrm{Ext}^0(\overline{T}_m^{(2)} \otimes B_{m+1})$ is the $A(m+1)$-module generated by

$$\left\{ \widehat{\beta}'_{i/t} \colon 0 < t \leq \min(i,p) \right\} \cup \left\{ \widehat{\beta}_{p^2/t} \colon p < t \leq p^2 \right\} \tag{7.1}$$

by Theorem 6.1. Then the spectral sequence collapses from E_2. We can compute d_1 on elements (7.1) using Proposition B.2: The action of $\widehat{r}_p$ on $\mathrm{Ext}^0(\overline{T}_m^{(2)} \otimes B_{m+1})$ is given by $\widehat{r}_p \left(\widehat{\beta}'_{i/e_1} \right) = \widehat{\beta}_{i-1/e_1-1}$ and $\widehat{r}_p \left(\widehat{\beta}_{pi/e_1} \right) = 0$, and the action of $\widehat{r}_{qp}$ is obtained by composing $\widehat{r}_p$ up to unit scalar. In order to understand the behavior of d_1, the following picture for $p = 3$ may be helpful.



(7.2)

Here each arrow represents the action of the Quillen operation $\widehat{r}_3$ up to unit scalar. For a general prime p, the analogous picture would show a directed graph with $2p$ components, two of which have p vertices, and in which the arrow shows the action of the Quillen operation $\widehat{r}_p$ up to unit scalar. Each component corresponds to an $A(m+1)$-summand of the E_2-term, with the caveat that $p\widehat{\beta}'_{p/e_1} = \widehat{\beta}_{p/e_1}$ and $v_1\widehat{\beta}'_{i/e} = \widehat{\beta}'_{i/e-1}$. Notice that the entire configuration is $\widehat{v}_2^p$-periodic. Corresponding to the diagonal containing $\widehat{\beta}_1$ in (7.2), the subgroup of E_1 generated by

$$\left\{\widehat{\beta}_1,\, \widehat{\beta}_{2/2},\, \widehat{\beta}'_{3/3}\right\} \otimes E(\widehat{h}_{1,1}) \otimes P(\widehat{b}_{1,1})$$

reduces on passage to E_2 to simply $\{\widehat{\beta}_1\}$. Similarly, the subset

$$\left\{\widehat{\beta}_2,\, \widehat{\beta}'_{3/2}\right\} \otimes E(\widehat{h}_{1,1}) \otimes P(\widehat{b}_{1,1})$$

reduces to $\left\{\widehat{\beta}_2,\, \widehat{\beta}'_{3/2}\widehat{h}_{1,1}\right\} \otimes P(\widehat{b}_{1,1})$, where

$$\begin{aligned} \widehat{\beta}'_{3/2}\widehat{h}_{1,1} &= \langle \widehat{h}_{1,1},\, \widehat{h}_{1,1},\, \widehat{\beta}_2\rangle \\ \text{and} \qquad \widehat{h}_{1,1}(\widehat{\beta}'_{3/2}\widehat{h}_{1,1}) &= \widehat{h}_{1,1}\langle \widehat{h}_{1,1},\, \widehat{h}_{1,1},\, \widehat{\beta}_2\rangle \;=\; \langle \widehat{h}_{1,1},\, \widehat{h}_{1,1},\, \widehat{h}_{1,1}\rangle\widehat{\beta}_2 \;=\; \widehat{b}_{1,1}\widehat{\beta}_2. \end{aligned}$$

These observations give us the following result.

Proposition 7.3. **Structure of** $\mathrm{Ext}(\overline{T}_m^{(1)} \otimes B_{m+1})$**.** *In dimensions less than* $|\widehat{v}_2^{p^2+1}/v_1^{p^2}|$, $\mathrm{Ext}(\overline{T}_m^{(1)} \otimes B_{m+1})$ *is a free module over* $A(m+1)/I_2$ *with basis*

$$\left\{\widehat{\beta}_{1+pi}, \widehat{\beta}_{p+pi};\, \widehat{\beta}_{p^2/k}\right\} \oplus P(\widehat{b}_{1,1}) \otimes \begin{cases} \left\{\widehat{\beta}'_{pi+s};\, \widehat{\beta}_{pi+p/s};\, \widehat{\beta}_{p^2/\ell}\right\} \\ \oplus \\ \widehat{h}_{1,1}\left\{\widehat{\beta}'_{pi+p/t};\, \widehat{\beta}_{pi+r/p};\, \widehat{\beta}_{p^2/\ell}\right\}, \end{cases}$$

where $0 \le i < p$, $1 \le k \le p^2-p+1$, $p^2-p+2 \le \ell \le p^2$, $2 \le s \le p$, $1 \le t \le p-1$ *and* $p \le u \le 2p-2$, *subject to the caveat that* $v_1\widehat{\beta}_{p/e} = \widehat{\beta}_{p/e-1}$ *and* $p\widehat{\beta}'_{p/e} = \widehat{\beta}_{p/e}$. *In particular* $\mathrm{Ext}^0(\overline{T}_m^{(1)} \otimes B_{m+1})$ *has basis*

$$\left\{\widehat{\beta}'_{1+pi}, \ldots, \widehat{\beta}'_{p+pi};\, \widehat{\beta}_{p+pi/p}, \ldots, \widehat{\beta}_{p+pi/1};\, \widehat{\beta}_{p^2/p^2}, \ldots, \beta_{p^2/1}\right\}.$$

Note that for $m > 0$, this range of dimensions exceeds $p|\widehat{v}_3|$.

Appendix A. *Some results on binomial coefficients*

Fix a prime number p.

Definition A.1. $\alpha(n)$**, the sum of the** p**-adic digits of** n**.** *For a nonnegative integer* n, $\alpha(n)$ *denotes sum of the digits in the* p*-adic expansion of* n, *i.e., for* $n = \sum_{i\ge 0} a_i p^i$ *with* $0 \le a_i \le p-1$, *we define* $\alpha(n) = \sum_{i\ge 0} a_i$.

As before, let $\nu_p(n)$ denote the p-adic valuation of n, i.e., the exponent that makes n a p-local unit multiple of $p^{\nu_p(n)}$. When the integer is a binomial coefficient $\binom{i}{j}$, we will write $\nu_p\binom{i}{j}$ instead of $\nu_p\left(\binom{i}{j}\right)$. Then we have

Lemma A.2. **p-adic valuation of a binomial coefficient.**

$$q\nu_p\begin{pmatrix} n \\ k \end{pmatrix} = \alpha(k) + \alpha(n-k) - \alpha(n)$$

where $q = p-1$. In particular,

$$q\nu_p\begin{pmatrix} n \\ p^j \end{pmatrix} = 1 + \alpha(n-p^j) - \alpha(n).$$

Proof. Recall that $q\nu_p(n!) = n - \alpha(n)$, and observe that

$$\begin{aligned} q\nu_p\begin{pmatrix} n \\ k \end{pmatrix} &= q\nu_p\left(\frac{n!}{(n-k)!k!}\right) \\ &= q\left(\nu_p(n!) - \nu_p((n-k)!) - \nu_p(k!)\right) \\ &= n - \alpha(n) - (n-k) + \alpha(n-k) - k + \alpha(k) \\ &= -\alpha(n) + \alpha(n-k) + \alpha(k) \end{aligned}$$

□

Using this lemma we can determine the number how many times a binomial coefficient is divisible by a prime p. For example, we have

Lemma A.3. **Divisibility of a binomial coefficient.** *Assume that $p \nmid a$ and $0 < n \leq \ell$. Then the binomial coefficient $\begin{pmatrix} ap^\ell + n - 1 \\ n \end{pmatrix}$ is divisible by $p^{\ell+1-n}$.*

Proof. Since $a \not\equiv 0 \bmod (p)$, we have $\alpha(a-1) = \alpha(a) - 1$. Let $m = \nu_p(n)$ and $n = n'p^m$. Then $\alpha(n'-1) = \alpha(n') - 1$, and we have

$$\begin{aligned} q\nu_p\begin{pmatrix} ap^\ell + n - 1 \\ n \end{pmatrix} &= q\nu_p\begin{pmatrix} ap^\ell + n'p^m - 1 \\ n'p^m \end{pmatrix} \\ &= \alpha(n'p^m) + \alpha(ap^\ell - 1) - \alpha(ap^\ell + n'p^m - 1) \\ &= \alpha(n') + \alpha(a-1) + q\ell - \alpha(ap^{\ell-m} + n' - 1) - qm \\ &= \alpha(n') + \alpha(a-1) + q\ell - \alpha(a) - \alpha(n'-1) - qm \\ &= q(\ell - m) \geq q(\ell + 1 - n). \end{aligned}$$

□

We consider this type of binomial coefficients in Proposition 4.4. The other types we need are the followings:

Lemma A.4. **Divisibility of another binomial coefficient.** *Assume that p is a prime and that a positive integer s is expressed as $s = s_1p^\ell + s_0 > 0$ with $0 \leq s_0 < p^\ell$. Then we have $\nu_p\begin{pmatrix} s \\ p^\ell \end{pmatrix} = \nu_p(s_1)$. In particular, we have $p^n \mid \begin{pmatrix} s \\ p^\ell \end{pmatrix}$ iff $s \equiv 0, 1, \ldots, p^\ell - 1 \bmod (p^{n+\ell})$.*

Proof. Observe that

$$q\nu_p\begin{pmatrix} s \\ p^\ell \end{pmatrix} = \alpha(p^\ell) + \alpha(s - p^\ell) - \alpha(s)$$

$$\begin{aligned} &= 1 + \alpha((s_1 - 1)p^\ell + s_0) - \alpha(s_1 p^\ell + s_0) \\ &= \alpha(1) + \alpha(s_1 - 1) - \alpha(s_1) \\ &= q\nu_p(s_1). \end{aligned}$$

This implies that $\nu_p \begin{pmatrix} s \\ p^\ell \end{pmatrix} = n$ iff $s \equiv s_0 \bmod (p^{n+\ell})$. □

In Appendix B it is required to know how many times the binomial coefficient $\begin{pmatrix} i-1 \\ p^{j-1}-1 \end{pmatrix}$ is divisible by p.

For $0 < i < p^{j-1}$ it is clear that $\begin{pmatrix} i-1 \\ p^{j-1}-1 \end{pmatrix} = 0$. For $i \geq p^{j-1}$, the number $\nu_p \begin{pmatrix} i-1 \\ p^{j-1}-1 \end{pmatrix}$ can be determined explicitly in the following results.

Proposition A.5. **A third divisibility statement.** *For $i \geq p^{j-1}$, define non-negative integers i_0 and i_1 by*

$$i = i_1 p^{j-1} + i_0 \quad (\ i_1 > 0 \text{ and } 0 \leq i_0 < p^{j-1}\ ). \tag{A.6}$$

Then we have

(i) $\begin{pmatrix} i-1 \\ p^{j-1}-1 \end{pmatrix}$ *is divisible by p iff $i_0 \neq 0$;*

(ii) *More generally,* $\begin{pmatrix} i-1 \\ p^{j-1}-1 \end{pmatrix}$ *is divisible by p^{j-k} $(0 \leq k < j)$ iff*

$$\nu_p(i_0) \leq k - 1 + \nu_p(i_1). \tag{A.7}$$

or equivalently $i_0 \neq 0$ and $p^{k+\nu_p(i_1)} \nmid i_0$.

In particular, the inequality (A.7) is automatically satisfied if $\nu_p(i_1) \geq j-k-1$.

Proof. Observe that

$$\begin{aligned} \nu_p \begin{pmatrix} i-1 \\ p^{j-1}-1 \end{pmatrix} &= \nu_p(p^{j-1}) + \nu_p \begin{pmatrix} i \\ p^{j-1} \end{pmatrix} - \nu_p(i) \\ &= (j-1) + \nu_p(i_1) - \begin{cases} (j-1+\nu_p(i_1)) & \text{if } i_0 = 0 \\ \nu_p(i_0) & \text{if } i_0 \neq 0 \end{cases} \quad \text{by Lemma A.4} \\ &= \begin{cases} 0 & \text{if } i_0 = 0 \\ j-1+\nu_p(i_1) - \nu_p(i_0) & \text{if } i_0 \neq 0 \end{cases}. \end{aligned}$$

If $i_0 \neq 0$, then we have $j - 1 + \nu_p(i_1) - \nu_p(i_0) > 0$ since $\nu_p(i_0) \leq j-2$, and so the binomial coefficient is divisible by p. Since $i_0 = 0$ is equivalent to $p^{j-1} \mid i$, the statement ((i)) follows.

The condition $p^{j-k} \mid \begin{pmatrix} i-1 \\ p^{j-1}-1 \end{pmatrix}$ is equivalent to the inequality $\nu_p \begin{pmatrix} i-1 \\ p^{j-1}-1 \end{pmatrix} \geq j-k$, and if we suppose that $j-k > 0$ then this inequality gives (A.7).

Note that (A.7) is always satisfied if $\nu_p(i_1) \geq j-k-1$ since $\nu_p(i_0) \leq j-2$ by definition. □

The following is the obvious translation of Proposition A.5.

Corollary A.8. **A fourth divisibility statement.** *Let i_0 and i_1 be as in (A.6) and assume that $p^{j-1} < i \le p^{j-1+m}$. Then, we have $p^{j-k} \mid \binom{i-1}{p^{j-1}-1}$ for $0 \le k < j$ iff*

$$\nu_p(i_0) \le k-1+\nu_p(i_1) \quad \text{with } 0 \le \nu_p(i_1) \le m.$$

Proof. The given range $p^{j-1} < i \le p^{j-1+m}$ means that $0 \le \nu_p(i_1) \le m$ and the result follows from Proposition A.5. □

Appendix B. *Quillen operations on β-elements*

In this section we discuss the action of the Quillen operations $\widehat{r}_{p^j}$ for $j > 0$ on the β-elements.

First we consider the following easy cases.

Proposition B.1. **Primitive β-elements.** *For $i > 0$, the elements $\widehat{\beta}_{i/t}$ are primitive if $0 < t \le p^{\nu_p(i)}$, i.e., it satisfies $\widehat{r}_\ell(\widehat{\beta}_{i/t}) = 0$ for all $\ell \ge 0$.*

Proof. Set $\nu_p(i) = n$ and $i = i'p^n$. By direct calculations we have

$$\eta_R\left(\frac{\widehat{v}_2^i}{pv_1^t}\right) = \frac{(\widehat{v}_2^{p^n} + v_1^{p^n}\widehat{t}_1^{p^{n+1}} - v_1^{p^{n+1}\omega}\widehat{t}_1^{p^n})^{i'}}{pv_1^t} = \frac{\widehat{v}_2^i}{pv_1^t}.$$

□

For the other cases, the Quillen operation $\widehat{r}_{p^j}$ is computed as follows:

Proposition B.2. **Quillen operations on β-elements.** *When $j > 0$, we have*

$$\widehat{r}_{p^j}(\widehat{\beta}'_{i/t}) = \binom{i-1}{p^{j-1}}\widehat{\beta}'_{i-p^{j-1}/t-p^{j-1}} \quad \text{for } t < p^{j-1} + p^{m+2}.$$

Proof. First assume that $m > 0$. Observe that

$$\begin{aligned}
\eta_R(\widehat{\beta}'_{i/t}) = \eta_R\left(\frac{\widehat{v}_2^i}{ipv_1^t}\right) &= \frac{\left(\widehat{v}_2 + v_1\widehat{t}_1^p - v_1^{p\omega}\widehat{t}_1\right)^i}{ipv_1^t} \\
&= \sum_{0\le k\le \ell\le i} (-1)^k \binom{i}{\ell}\binom{\ell}{k} \frac{\widehat{v}_2^{i-\ell}\left(v_1\widehat{t}_1^p\right)^{\ell-k}\left(v_1^{p\omega}\widehat{t}_1\right)^k}{ipv_1^t} \\
&= \sum_{0\le k\le \ell\le i} (-1)^k \binom{i-1}{\ell}\binom{\ell}{k} \frac{\widehat{v}_2^{i-\ell}\widehat{t}_1^{p(\ell-k)+k}}{(i-\ell)pv_1^{t-\ell+k-p\omega k}}.
\end{aligned}$$

Since $\widehat{r}_{p^j}(\widehat{\beta}'_{i/t})$ is the coefficient of $\widehat{t}_1^{p^j}$ in the above, we need to consider the terms satisfying $p(\ell-k)+k = p^j$. Note that k must be divisible by p and that we may set $k = pn$. Thus we have

$$p^j = p(\ell - pn) + pn.$$

Now let

$$\ell(n) = \ell \ = \ p^{j-1} + qn \qquad \text{where } q = p-1$$

$$\begin{aligned} \text{and} \qquad g(n) &= t - \ell + k - p\omega k \\ &= t - p^{j-1} - qn + pn - p^{m+2}n \\ &= t - p^{j-1} - n(p^{m+2} - 1). \end{aligned}$$

Then we have

$$\widehat{r}_{p^j}(\widehat{\beta}'_{i/t}) = \sum_{0 \le n \le p^{j-1}} (-1)^{pn} \binom{i-1}{\ell(n)} \binom{\ell(n)}{np} \frac{\widehat{v}_2^{i-\ell(n)}}{(i-\ell(n))pv_1^{g(n)}}.$$

Given our assumption about t, the only value of n satisfying $g(n) > 0$ is $n = 0$, which gives

$$\widehat{r}_{p^j}(\widehat{\beta}'_{i/t}) = \binom{i-1}{p^{j-1}} \frac{\widehat{v}_2^{i-p^{j-1}}}{(i-p^{j-1})pv_1^{t-p^{j-1}}}.$$

The proof for $m = 0$ is more complicated. Observe that

$$\psi(\beta'_{i/t}) = \sum_{0 \le k \le \ell \le i} \sum_{r \ge 0} (-1)^{k+r} \binom{i-1}{\ell} \binom{\ell}{k} \binom{t+r-1}{r} p^r \frac{v_2^{i-\ell} t_1^{p(\ell-k)+k+r}}{(i-\ell)pv_1^{t+r-\ell+k-pk}},$$

which shows that $\widehat{r}_{p^j}(\beta'_{i/t})$ is equal to

$$\sum_{0 \le n \le p^{j-1}} \sum_{0 \le r \le np} (-1)^{np} \binom{i-1}{\ell(n,r)-1} \binom{\ell(n,r)-1}{np-r-1} \binom{t+r-1}{r} \frac{p^r v_2^{i-\ell(n,r)}}{(np-r)pv_1^{g(n,r)}},$$

where $\ell(n,r) = p^{j-1} + nq - r$ and $g(n,r) = t - p^{j-1} - n(p^2-1) + r(p+1)$. If $p^r \mid (np-r)$ for a positive r, then we may put $r = sp$ and $n \ge p^{sp-1} + s$ for a positive s and the exponent of v_1 is not positive since

$$\begin{aligned} g(n,r) &\le t - p^{j-1} - (p^{sp-1} + s)(p^2-1) + sp(p+1) \\ &= t - p^{j-1} - (p+1)(p^{sp} - p^{sp-1} - s) \\ &\le t - p^{j-1} - (p+1)(p^p - p^{p-1} - 1) \\ &\le t - p^{j-1} - (p^2-1). \end{aligned}$$

Thus, the nontrivial term arises only when $r = 0$. We can see that it is also required that $n = 0$ by the same reason as the $m > 0$ case, and the result follows. □

To know the condition of triviality of $\widehat{r}_{p^j}$ in Proposition B.2, we need the results on the p-adic valuation of binomial coefficients obtained in Appendix A. In particular, we have

PROPOSITION B.3. **Some trivial actions of Quillen operations.** *Assume that $p^{j-1} < i \le p^{j+1}$ and $t < p^{j-1} + p^{m+2}$. Then we have the following trivial Quillen operations:*

(i) $\widehat{r}_{p^\ell}(\widehat{\beta}'_{i/t})$ $(\ell \ge j)$ *for* $0 < t \le \min(i, p^{j-1})$;

(ii) $\widehat{r}_{p^\ell}(\widehat{\beta}_{ap^j+b/t})$ $(\ell \ge j)$ *for* $p^{j-1} < t \le p^j$ *and* $0 \le b < p^{j-1}$.

Proof. We will show the following Quillen operations on $p^k\widehat{\beta}'_{i/t}$ are trivial:

a $\widehat{r}_{p^\ell}$ $(\ell \ge j)$ for $0 < t \le \min(i, p^{j-1})$ and $k \ge 0$;

b $\widehat{r}_{p^\ell}$ $(\ell \ge j)$ for $p^{j-1} < t \le p^j$, $i = ap^j + bp^k$ with $p \nmid a$, $p \nmid b$ and $0 \le k < j-1$;

c $\widehat{r}_{p^\ell}$ $(\ell \ge 0)$ for $p^{j-1} < t \le p^j$, $i = ap^j$ with $0 < a \le p$ and $j = k$.

For the case ((i)), note that

$$\widehat{r}_{p^j}(p^k \widehat{\beta}'_{i/t}) = \begin{pmatrix} i-1 \\ p^{j-1}-1 \end{pmatrix} \frac{\widehat{v}_2^{i-p^{j-1}}}{p^{j-k} v_1^{t-p^{j-1}}}.$$

by Proposition B.2, which is clearly trivial when $0 < t \leq p^{j-1} (\leq p^{\ell-1})$. Even if $p^{j-1} < t \leq i$, it is trivial when the binomial coefficient $\begin{pmatrix} i-1 \\ p^{j-1}-1 \end{pmatrix}$ is divisible by p^{j-k}, or equivalently when the inequality (A.7) holds.

When $0 < k < j$, by the assumption we have

$$p^{j-1} < i_1 p^{j-1} + i_0 \leq p^{j+1}$$

(where $\nu_p(i_0) < j-1$ by definition) and $\nu_p(i_1) \leq 2$. Note that if $k > 0$ and $p^k \not| \, i$ then $p^k \widehat{\beta}'_{i/t}$ itself is trivial and that we may assume that $\nu_p(i) \geq k$. These observations suggest that the only case satisfying the inequality (A.7) is $(\nu_p(i_1), \nu_p(i_0)) = (1, k)$, which gives the case (b).

When $j = k$, the Quillen operation $\widehat{r}_{p^j}(p^j \widehat{\beta}'_{i/t})$ is clearly trivial and $p^j \widehat{\beta}'_{i/t}$ is nontrivial only if $p^j \mid i$, which gives the case (c).

For the case (b) and (c), observe that the Quillen operation $\widehat{r}_{p^{j+1}}(p^k \widehat{\beta}'_{i/t})$ is a unit scalar multiple of $\widehat{\beta}_{i-p^j/t-p^j}$ and $p^k \widehat{\beta}'_{i/t}$ is not in $L_j(B_{m+1})$, which means that the condition $t \leq p^j$ is required. Conbining (b) and (c) gives the case ((ii)).

Note that no linear combination of β-elements can be killed by $\widehat{r}_{p^j}$ since the $\widehat{r}_{p^j}$-image has different exponents of $\widehat{v}_2$ or v_1 if $\widehat{\beta}'_{i_1/t_1} \neq \widehat{\beta}'_{i_2/t_2}$. □

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