CHAPTER 4

BP-Theory and the Adams–Novikov Spectral Sequence

In this chapter we turn to the main topic of this book, the Adams–Novikov spectral sequence. In Section 1 we develop the basic properties of MU and the Brown–Peterson spectrum BP, using the calculation of $\pi_*(MU)$ (3.1.5) and the algebraic theory of formal group laws as given in Appendix 2. The main result is 4.1.19, which describes $BP_*(BP)$, the BP-theoretic analog of the dual Steenrod algebra.

Section 2 is a survey of other aspects of BP-theory not directly related to this book.

In Section 3 we study $BP_*(BP)$ more closely and obtain some formulas, notably 4.3.13, 4.3.18, 4.3.22, and 4.3.33, which will be useful in subsequent calculations.

In Section 4 we set up the Adams–Novikov spectral sequence and use it to compute the stable homotopy groups of spheres through a middling range of dimensions, namely ≤ 24 for $p = 2$ and $\leq 2p^3 - 2p - 1$ for $p > 2$.

1. Quillen's Theorem and the Structure of $BP_*(BP)$

In this section we will construct the Brown–Peterson spectrum BP and determine the structure of its Hopf algebroid of cooperations, $BP_*(BP)$, i.e., the analog of the dual Steenrod algebra. This will enable us to begin computing with the Adams–Novikov spectral sequence (ANSS) in Section 4. The main results are Quillen's theorem 4.1.6, which identifies $\pi_*(MU)$ with the Lazard ring L (A2.1.8); the Landweber–Novikov theorem 4.1.11, which describes $MU_*(MU)$; the Brown–Peterson theorem 4.1.12, which gives the spectrum BP; and the Quillen–Adams theorem 4.1.19, which describes $BP_*(BP)$.

We begin by informally defining the spectrum MU. For more details see Milnor and Stasheff [**?**]. Recall that for each $n \geq 0$ the group of complex unitary $n \times n$ matrices $U(n)$ has a classifying space $BU(n)$. It has a complex n-plane bundle γ_n over it which is universal in the sense that any such bundle ξ over a paracompact space X is the pullback of γ_n, induced by a map $f\colon X \to BU(n)$. Isomorphism classes of such bundles ξ are in one-to-one correspondence with homotopy classes of maps from X to $BU(n)$. Any $\mathbf{C}^n$-bundle ξ has an associated disc bundle $D(\xi)$ and sphere bundle $S(\xi)$. The *Thom space* $T(\xi)$ is the quotient $D(\xi)/S(\xi)$. Alternatively, for compact X, $T(\xi)$ is the one-point compactification of the total space of ξ.

$MU(n)$ is $T(\gamma_n)$, the Thom space of the universal n-plane bundle γ_n over $BU(n)$. The inclusion $U(n) \to U(n+1)$ induces a map $BU(n) \to BU(n+1)$. The pullback of γ_{n+1} under this map has Thom space $\Sigma^2 MU(n)$. Thom spaces are functorial so we have a map $\Sigma^2 MU(n) \to MU(n+1)$. Together these maps give the spectrum MU.

It follows from the celebrated theorem of Thom [?] that $\pi_*(MU)$ is isomorphic to the complex cobordisrn ring (see Milnor [?]) which is defined as follows. A *stably complex manifold* is one with a complex structure on its stable normal bundle. (This notion of a complex manifold is weaker than others, e.g., algebraic, analytic, and almost complex.) All such manifolds are oriented. Two closed stably complex manifolds M_1 and M_2 are *cobordant* if there is a stably complex manifold W whose boundary is the disjoint union of M_1 (with the opposite of the given orientation) and M_2. Cobordism, i.e., being cobordant, is an equivalence relation and the set of equivalence classes forms a ring (the complex cobordism ring) under disjoint union and Cartesian product. Milnor and Novikov's calculation of $\pi_*(MU)$ (3.1.5) implies that two such manifolds are cobordant if they have the same Chern numbers. For the definition of these and other details of the above we refer the reader to Milnor and Stasheff [?] or Stong [?].

This connection between MU and complex manifolds is, however, not relevant to most of the applications we will discuss, nor is the connection between MU and complex vector bundles. On the other hand, the connection with formal group laws (A2.1.1) discovered by Quillen [?] (see 4.1.6) is essential to all that follows. This leads one to suspect that there is some unknown formal group theoretic construction of MU or its associated infinite loop space. For example, many well-known infinite loop spaces have been constructed as classifying spaces of certain types of categories (see Adams [?], section 2.6), but to our knowledge no such description exists for MU. This infinite loop space has been studied in Ravenel and Wilson [?].

In order to construct BP and compute $BP_*(BP)$ we need first to analyze MU. Our starting points are 3.1.4, which describes its homology, and the Milnor–Novikov theorem 3.1.5, which describes its homotopy and the behavior of the Hurewicz map. The relevant algebraic information is provided by A2.1, which describes universal formal group laws and related concepts and which should be read before this section. The results of this section are also derived in Adams [?].

Before we can state Quillen's theorem (4.1.6), which establishes the connection between formal group laws and complex cobordism, we need some preliminary discussion.

4.1.1. Definition. *Let E be an associative commutative ring spectrum. A* complex orientation *for E is a class $x_E \in \widetilde{E}^2(\mathbf{C}P^\infty)$ whose restriction to*

$$\widetilde{E}(\mathbf{C}P^1) \simeq \widetilde{E}^2(S^2) \cong \pi_0(E)$$

is 1, where $\mathbf{C}P^n$ denotes n-dimensional complex projective space. $\square$

This definition is more restrictive than that given in Adams [?] (2.1), but it is adequate for our purposes.

Of course, not all ring spectra (e.g., *bo*) are orientable in this sense. Two relevant examples of oriented spectra are the following.

4.1.2. Example. Let $E = H$, the integral Eilenberg–Mac Lane spectrum. Then the usual generator of $H^2(\mathbf{C}P^\infty)$ is a complex orientation x_H.

4.1.3. Example. Let $E = MU$. Recall that MU is built up out of Thom spaces $MU(n)$ of complex vector bundles over $BU(n)$ and that the map $BU(n) \to MU(n)$ is an equivalence when $n = 1$. The composition

$$\mathbf{C}P^\infty = BU(1) \xrightarrow{\simeq} MU(1) \to MU$$

gives a complex orientation $x_{MU} \in MU^2(\mathbf{C}P^\infty)$. Alternatively, x_{MU} could be defined to be the first Conner–Floyd Chern class of the canonical complex line bundle over $\mathbf{C}P^\infty$ (see Conner and Floyd [**?**]).

4.1.4. LEMMA. *Let E be a complex oriented ring spectrum.*
(a) $E^*(\mathbf{C}P^\infty) = E^*(\mathrm{pt})[[x_E]]$.
(b) $E^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) = E^*(\mathrm{pt})[[x_E \otimes 1, 1 \otimes x_E]]$.
(c) *Let $t\colon \mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$ be the H-space structure map, i.e., the map corresponding to the tensor product of complex line bundles, and let $F_E(x,y) \in E^*(\mathrm{pt})[[x,y]]$ be defined by $t^*(x_E) = F_E(x_E \otimes 1, 1 \otimes_{x_E})$. Then F_E is a formal group law* (A2.1.1) *over $E^*(\mathrm{pt})$.*

PROOF. For (c), the relevant properties of F_E follow from the fact that $\mathbf{C}P^\infty$ is an associative, commutative H-space with unit.

For (a) and (b) one has the Atiyah–Hirzebruch spectral sequence (AHSS) $H^*(X; E^*(\mathrm{pt})) \Rightarrow E^*(X)$ (see section 7 of Adams [**?**]). For $X = \mathbf{C}P^\infty$ the class x_E represents a unit multiple of $x_H \in H^2(\mathbf{C}P^\infty)$. Hence x_H and all of its powers are permanent cycles so the spectral sequence collapses and (a) follows. The argument for (b) is similar. □

Hence a complex orientation x_E leads to a formal group law F_E over $E^*(\mathrm{pt.})$. Lazard's theorem A2.1.8 asserts that F_E is induced by a homomorphism $\theta_E\colon L \to E^*(\mathrm{pt.})$, where L is a certain ring over which a universal formal group law is defined. Recall that $L = \mathbf{Z}[x_1, x_2, \dots]$, where x_i has degree $2i$. There is a power series over $L \otimes \mathbf{Q}$

$$\log(x) = \sum_{i \geq 0} m_i x^{i+1}$$

where $m_0 = 1$ such that

$$L \otimes \mathbf{Q} = \mathbf{Q}[m_1, m_2 \dots]$$

and

$$\log(F(x,y)) = \log(x) + \log(y)$$

This formula determines the formal group law $F(x,y)$.

The following geometric description of θ_{MU}, while interesting, is not relevant to our purposes, so we refer the reader to Adams [**?**, Theorem 9.2] for a proof.

4.1.5. THEOREM (Mischenko [**?**]). *The element $(n+1)\theta_{MU}(m_n) \in \pi_*(MU)$ is represented by the complex manifold $\mathbf{C}P^n$.* □

4.1.6. THEOREM (Quillen [**?**]). *θ_{MU} is an isomorphism.* □

We will prove this with the help of the diagram

$$\begin{tikzcd} L \arrow[r, "j"] \arrow[d, "\theta_{MU}"'] & M \arrow[d, "\phi"] \\ \pi_*(MU) \arrow[r, "h"] & H_*(MU) \end{tikzcd}$$

where $M = \mathbf{Z}[m_1, m_2 \dots]$ is defined in A2.1.9(b) and contains L. The map ϕ will be constructed below. Recall [A2.1.10(b)] that modulo decomposables in M, the image of j is generated by

$$\begin{cases} pm_i & \text{if } i = p^k - 1 \text{ for some prime } p, \\ m_i & \text{otherwise.} \end{cases}$$

Recall also that $H_*(MU) = \mathbf{Z}[b_1, b_2, \dots]$ [3.1.4(a)] and that modulo decomposables in $H_*(MU)$, the image of h is generated by

$$\begin{cases} -pb_i & \text{if } i = p^k - 1 \text{ for some prime } p, \\ -b_i & \text{otherwise} \end{cases}$$

Hence it suffices to construct ϕ and show that it is an isomorphism.

Before doing this we need two lemmas.

First we must compute $E_*(MU)$. It follows easily from 4.1.4(a) that $E_*(\mathbf{C}P^\infty)$ is a free $\pi_*(E)$ module on elements β_i^E dual to x_E^i. We have a stable map $\mathbf{C}p^\infty \to \Sigma^2 MU$ and we denote by b_i^E the image of β_{i+1}^E.

4.1.7. Lemma. *If E is a complex oriented ring spectrum then*

$$E_*(MU) = \pi_*(E)[b_1^E, b_2^E \dots].$$

Proof. We use the Atiyah–Hirzebruch spectral sequence $H_*(MU, \pi_*(E)) \to E_*(MU)$. The b_i^E represent unit multiples of $b_i \in H_{2i}(MU)$ [3.1.4(a)], so the b_i are permanent cycles and the Atiyah–Hirzebruch spectral sequence collapses. $\square$

If E is complex oriented so is $E \wedge MU$. The orientations x_E and x_{MU} both map to orientations for $E \wedge MU$ which we denote by $\hat{x}_E$ and $\hat{x}_{MU}$, respectively. We also know by 4.1.7 that

$$\pi_*(E \wedge MU) = E_*(MU) = \pi_*(E)[b_i^E]$$

4.1.8. Lemma. *Let E be a complex oriented ring spectrum. Then in $(E \wedge MU)^2(\mathbf{C}P^\infty)$.*

$$\hat{x}_{MU} = \sum_{i \geq 0} b_i^E \hat{x}_E^{i+1},$$

where $b_0 = 1$. This power series will be denoted by $g_E(\hat{x}_E)$.

Proof. We will show by induction on n that after restricting to $\mathbf{C}P^n$ we get

$$\hat{x}_{MU} = \sum_{0 \leq i < n} b_i^E \hat{x}_E^{i+1}.$$

For $n = 1$ this is clear since x_E and x_{MU} restrict to the canonical generators of $\widetilde{E}^*(\mathbf{C}P^1)$ and $MU^*(\mathbf{C}P^1)$. Now notice that x_E^n is the composite

$$\mathbf{C}P^n \to S^{2n} \to \Sigma^{2n} E$$

where the first map is collapsing to the top cell and the second map is the unit. Also b_{n-1}^E is by definition the composite

$$S^{2n} \xrightarrow{\beta_n^E} \mathbf{C}P^n \wedge E \xrightarrow{x_{MU} \wedge E} \Sigma^2 MU \wedge E.$$

Hence we have a diagram

$$\begin{tikzcd}
\mathbf{C}P^{n-1} \arrow[r] \arrow[d] & \mathbf{C}P^{n-1} \wedge E \arrow[dr, "x_{MU}\wedge E"] & \\
\mathbf{C}P^{n} \arrow[r] \arrow[ddr, "x_E^n"'] \arrow[dd] & \mathbf{C}P^{n} \wedge E \arrow[r, "x_{MU}\wedge E"] \arrow[u, "g"'] & \Sigma^2 MU \wedge E \\
 & \mathbf{C}P^{n} \wedge E \wedge E \arrow[r, "x_{MU}\wedge E\wedge E"] \arrow[u, "\mathbf{C}P^n\wedge m"'] & \Sigma^2 MU \wedge E \wedge E \arrow[u, "\Sigma^2 MU\wedge m"'] \\
S^{2n} \arrow[r] & S^{2n} \wedge E \arrow[u, "\beta_n^E\wedge E"'] \arrow[ur, "b_{n-1}^E\wedge E"'] &
\end{tikzcd}$$

where $m\colon E \wedge E \to E$ is the multiplication and g is the cofiber projection of $(\mathbf{C}P^n \wedge m)\,(\beta_n^E \wedge E)$. is now split as $(\mathbf{C}P^{n-1} \wedge E) \vee (S^{2n} \wedge E)$ and $x_{MU} \wedge E\colon \mathbf{C}P \wedge E \to \Sigma^2 MU \wedge E$ is the sum of $(x_{MU} \wedge E)g$ and the map from $S^{2n} \wedge E$. Since x_{MU} is the composition

$$\mathbf{C}P^n \to \mathbf{C}P^n \wedge E \xrightarrow{X_{MU}\wedge E} \Sigma^2 MU \wedge E$$

and the lower composite map from $\mathbf{C}P^n$ to $\Sigma^2 MU \wedge E$ is $b_{n-1}^E x_E^n$, the inductive step and the result follow. □

4.1.9. Corollary. *In* $\pi_*(E \wedge MU)[[x, y]]$,

$$F_{MU}(x, y) = g_E(F_E(g_E^{-1}(x), g_E^{-1}(y))).$$

Proof. In $(E \wedge MU)^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty)$,

$$\begin{aligned}
F_{MU}(\hat{x}_{MU} \otimes 1, 1 \otimes \hat{x}_{MU}) &= t^*(x_{MU}) \\
&= g_E(t^*(\hat{x}_E)) \\
&= g_E(F_E(\hat{x}_E \otimes 1, 1 \otimes \hat{x}_E)) \\
&= g_E(F_E(g_E^{-1}(x_{MU}) \otimes 1, 1 \otimes g_E^{-1}(\hat{x}_{MU}))).
\end{aligned}$$

□

Now we are ready to prove 4.1.6. The map ϕ in 4.1.6 exists if the logarithm of the formal group law defined over $H_*(MU)$ by $h\theta_{MU}$ is integral, i.e., if the formal group law is isomorphic to the additive one. For $E = H$, $F_E(x, y) = x + y$, so the formal group law over $H_*(MU) = \pi_*(H \wedge MU)$ is indeed isomorphic to the additive one, so ϕ exists. Moreover, $\log_E(\hat{x}_E) = \hat{x}_E$, so

$$\hat{x}_E = \sum \phi(m_i)\hat{x}_{MU}^{i-1} = g_H^{-1}(\hat{x}_{MU})$$

by 4.1.9. It follows that $\sum \phi(m_i)x^{i+1}$ is the functional inverse of $\sum b_i x^{i+1}$, i.e.,

$$h\theta_{MU} \exp(x) = \sum_{i \geq 0} b_i x^{i+1}, \tag{4.1.10}$$

where exp is the functional inverse of the logarithm (A2.1.5), so $\phi(m_i) \equiv -b_i$, modulo decomposables in $H^*(MU)$ and 4.1.6 follows.

Now we will determine the structure of $MU_*(MU)$. We know it as an algebra by 4.1.7. In particular, it is a free $\pi_*(MU)$ module, so MU is a flat ring spectrum. Hence by 2.2.8 ($\pi_*(MU)$, $MU_*(MU)$) is a Hopf algebroid (A1.1.1). We will show that it is isomorphic to (L, LB) of A2.1.16. We now recall its structure. As an algebra, $LB = L[b_1, b_2 \ldots]$ with $\deg b_i = 2i$. There are structure maps $\varepsilon\colon LB \to L$ (augmentation), $\eta_L, \eta_R\colon L \to LB$ (left and right units), $\Delta\colon LB \to LB \otimes_L LB$ (coproduct), and $c\colon LB \to LB$ (conjugation) satisfying certain identities listed in A1.1.1.

$\varepsilon\colon LB \to L$ is defined by $\varepsilon(b_i) = 0$; $\eta_L\colon L \to LB$ is the standard inclusion, while $\eta_R\colon L \otimes \mathbf{Q} \to LB \otimes \mathbf{Q}$ is given by

$$\sum_{i\ge 0} \eta_R(m_i) = \sum_{i\ge 0} m_i \left(\sum_{j\ge 0} c(b_j)\right)^{i+1},$$

where $m_0 = b_0 = 1$;

$$\sum_{i\ge 0} \Delta(b_i) = \sum_{j\ge 0} \left(\sum_{i\ge 0} b_i\right)^{j+1} \otimes b_j;$$

and $c\colon LB \to LB$ is determined by $c(m_i) = \eta_R(m_i)$ and

$$\sum_{i\ge 0} c(b_i) \left(\sum_{j\ge 0} b_j\right)^{i+1} = 1.$$

Note that the maps η_L and η_R, along with the identities of A1.1.1, determine the remaining structure maps ε, Δ, and c.

The map θ_{MU} of 4.1.6 is an isomorphism which can be extended to LB by defining $\theta_{MU}(b_i)$ to be $b_i^{MU} \in MU_{2i}(MU)$ (4.1.8).

4.1.11. THEOREM (Novikov [**?**], Landweber [**?**]). *The map $\theta_{MU}\colon LB \to MU_*(MU)$ defined above gives a Hopf algebroid isomorphism $(L, LB) \to (\pi_*(MU), MU_*(MU))$.*

PROOF. Recall that the Hopf algebroid structure of (L, LB) is determined by the right unit $\eta_R\colon L \to LB$. Hence it suffices to show that θ_{MU} respects η_R. Now the left and right units in $MU_*(MU)$ are induced by $MU \wedge S^0 \to MU \wedge MU$ and $S^0 \wedge MU \to MU \wedge MU$, respectively. These give complex orientations x_L and x_R for $MU \wedge MU$ and hence formal group laws (4.1.4) F_R and F_L over $MU_*(MU)$. The b_i in LB are the coefficients of the power series of the universal isomorphism between two universal formal group laws. Hence it suffices to show that $x_R = \sum_{i\ge 0} b_i^{MU} x_R^{i+1}$, but this is the special case of 4.1.9 where $E = MU$. $\square$

Our next objective is

4.1.12. THEOREM. *[Brown and Peterson [**?**], Quillen [**?**]] For each prime p there is a unique associative commutative ring spectrum BP which is a retract of $MU_{(p)}$* (2.1.12) *such that the map $g\colon MU_{(p)} \to BP$ is multiplicative,*

(a) $\pi_*(BP) \otimes \mathbf{Q} = \mathbf{Q}[g_*(m_{p^k-1})\colon k > 0]$ *with $g_*(m_n) = 0$ for $n \ne p^k - 1$;*

(b) $H_*(BP\colon \mathbf{Z}/(p)) = P_*$ (3.1.6) *as comodule algebras over the dual Steenrod algebra A_** (3.1.1); *and*

(c) $\pi_*(BP) = \mathbf{Z}_{(p)}[v_1, v_2 \ldots]$ *with $v_n \in \pi_{2(p^n-1)}$ and the composition $\pi_*(g)\theta_{MU_{(p)}}$ factors through the map $L \times \mathbf{Z}_{(p)} \to V$ of* A2.1.25, *giving an isomorphism from V to $\pi_*(BP)$.* $\square$

The spectrum BP is named after Brown and Peterson, who first constructed it via its Postnikov tower. Recall (3.1.9) that $H_*(MU; \mathbf{Z}/(p))$ splits as an A_*-comodule into many copies of P_*. Theorem 4.1.12 implies that there is a corresponding splitting of $MU_{(p)}$. Since P_* is dual to a cyclic A-module, it is clear that BP cannot be split any further. Brown and Peterson [**?**] also showed that BP can be constructed from H (the integral Eilenberg–Mac Lane spectrum) by killing all of the torsion in its integral homology with Postnikov fibrations. More recently, Priddy [**?**] has shown that BP can be constructed from $S^0_{(p)}$ by adding local cells to kill off all of the torsion in its homotopy.

The generators v_n of $\pi_*(BP)$ will be defined explicitly below.

Quillen [**?**] constructed BP in a more canonical way which enabled him to determine the structure of $BP_*(BP)$. BP bears the same relation to p-typical formal group laws (A2.1.17) that MU bears to formal group laws as seen in 4.1.6. The algebraic basis of Quillen's proof of 4.1.12 is Cartier's theorem A2.1.18, which states that any formal group law over a $\mathbf{Z}_{(p)}$-algebra is canonically isomorphic to p-typical one. Accounts of Quillen's work are given in Adams [**?**] and Araki [**?**].

Following Quillen [**?**], we will construct a multiplicative map $g\colon MU_{(p)} \to MU_{(p)}$ which is idempotent, i.e., $g^2 = g$. This map will induce an idempotent natural transformation or cohomology operation on $MU^*_{(p)}(-)$. The image of this map will be a functor satisfying the conditions of Brown's representability theorem (see Brown [**?**] or, in terms of spectra, 3.12 of Adams [**?**]) and will therefore be represented by a spectrum BP. The multiplicativity of BP and its other properties will follow from the corresponding properties of g.

To construct g we need two lemmas.

4.1.13. LEMMA. *Let E be an oriented ring spectrum. Then orientations of E are in one-to-one correspondence with multiplicative maps from MU to E; i.e., given an orientation $y_E \in E^2(\mathbf{C}P^\infty)$, there is a unique multiplicative map $g\colon MU \to E$ such that $g^*(x_{MU}) = y_E$ and vice versa.*

PROOF. By 4.1.4, $E^*(\mathbf{C}P^\infty) = \pi_*(E)[[x_E]]$ so we have

$$y_E = f(x_E) = \sum_{i\geq 0} f_i x^{i+1}$$

with $f_0 = 1$ and $f_i \in \pi_{2i}(E)$. Using arguments similar to that of 4.1.8 and 4.1.6 one shows

$$E^*(MU) \cong \mathrm{Hom}_{\pi_*(E)}(E_*(MU), \pi_*(E)) \tag{4.1.14}$$

and

$$E^*(\mathbf{C}P^\infty) \cong \mathrm{Hom}_{\pi_*(E)}(E_*(\mathbf{C}P^\infty), \pi_*(E)).$$

A diagram chase shows that a map $MU \to E$ is multiplicative if the corresponding map $E_*(MU) \to \pi_*(E)$ is a $\pi_*(E)$-algebra map. The map y_E corresponds to the map which sends β^E_{i+1} to f_i and β^E_{i+1} by definition maps to $b^E_i \in E_{2i}(MU)$, so we let g be the map which sends b^E_i to f_i. □

4.1.15. LEMMA. *A map $g\colon MU_{(p)} \to MU_{(p)}$ (or $MU \to MU$) is determined up to homotopy by its behavior on π_*.*

PROOF. We do the MU case first. By 4.1.14,

$$MU^*(MU) = \mathrm{Hom}_{\pi_*(MU)}(MU_*(MU), \pi_*(MU)).$$

This object is torsion-free so we lose no information by tensoring with $\mathbf{Q}$. It follows from 4.1.11 that $MU_*(MU) \otimes \mathbf{Q}$ is generated over $\pi_*(MU) \otimes \mathbf{Q}$ by the image of η_R, which is the Hurewicz map. Therefore the map

$$MU^*(MU) \otimes \mathbf{Q} \to \mathrm{Hom}_{\mathbf{Q}}(\pi_*(MU) \otimes \mathbf{Q}, \pi_*(MU) \otimes \mathbf{Q})$$

is an isomorphism, so the result follows for MU.

For the $MU_{(p)}$ case we need to show

$$MU^*_{(p)}(MU_{(p)}) = MU^*(MU) \otimes \mathbf{Z}_{(p)}. \tag{4.1.16}$$

This will follow from 4.1.13 if we can show that the map

$$MU^*_{(p)}(MU) \to MU^*_{(p)}(MU_{(p)}) \tag{4.1.17}$$

is an isomorphism, i.e., that $MU^*_{(p)}(C) = 0$, where C is the cofiber of $MU \to MU_{(p)}$. Now C is trivial when localized at p, so any p-local cohomology theory vanishes on it. Thus 4.1.15 and the $MU_{(p)}$ case follow. □

We are now ready to prove 4.1.12. By 4.1.13 and 4.1.15 a multiplicative map $g\colon MU_{(p)} \to MU_{(p)}$ is determined by a power series $f(x)$ over $\pi_*(MU_{(p)})$. We, take $f(x)$ to be as defined by A2.1.23. By 4.1.15 the corresponding map g is idempotent if $\pi_*(g) \otimes \mathbf{Q}$ is. To compute the latter we need to see the effect of g^* on

$$\log(x_{MU}) = \sum m_i x_{MU}^{i+1} \in MU^2(\mathbf{C}P^\infty) \otimes \mathbf{Q}.$$

Let $F'_{MU_{(p)}}$ be the formal group law associated with the orientation $f(x_{MU})$, and let $\mathrm{mog}(x)$ be its logarithm (A2.1.6). The map g^* preserves formal group laws and hence their logarithms, so we have $g^*(\log(x_{MU})) = \mathrm{mog}(f(x_{MU}))$. By A2.1.24 $\mathrm{mog}(x) = \sum_{k\geq 0} m_{p^k-1} x^{p^k}$ and it follows that $\pi_*(g)$ has the indicated behavior; i.e., we have proved 4.1.12(a).

For (b), we have $H_*(BP; \mathbf{Q}) = \pi_*(BP) \otimes \mathbf{Q}$, and $H_*(BP; \mathbf{Z}_{(p)})$ is torsionfree, so $H_*(BP; \mathbf{Z}/(p)) = P_*$ as algebras. Since BP is a retract of $MU_{(p)}$ its homology is a direct summand over A_* and (b) follows.

For (c) the structure of $\pi_*(BP)$ follows from (a) and the fact that BP is a retract of $MU_{(p)}$. For the isomorphism from V we need to complete the diagram

$$\begin{tikzcd} L \otimes \mathbf{Z}_{(p)} \arrow[r] \arrow[d, "\theta_{MU_{(p)}}"'] & V \arrow[d, dashed] \\ \pi_*(MU_{(p)}) \arrow[r, "g_*"] & \pi_*(BP) \end{tikzcd}$$

The horizontal maps are both onto and the left-hand vertical map is an isomorphism so it suffices to complete the diagram tensored with $\mathbf{Q}$. In this case the result follows from (a) and A2.1.25. This completes the proof of 4.1.12.

Our last objective in this section is the determination of the Hopf algebroid (A1.1.1) $(\pi_*(BP), BP_*(BP))$. (Proposition 2.2.8 says that this object is a Hopf algebroid if BP is flat. It is since $MU_{(p)}$ is flat.) We will show that it is isomorphic to (V, VT) of A2.1.27, which bears the same relation to p-typical formal group laws that (L, LB) (A2.1.16 and 4.1.11) bears to ordinary formal group laws. The ring V (A2.1.25), over which the universal p-typical formal group law is defined, is isomorphic to $\pi_*(BP)$ by 4.1.12(c). $V \otimes \mathbf{Q}$ is generated by m_{p^i-1} for $i \geq 0$, and we denote this element by λ_i. Then from A2.1.27 we have

4.1.18. THEOREM. *In the Hopf algebroid* (V, VT) (*see* A1.1.1)

(a) $V = Z_{(p)}[v_1, v_2, \dots]$ *with* $|v_n| = 2(p^n - 1)$,

(b) $VT = V[t_1, t_2, \dots]$ *with* $|t_n| = 2(p^n - 1)$, *and*

(c) $\eta_L \colon V \to VT$ *is the standard inclusion and* $\varepsilon \colon VT \to V$ *is defined by* $\varepsilon(t_i) = 0$, $\varepsilon(v_i) = v_i$.

(d) $\eta_R \colon V \to VT$ *is determined by* $\eta_R(\lambda_n) = \sum_{0 \le i \le n} \lambda_i t_{n-i}^{p^i}$, *where* $\lambda_0 = t_0 = 1$,

(e) Δ *is determined by*

$$\sum_{i,j \ge 0} \lambda_i \Delta(t_j)^{p^i} = \sum_{i,u,k \ge 0} \lambda_i t_j^{p^i} \otimes t_j^{p^{i+j}},$$

and

(f) c *is determined by*

$$\sum_{i,j,k \ge 0} \lambda_i t_j^{p^i} c(t_k)^{p^{i+j}} = \sum_{i \ge 0} \lambda_i.$$

(g) *The forgetful functor from p-typical formal group laws to formal group laws induces a surjection of Hopf algebroids* (A1.1.19)

$$(L \otimes \mathbf{Z}_{(p)}, LB \otimes \mathbf{Z}_{(p)}) \to (V, VT). \qquad \square$$

4.1.19. THEOREM (Quillen [**?**], Adams [**?**]). *The Hopf algebroid* $(\pi_*(BP), BP_*(BP))$ *is isomorphic to* (V, VT) *described above.*

PROOF. Consider the diagram

$$\begin{tikzcd}
(L, LB) \otimes \mathbf{Z}_{(p)} \arrow[r] \arrow[d, "\theta_{MU}"] & (V, VT) \arrow[d, dashed] \\
MU_*(MU) \otimes \mathbf{Z}_{(p)} \arrow[r, "g_*"] & (\pi_*(BP), BP_*(BP)).
\end{tikzcd}$$

The left-hand map is an isomorphism by 4.1.11 and the horizontal maps are both onto by (g) above and by 4.1.12. Therefore it suffices to complete the diagram with an isomorphism over $\mathbf{Q}$. One sees easily that $VT \otimes \mathbf{Q}$ and $BP_*(BP) \otimes \mathbf{Q}$ are both isomorphic to $V \otimes V \otimes \mathbf{Q}$. $\square$

2. A Survey of BP-Theory

In this section we will give an informal survey of some aspects of complex cobordism theory not directly related to the Adams–Novikov spectral sequence. (We use the terms complex cobordism and BP interchangeably in light of 4.1.12.) Little or no use of this material will be made in the rest of the book. This survey is by no means exhaustive.

The history of the subject shows a movement from geometry to algebra. The early work was concerned mainly with applications to manifold theory, while more recent work has dealt with the internal algebraic structure of various cohomology theories and their applictions to homotopy theory. The present volume is, of course, an example of the latter. The turning point in this trend was Quillen's theorem 4.1.6, which established a link with the theory of formal groups treated in Appendix 2. The influential but mostly unpublished work of Jack Morava in the early 1970s was concerned with the implications of this link.

Most geometric results in the theory, besides the classification of closed manifolds up to cobordism, rest on the notion of the *bordism groups* $\Omega_*(X)$ of a space X, first defined by Conner and Floyd [**?**]. $\Omega_n(X)$ is the group (under disjoint union) of equivalence classes of maps from closed n-dimensional manifolds (possibly with some additional structure such as an orientation or a stable complex structure) to X. Two such maps $f_i\colon M_i \to X$ $(i = 1, 2)$ are equivalent if there is a map $f\colon W \to X$ from a manifold whose boundary is $M_1 \cup M_2$ with f extending f_1 and f_2. It can be shown (Conner and Floyd [**?**]) that the functor $\Omega_*(-)$ is a generalized homology theory and that the spectrum representing it is the appropriate Thom spectrum for the manifolds in question. For example, if the manifolds are stably complex (see the beginning of Section 1) the bordism theory, denoted by $\Omega_*^U(-)$, coincides with $MU_*(-)$, the generalized homology theory represented by the spectrum MU, i.e., $\Omega_n^U(X) = \pi_n(MU \wedge X)$. The notation $\Omega_*(-)$ with no superscript usually denotes the oriented bordism group, while the unoriented bordism group is usually denoted by $N_*(-)$.

These bordism groups are usually computed by algebraic methods that use properties of the Thom spectra. Thom [**?**] showed that MO, the spectrum representing unoriented bordism, is a wedge of mod (2) Eilenberg–Mac Lane spectra, so $N_*(X)$ is determined by $H_*(X; \mathbf{Z}/(2))$. MSO (which represents oriented bordism) when localized at the prime 2 is known (Stong [**?**, p. 209]) to be a wedge of integral and mod (2) Eilenberg–Mac Lane spectra, so $\Omega_*(X)_{(2)}$ is also determined by ordinary homology. Brown and Peterson [**?**] showed that when localized at any odd prime the spectra MSO, MSU, and MSp as well as MU are wedges of various suspensions of BP, so the corresponding bordism groups are all determined by $BP_*(X)$. Conner and Floyd [**?**] showed effectively that $BP_*(X)$ is determined by $H_*(X; \mathbf{Z}_{(p)})$ when the latter is torsion-free.

For certain spaces the bordism groups have interesting geometric interpretations. For example, $\Omega_*(BO)$ is the cobordism group of vector bundles over oriented manifolds. Since $H_*(BSO)$ has no odd torsion, it determines this group. If X_n is the nth space in the Ω-spectrum for MSO, then $\Omega_*(X_n)$ is the cobordism group of maps of codimension n between oriented manifolds. The unoriented analog was treated by Stong [**?**] and the complex analog by Ravenel and Wilson [**?**].

For a finite group G, $\Omega_*(BG)$ is the cobordism group of oriented manifolds with free G-actions, the manifolds mapped to BG being the orbit spaces. These groups were studied by Conner and Floyd [**?**] and Conner [**?**]. Among other things they computed $\Omega_*(BG)$ for cyclic G. In Landweber [**?**] it was shown that the map $MU_*(BG) \to H_*(BG)$ is onto iff G has periodic cohomology. In Floyd [**?**] and tom Dieck [**?**] it is shown that the ideal of $\pi_*(MU)$ represented by manifolds on which an abelian p-group with n cyclic summands can act without stationary points is the prime ideal I_n defined below. The groups $BP_*(BG)$ for $G = (\mathbf{Z}/(p))^n$ have been computed by Johnson and Wilson [**?**].

We now turn to certain other spectra related to MU and BP. These are constructed by means of either the Landweber exact functor theorem (Landweber [**?**]) or the *Sullivan–Baas construction* (Baas [**?**]), which we now describe. Dennis Sullivan (unpublished, circa 1969) wanted to construct "manifolds with singularities" (admittedly a contradiction in terms) with which any ordinary homology class could be represented; i.e., any element in $H_*(X; \mathbf{Z})$ could be realized as the image of the fundamental homology class of such a "manifold" M under some map $M \to X$.

It was long known that not all homology classes were representable in this sense by ordinary manifolds, the question having been originally posed by Steenrod. (I heard Sullivan begin a lecture on the subject by saying that homology was like the weather; everybody talks about it but nobody does anything about it.)

In terms of spectra this nonrepresentability is due to the fact that MU (if we want our manifolds to be stably complex) is *not* a wedge of Eilenberg–Mac Lane spectra. The Sullivan–Baas construction can be regarded as a way to get from MU to H.

Let $y \in \pi_k(MU)$ be represented by a manifold X. A closed n-dimensional manifold with *singularity of type* (y) $(n > k)$ is a space W of the form $A \cup (B \times CM)$, where CM denotes the cone on a manifold M representing y, B is a closed $(n-k-1)$-dimensional manifold, A is an n-dimensional manifold with boundary $B \times M$, and A and $B \times CM$ are glued together along $B \times M$. It can be shown that the bordism group defined using such objects is a homology theory represented by a spectrum $C(y)$ which is the cofiber of

$$\Sigma^k MU \xrightarrow{y} C(y), \quad \text{so} \quad \pi_* C(Y) = \pi_*(MU)/y.$$

This construction can be iterated any number of times. Given a sequence $y_1, y_2, \dots$ of elements in $\pi_*(MU)$ we get spectra $C(y_1, y_2, \dots y_n)$ and cofibrations

$$\Sigma^{|y_n|} C(y_1, \dots, y_{n-1}) \to C(y_1, \dots, y_{n-1}) \to C(y_1, \dots, y_n).$$

If the sequence is regular, i.e., if y_n is not a zero divisor in $\pi_*(MU)/(y_1, \dots, y_{n-1})$, then each of the cofibrations will give a short exact sequence in homotopy, so we get

$$\pi_*(C(y_1, \dots, y_n)) = \pi(MU)/(y_1, \dots, y_n).$$

In this way one can kill off any regular ideal in $\pi_*(MU)$. In particular, one can get H by killing $(x_1, x_2, \dots)$. Sullivan's idea was to use this to show that any homology class could be represented by the corresponding type of manifold with singularity. One could also get BP by killing the kernel of the map $\pi_*(MU) \to \pi_*(BP)$ and then localizing at p. This approach to BP does not reflect the splitting of $MU_{(p)}$.

Much more delicate arguments are needed to show that the resulting spectra are multiplicative (Shimada and Yagita [**?**], Morava [**?**], Mironov [**?**]), and the proof only works at odd primes. Once they are multiplicative, it is immediate that they are orientable in the sense of 4.1.1.

The two most important cases of this construction are the *Johnson–Wilson spectra* $BP\langle n\rangle$ (Johnson and Wilson [**?**]) and the Morava K-theories $K(n)$ (Morava's account remains unpublished; see Johnson and Wilson [**?**]).

$BP\langle n\rangle$ is the spectrum obtained from BP (one can start there instead of MU since BP itself is a product of the Sullivan–Baas construction) by killing $(v_{n+1}, v_{n+2}, \dots) \subset \pi_*(BP)$. One gets

$$\pi_*(BP\langle n\rangle) = \mathbf{Z}_{(p)}[v_1, \dots, v_n]$$

and

$$H_*(BP\langle n\rangle, \mathbf{Z}/(p)) = P_* \otimes E(\tau_{n+1}, \tau_{n+2}, \dots).$$

(It is an easy exercise using the methods of Section 3.1 to show that a connective spectrum with that homology must have the indicated homotopy.) One has fibrations

$$\Sigma^{2(p^n-1)} BP\langle n\rangle \xrightarrow{v_n} BP\langle n\rangle \to BP\langle n-1\rangle.$$

$BP\langle 0\rangle$ is $H_{(p)}$ and $BP\langle 1\rangle$ is a summand of $bu_{(p)}$, the localization at p of the spectrum representing connective complex K-theory. One can iterate the map

$$v_n\colon \Sigma^{2(p^n-1)}BP\langle n\rangle \to BP\langle n\rangle$$

and form the direct limit

$$E(n) = \varinjlim_{v_n} \Sigma^{-2i(p^n-1)}BP\langle n\rangle.$$

$E(1)$ is a summand of periodic complex K-theory localized at p. Johnson and Wilson [?] showed that

$$E(n)_*(X) = BP_*(X) \otimes_{BP_*} E(n)_*.$$

$E(n)$ can also be obtained by using the Landweber exact functor theorem below.

The $BP\langle n\rangle$ are interesting for two reasons. First, the fibrations mentioned above split unstably; i.e., if $BP\langle n\rangle_k$ is the kth space in the Q-spectrum for $BP\langle n\rangle$ (i.e., the space whose homotopy starts in dimension k) then

$$BP\langle n\rangle_k \simeq BP\langle n-1\rangle_k \times BP\langle n\rangle_{k+2(p^n-1)}$$

for $k \leq 2(p^n-1)/(p-1)$ (Wilson [?]). This means that if X is a finite complex then $BP_*(X)$ is determined by $BP\langle n\rangle_*(X)$ for an appropriate n depending on the dimension of X.

The second application of $BP\langle n\rangle$ concerns $\operatorname{Hom}\dim BP_*(X)$, the projective dimension of $BP_*(X)$ as a module over $\pi_*(BP)$, known in some circles as the ugliness number. Johnson and Wilson [?] show that the map $BP_*(X) \to BP\langle n\rangle_*(X)$ is onto iff $\operatorname{Hom}\dim BP_*(X) \leq n+1$. The cases $n=0$ and $n=1$ of this were obtained earlier by Conner and Smith [?].

We now turn to the *Morava K-theories* $K(n)$. These spectra are periodic, i.e., $\Sigma^{2(p^n-1)}K(n) = K(n)$. Their connective analogs $k(n)$ are obtained from BP by killing $(p, v_1, \dots, v_{n-1}, v_{n+1}, v_{n+2}, \dots)$. Thus one has $\pi_*(k(n)) = \mathbf{Z}/(p)[v_n]$ and $H_*(k(n), \mathbf{Z}/(p)) = A/(Q_n)_*$. One has fibrations

$$\Sigma^{2(p^n-1)}k(n) \xrightarrow{v_n} k(n) \to H\mathbf{Z}/(p),$$

and one defines

$$K(n) = \varinjlim_{v_n} \Sigma^{-2i(p^n-1)}k(n).$$

$K(1)$ is a summand of mod p complex K-theory and it is consistent to define $K(0)$ to be HQ, rational homology.

The coefficient ring $\pi_*(K(n)) = \mathbf{F}_p[v_n, v_n^{-1}]$ is a graded field in the sense that every graded module over it is free. One has a Künneth isomorphism

$$K(n)_*(X \times Y) = K(n)_*(X) \otimes_{\pi_*(K(n))} K(n)_*(Y).$$

This makes $K(n)_*(-)$ much easier to compute with than any of the other theories mentioned here. In Ravenel and Wilson [?] we compute the Morava K-theories of all the Eilenberg–Mac Lane spaces, the case $n=1$ having been done by Anderson and Hodgkin [?]. We show that for a finite abelian group G, $K(n)_*(K(G,m))$ is finite-dimensional over $\pi_*(K(n))$ for all m and n, and is isomorphic to $K(n)_*(\mathrm{pt})$ if $m > n$. $K(n+1)^*K(\mathbf{Z}, m+2)$ for $m, n \geq 0$ is a power series ring on $\binom{n}{m}$ variables. In all cases the $K(n)$-theory is concentrated in even dimensions. These calculations enabled us to prove the conjecture of Conner and Floyd [?] which concerns $BP_*(B(\mathbf{Z}/p)^n)$.

To illustrate the relation between the $K(n)$'s and BP we must introduce some more theories. Let $I_n = (p, v_1, \dots, v_{n-1}) \subset \pi_*(BP)$ (see 4.3.2) and let $P(n)$ be the spectrum obtained from BP by killing I_n. Then one has fibrations

$$\Sigma^{2(p^n-1)}P(n) \xrightarrow{v_n} P(n) \to P(n+1)$$

and we define

$$B(n) = \varinjlim_{v_n} \Sigma^{-2i(p^n-1)}P(n).$$

$P(n)_*(X)$ is a module over $\mathbf{F}_p[v_n]$ and its torsion-free quotient maps monomorphically to $B(n)_*(X)$. In Johnson and Wilson [**?**] it is shown that $B(n)_*(X)$ is determined by $K(n)_*(X)$. In Würgler [**?**] it is shown that a certain completion of $B(n)$ splits into a wedge of suspensions of $K(n)$.

This splitting has the following algebraic antecedent. The formal group law associated with $K(n)$ (4.1.4) is essentially the standard height n formal group law F_n of A2.2.10, while $\pi_*(B(n)) = \mathbf{F}_p[v_n, v_n^{-1}, v_{n+1}, \dots]$ is the universal ring for all p-typical formal group laws of height n (A2.2.7). In A2.2.11 it is shown that over the algebraic closure of $\mathbf{F}_p$ any height n formal group law is isomorphic to the standard one. Heuristically this is why $B(n)_*(X)$ is determined by $K(n)_*(X)$.

This connection between $K(n)$ and height n formal group laws also leads to a close relation between $K(n)_*(K(n))$ and the endomorphism ring of F_n (A2.2.17). An account of $K(n)_*(K(n))$ is given in Yagita [**?**]. The reader should be warned that $K(n)_*(K(n))$ is not the Hopf algebroid $K(n)_*K(n)$ of Ravenel [**?, ?**], which is denoted herein by $\Sigma(n)$; in fact, $K(n)_*(K(n)) = \Sigma(n) \otimes E(\tau_0, \tau_1, \dots, \tau_{n-1})$, where the τ_i are analogous to the τ_i in A_*.

Most of the above results on $K(n)$ (excluding the results about Eilenberg–Mac Lane spaces) were known to Morava and communicated by him to the author in 1973.

The invariance of the I_n (4.3.2) under the BP-operations makes it possible to construct the spectra $P(n)$, $B(n)$, and $K(n)$ and to show that they are ring spectra for $p > 2$ by more algebraic means, i.e., without using the Sullivan–Baas construction. This is done in Würgler [**?**], where the structure of $P(n)_*(P(n))$ is also obtained. $k(n)_*(k(n))$ is described in Yagita [**?**].

We now turn to the important work of Peter Landweber on the internal algebraic structure of MU- and BP-theories. The starting point is the invariant prime ideal theorem 4.3.2, which first appeared in Landweber [**?**], although it was probably first proved by Morava. It states that the only prime ideals in $\pi_*(BP)$ which are invariant (A1.1.21), or, equivalently, which are subcomodules over $BP_*(BP)$, are the $I_n = (p, v_1, v_2, \dots, v_{n-1})$ for $0 \le n \le \infty$. In Conner and Smith [**?**] it is shown that for a finite complex X, $BP_*(X)$ is finitely presented as a module over $\pi_*(BP)$. [The result there is stated in terms of $MU_*(X)$, but the two statements are equivalent.] From commutative algebra one knows that such a module over such a ring has a finite filtration in which each of the successive subquotients is isomorphic to the quotient of the ring by some prime ideal. Of course, as anyone who has contemplated the prospect of algebraic geometry knows, a ring such as $\pi_*(BP)$ has a very large number of prime ideals. However, Landweber [**?**] shows that the coaction of $BP_*(BP)$ implies that the filtration of $BP_*(X)$ [or of any $BP_*(BP)$-comodule which is finitely presented as a module over $\pi_*(BP)$] can be chosen so that each successive subquotient has the form $\pi_*(BP)/I_n$ for some finite n. {The corresponding statement about $MU_*(X)$ appeared earlier in Landweber [**?**].} The submodules in the filtration can be taken to be subcomodules and n (the number of generators of the prime ideal) never exceeds the projective dimension of the module. This useful result is known as the *Landweber filtration theorem*.

It leads to the *Landweber exact functor theorem*, which addresses the following question. For which $\pi_*(BP)$-modules M is the functor $BP_*(-)\otimes_{\pi_*(BP)}M$ a generalized homology theory? Such a functor must be exact in the sense that it converts cofiber sequences into long exact sequences of modules. This will be the case if M is flat, i.e., if $\mathrm{Tor}_1^{\pi_*(BP)}(M,N)=0$ for all modules N. However, in view of the filtration theorem it suffices for this Tor group to vanish only for $N=\pi_*(BP)/I_n$ for all n. This weaker (than flatness) condition on M can be made more explicit as follows. For each n, multiplication by v_n in $M\otimes_{\pi_*(BP)}\pi_*(BP)/I_n$ is monic. Thus Landweber [**?**] shows that any M satisfying this condition gives a homology theory.

For example, the spectrum $E(n)$ mentioned above (in connection with Johnson–Wilson spectra) can be so obtained since

$$\pi_*(E(n))=\mathbf{Z}_{(p)}[v_1,v_2,\ldots,v_n,v_n^{-1}]$$

satisfies Landweber's condition. [Multiplication by v_i is monic in $\pi_*(E(n))$ itself for $i\le n$, while for $i>n$, $\pi_*(E(n))\otimes_{\pi_*(BP)}\pi_*(BP)/I_i=0$ so the condition is vacuous.]

As remarked earlier, $E(1)$ is a summand of complex K-theory localized at p. The exact functor theorem can be formulated globally in terms of MU-theory and $\pi_*(K)$ [viewed as a $\pi_*(MU)$-module via the Todd genus $td\colon \pi_*(MU)\to\mathbf{Z}$] satisfies the hypotheses. Thereby one recovers the *Conner–Floyd isomorphism*

$$K_*(X)=MU_*(X)\otimes_{\pi_*(MU)}\pi_*(K)$$

and similarly for cohomology. In other words, complex K-theory is determined by complex cobordism. This result was first obtained by Conner and Floyd [**?**], whose proof relied on an explicit K-theoretic orientation of a complex vector bundle. Using similar methods they were able to show that real K-theory is determined by symplectic cobordism.

Landweber's results have been generalized as follows. Let $J\subset\pi_*(BP)$ be an invariant regular ideal (see Landweber [**?**]), and let BPJ be the spectrum obtained by killing J; e.g., $P(n)$ above is BPI_n. Most of the algebra of BP-theory carries over to these spectra, which are studied systematically in a nice paper by Johnson and Yosimura [**?**]. The case $J=I_n$ was treated earlier by Yagita [**?**] and Yosimura [**?**]. The mod I_n version of the exact functor enables one to get $K(n)$ from $P(n)$.

Johnson and Yosimura [**?**] also prove some important facts about $\pi_*(BP)$ modules M which are comodules over $BP_*(BP)$. They show that if an element $m\in M$ is v_n-torsion (i.e., it is annihilated by some power of v_n) then it is v_{n-1}-torsion. If all of the primitive elements in M [i.e., those with $\psi(m)=1\otimes m$] are v_n-torsion, then so is every element, and, if none is, then M is v_n-torsion free. If M is a v_{n-1}-torsion module, then $v_n^{-1}M$ is still a comodule over $BP_*(BP)$. Finally, they show that $v_n^{-1}BP_*(X)=0$ if $E(n)_*(X)=0$.

This last result may have been prompted by an erroneous claim by the author that the spectrum $v_n^{-1}BP$ splits as a wedge of suspensions of $E(n)$. It is clear from the methods of Würgler [**?**] that one must complete the spectra in some way before such a splitting can occur. Certain completions of MU are studied in Morava [**?**].

We now turn to the last topic of this section, the applications of BP-theory to unstable homotopy theory. This subject began with Steve Wilson's thesis (Wilson [**?, ?**]) in which he studied the spaces in the Ω-spectra for MU and BP. He obtained the splitting mentioned above (in connection with the Johnson–Wilson spectra) and showed that all of the spaces in question have torsion-free homology. Both the homology and cohomology of each space are either an exterior algebra on odd-dimensional generators or a polynomial algebra on even-dimensional generators.

These spaces were studied more systematically in Ravenel and Wilson [**?**]. There we found it convenient to consider all of them simultaneously as a graded space. The mod (p) homology of such an object is a bigraded coalgebra. The fact that this graded space represents a multiplicative homology theory implies that its homology is a ring object in the category of bigraded coalgebras; we call such an object a *Hopf ring*. We show that the one in question has a simple set of generators and relations which are determined by the structure of $MU^*(\mathbf{C}P^\infty)$, i.e., by $\pi_*(MU)$ and the associated formal group law. We obtain similar results for the value on this graded space of any complex oriented (4.1.1) generalized homology theory.

As mentioned earlier, the complex bordism of the graded space associated with MU is the cobordism group of maps between stably complex manifolds. We show that it is a Hopf ring generated by maps from a manifold to a point and the linear embeddings of $\mathbf{C}P^n$ in $\mathbf{C}P^{n+1}$.

The Hopf ring point of view is also essential in Ravenel and Wilson [**?**], where we calculate $K(n)_*(K(G,m))$. We show that the Hopf ring $K(n)_*(K(\mathbf{Z}/(p^j),{}^*))$ is a certain type of free object on $K(n)_*(K(\mathbf{Z}/(p^j),1))$. The ordinary homology of $K(\mathbf{Z}/(p^j),{}^*)$ can be described in similar terms and the methods of our paper may lead to simpler proofs of the classical theorems about it (see Wilson [**?**], section II.8).

Knowing the BP homology of the spaces in the BP spectrum is analogous to knowing the mod (p) homology of the mod (p) Eilenberg–Mac Lane spaces. This information, along with some ingenious formal machinery, is needed to construct the unstable Adams spectral sequence, i.e., a spectral sequence for computing the homotopy groups of a space X rather than a spectrum. This was done in the BP case by Bendersky, Curtis, and Miller [**?**]. Their spectral sequence is especially convenient for $X = S^{2n+1}$. In that case they get an E_1-term which is a subcomplex of the usual E_1-term for the sphere spectrum, i.e., of the cobar complex of A1.2.11. Their E_2-term is Ext in an appropriate category. For S^{2n+1} they compute Ext^1, which is a subgroup of the stable Ext^1, and get some corresponding information about $\pi_*(S^{2n+1})$.

In Bendersky [**?**] the spectral sequence is applied to the special unitary groups $SU(n)$. In Bendersky, Curtis, and Ravenel [**?**] the E_2-terms for various spheres are related by an analog of the EHP sequence.

3. Some Calculations in $BP_*(BP)$

In this section we will prove the Morava–Landweber theorem (4.3.2), which classifies invariant prime ideals in $\pi_*(BP)$. Then we will derive several formulas in $BP_*(BP)$ (4.1.18 and 4.1.19). These results are rather technical. Some of them are more detailed than any of the applications in this book require and they are included here only for possible future reference. The reader is advised to refer to this material only when necessary.

Theorem 4.3.3 is a list of invariant regular ideals that will be needed in Chapter 5. Lemma 4.3.8 gives some generalizations of the Witt polynomials. They are used to give more explicit formulas for the formal group law (4.3.9), the coproduct (4.3.13), and the right unit (4.3.18). We define certain elements, $b_{i,J}$ (4.3.14) and $c_{i,J}$ (4.3.19), which are used to give approximations (modulo certain prime ideals) of the coproduct (4.3.15) and right unit (4.3.20). Explicit examples of the right unit are given in 4.3.21. The coboundaries of $b_{i,J}$ and $c_{i,J}$ in the cobar complex are given in 4.3.22.

In 4.3.23 we define a filtration of $BP_*(BP)/I_n$ which leads to a May spectral sequence which will be used in Section 6.3. The structure of the resulting bigraded Hopf algebroid is given in 4.3.32–34.

From now on $\pi_*(BP)$ will be abbreviated by BP_*. Recall (A2.2.3) that we have two sets of generators for the ring BP_* given by Hazewinkel [**?**] (A2.2.1) and Araki [**?**] (A2.2.2). The behavior of the right unit $\eta_R\colon BP_* \to BP_*(BP)$ on the Araki generators is given by A2.2.5, i.e.,

$$\sum_{i,j\geq 0}^{F} t_i\eta_R(v_j)^{p^i} = \sum_{i,j\geq 0}^{F} v_i t_j^{p^i} \tag{4.3.1}$$

For the Hazewinkel generators this formula is true only mod (p).

This formula will enable us to define some invariant ideals in BP_*. In each case it will be easy to show that the ideal in question is independent of the choice of generators used. The most important result of this sort is the following.

4.3.2. Theorem (Morava [**?**], Landweber [**?**]). *Let $I_n = (p, v_1, \ldots v_{n-1}) \subset BP_*$.* (a) *$I_n$ is invariant.*

(b) *For $n > 0$,*

$$\operatorname{Ext}^0_{BP_*(BP)}(BP_*, BP_*/I_n) = \mathbf{Z}/(p)[v_n]$$

and

$$\operatorname{Ext}^0_{BP_*(BP)}(BP_*, BP_*) = \mathbf{Z}_{(p)}.$$

(c) $0 \to \Sigma^{2(p^n-1)}BP_*/I_n \xrightarrow{v_n} BP_*/I_n \to BP_*/I_{n+1} \to 0$ *is a short exact sequence of comodules.*

(d) *The only invariant prime ideals in BP_* are the I_n for $0 \leq n \leq \infty$.*

Proof. Part (a) follows by induction on n, using (c) for the inductive step. Part (c) is equivalent to the statement that

$$v_n \in \operatorname{Ext}^0_{BP_*(BP)}(BP_*, BP_*/I_n)$$

and is therefore a consequence of (b). For (d) suppose J is an invariant prime ideal which properly contains some I_n. Then the smallest dimensional element of J not in I_n must be invariant modulo I_n, i.e., it must be in $\operatorname{Ext}^0_{BP_*(BP)}(BP_*(BP/I_n))$, so by (b) it must be a power of v_n (where $v_0 = p$). Since J is prime this element must be v_n itself, so $J \supset I_{n+1}$. If this containment is proper the argument can be repeated. Hence, if J is finitely generated, it is I_n for some $n < \infty$. If J is infinitely generated we have $J \supset I_\infty$, which is maximal, so (d) follows.

Hence it remains only to prove (b). It is clear from 4.3.1 that $\eta_R(v_n) \equiv v_n$ mod I_n, so it suffices to show that $\operatorname{Ext}^0_{BP_*(BP)}(BP_*, BP_*/I_n)$ is no bigger than indicated. From 4.3.1 we see that in $BP_*(BP)/I_n$,

$$\eta_R(v_{n+j}) \equiv v_{n+j} + v_n t_j^{p^n} - v_n^{p^j} t_j \mod (t_1, t_2, \dots, t_{j-1}),$$

so the set $\{v_{n+j}, \eta_R(v_{n+j}) \mid j > 0\} \cup \{v_n\}$ is algebraically independent. It follows that if $\eta_R(v) = v$ then v must be a polynomial in v_n. □

Now we will construct some invariant regular ideals in BP_*. Recall that an ideal $(x_0, x_1, \dots, x_{n-1})$ is regular if x_i is not a zero divisor in $BP_*/(x_0, \dots, x_{i-1})$ for $0 \le i < n$. This means that the sequence

$$0 \to BP_*/(x_0, \dots, x_{i-1}) \xrightarrow{x_i} BP_*/(x_0, \dots, x_{i-1}) \to BP_*/(x_0, \dots, x_i) \to 0$$

is exact. The regular sequence $(x_0, x_1, \dots)$ is invariant if the above is a short exact sequence of comodules. Invariant regular ideals have been studied systematically by Landweber [**?**]. He shows that an invariant regular ideal with n generators is primary with radical I_n, and that any invariant ideal with n generators and radical I_n is regular. Invariant ideals in general need not be regular, e.g., I_n^k for $k > 1$.

4.3.3. THEOREM. *Let $i_1, i_2, \dots$ be a sequence of positive integers such that for each $n > 0$, i_{n+i} is divisible by the smallest power of p not less than i_n, and let $k \ge 0$. Then for each $n > 0$, the regular ideal $(p^{1+k}, v_1^{i_1 p^k}, v_2^{i_2 p^{2k}}, \dots, v_n^{i_n p^{kn}})$ is invariant.* □

In order to prove this we will need the following.

4.3.4. LEMMA. *Let B, A_1, A_2, ... be ideals in a commutative ring. Then if*

$$x \equiv y \mod pB + \sum_i A_i,$$

then

$$x^{p^n} \equiv y^{p^n} \mod p^{n+1}B + \sum_{k=0}^{n} p^k \sum_i A_i^{p^{n-k}}.$$

PROOF. The case $n > 1$ follows easily by induction on n from the case $n = 1$. For the latter suppose $x = y + pb + \sum a_i$, with $b \in B$ and $a_i \in A_i$. Then

$$x^p = y^p + \sum_{0<j<p} \binom{p}{j} y^{p-j} \left(pb + \sum a_i\right)^j + \left(pb + \sum a_i\right)^p$$

and we have

$$\binom{p}{j} y^{p-j} \left(pb + \sum a_i\right)^j \in p^2 B + p \sum A_i$$

and

$$\left(pb + \sum a_i\right)^p \in p^2 B + p \sum A_i + \sum A_i^p.$$ □

PROOF OF 4.3.3. We have $v_n \equiv \eta_R(v_n) \bmod I_n$, so we apply 4.3.4 to the ring $BP_*(BP)$ by setting $B = (1)$, $A_i = (v_i)$. Then we get

$$v_n^{p^m} \equiv \eta_R(v_n)^{p^m} \mod (p^{m+1}) + \sum_{j=0}^{m} \sum_{i=1}^{n-1} (p^j v_i^{m-j}).$$

To prove the theorem we must show that the indicated power of v_n is invariant modulo the ideal generated by the first n elements. It suffices to replace this ideal by the smaller one obtained by replacing each of $i_1, \dots, i_{n-1}$ by the smallest power of p not less than it, i.e., by an ideal of the form

$$(p^{1+k}, v_1^{j_1}, v_2^{j_2}, \dots, v_{n-1}^{j_{n-1}}) = I$$

with $j_t = p^{tk+k_t}$ where $0 \le k_1 \le k_2 \cdots \le k_{n-1}$. Then the hypothesis on i_n is that it is divisible by $p^{k_{n-1}}$, so it suffices to assume that $i_n = p^{k_{n-1}}$. Hence we must show

$$v_n^{j_{n-1}p^k} \equiv \eta_R(v_n)^{j_{n-1}p^k} \mod I.$$

We have $v_n \equiv \eta_R(v_n) \bmod I_n$, so we apply 4.3.4 to the ring $BP_*(BP)$ by setting $B = (1)$, $A_i = (v_i)$. Then we get

$$v_n^{p^m} \equiv \eta_R(v_n)^{p_m} \mod (p^{m+1}) + \sum_{j=0}^{m}\sum_{i=0}^{n-1}(p^j v_i^{p^{m-j}}).$$

We are interested in the case $m = kn + k_{n-1}$. Careful inspection shows that the indicated ideal in this case is contained in I. □

Theorem 4.3.3 leads to a list of invariant regular ideals which one might hope is complete. Unfortunately, it is not. For example, it gives $\{(p^{k+1}, v^{ip^k}) \mid k \ge 0, i \ge 0\}$ as a list of I_2-primary regular ideals, and this list can be shown to be a complete for $p > 2$, but at $p = 2$ the ideal $(16, v_1^4 + 8v_1v_2)$ is regular and invariant but not in the list. Similarly, for $p > 2$ the ideal

$$(p, v_1^{p^2+p-1}, v_2^{2p^2} - 2v_1^{p^2}v_2^{p^2-p}v_3^p - 2v_1^{p^2-1}v_2^{2p^2-p+1})$$

is invariant, regular, and not predicted by 4.3.3. This example and others like it were used by Miller and Wilson [**?**] to produce unexpected elements in $\mathrm{Ext}^1_{BP_*(BP)}(BP_*, BP_*/I_n)$ (see Section 5.2).

Now we will make the structure of $BP_*(BP)$ (4.1.19) more explicit. We start with the formal group law.

Recall the lemma of Witt (see, e.g., Lang [**?**, pp. 234–235]) which states that there are symmetric integral polynomials $w_n = w_n(x_1, x_2, \dots)$ of degree p^n in any number of variables such that

$$w_0 = \sum x_t \quad \text{and} \quad \sum_t x_t^{p^n} = \sum_j p^j w_j^{p^{n-j}}. \tag{4.3.5}$$

For example,

$$w_1 = \Big(\sum (x_t^p) - \Big(\sum x_t\Big)^p\Big)\Big/ p \tag{4.3.6}$$

and for $p = 2$ with two variables,

$$w_2 = -x_1^3x_2 - 2x_1^2x_2^2 - x_1x_2^3.$$

Witt's lemma can be restated as follows. Let G be the formal group law with logarithm $\sum_{i\ge 0} x^{p^i}/p^i$. Then

$$\sum\nolimits^G x_t = \sum\nolimits^G w_n. \tag{4.3.7}$$

This formula is in some sense more explicit than the usual

$$\log\Big(\sum\nolimits^G x_t\Big) = \sum \log x_t.$$

We will derive a similar formula for the universal formal group law.

First we need some notation. Let $I = (i_1, i_2, \dots, i_m)$ be a finite (possibly empty) sequence of positive integers. Let $|I| = m$ and $\|I\| = \sum i_t$. For positive integers n let $\Pi(n) = p - p^{(p^n)}$ and define integers $\Pi(I)$ recursively by $\Pi(\phi) = 1$ and $\Pi(I) = \Pi(\|I\|)\Pi(i_1, \dots, i_{m-1})$. Note that $\Pi(I) \equiv p^{|I|} \bmod p^{|I|+1}$. Given sequences I and J let IJ denote the sequence $(i_1, \dots, i_m, j_1, \dots, j_n)$. Then we have $|IJ| = |I| + |J|$ and $\|IJ\| = \|I\| + \|J\|$. We will need the following analog of Witt's lemma (4.3.5), which we will prove at the end of this section.

4.3.8. Lemma.

(a) *For each sequence I as above there is a symmetric polynomial of degree $p^{\|I\|}$ in any number of variables with coefficients in $\mathbf{Z}_{(p)}$, $w_I = w_I(x_1, x_2, \dots)$ with $w_\phi = \sum_t x_t$ and*

$$\sum_t x_t^{p^{\|K\|}} = \sum_{IJ=K} \frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}}.$$

(b) *Let w_i be the polynomial defined by* 4.3.5. *Then*

$$w_I \equiv w_{|I|}^{p^{\|I\|-|I|}} \mod (p). \qquad \square$$

Now let v_i be Araki's generator and define v_I by $v_\phi = 1$ and $v_I = v_{i_1}(v_{I'})^{(p^{i_1})}$ where $I' = (i_2, i_3 \dots)$. Hence dim $v_I = 2(p^{\|I\|} - 1)$. Then our analog of 4.3.7 is

4.3.9. Theorem. *With notation as above,*

$$\sum_t^F x_t = \sum_I^F v_I w_I(x_1, x_2, \dots).$$

(*An analogous formula and proof in terms of Hazewinke's generators can be obtained, by replacing $\Pi(I)$ by $p^{|I|}$ throughout. This requires a different definition of w_J, which is still congruent to $w_{|J|}^{p^{\|J\|-|J|}}$ modulo p.*)

Proof. Araki's formula (A2.2.1) is

$$p\lambda_n = \sum_{0 \le i \le n} \lambda_i v_{n-i}^{p^i}$$

which can be written as

$$\Pi(n)\lambda_n = \sum_{0 \le i < n} \lambda_i v_{n-i}^{p^i}.$$

By a simple exercise this gives

$$\lambda_n = \sum_{\|I\|=n} \frac{v_I}{\Pi(I)},$$

i.e.,

$$\log(x) = \sum \frac{v_I x^{p^{\|I\|}}}{\Pi(I)}. \tag{4.3.10}$$

Therefore we have

$$\begin{aligned}
\log\Big(\sum_J^F v_J w_J\Big) &= \sum_J \log v_J w_J \\
&= \sum \frac{v_{IJ}}{\Pi(I)} w_J^{p^{\|I\|}} \\
&= \sum_{I,J} \frac{v_K}{\Pi(K)} \frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}} \quad \text{(where } K = IJ) \\
&= \sum_{t,K} \frac{v_K}{\Pi(K)} x_t^{p^{\|K\|}} \quad \text{by 4.3.8} \\
&= \sum_t \log x_t \quad \text{by 4.3.10} \\
&= \log \sum_t^F x_t.
\end{aligned}$$

□

In the structure formulas for $BP_*(BP)$ we encounter expressions of the form $\sum_{n,i}^F a_{n,i}$, where $a_{n,i}$ is in $BP_*(BP)$ or $BP_*(BP)\otimes_{BP_*} BP_*(BP)$ (or more generally in some commutative graded BP_* algebra D) and has dimension $2(p^n-1)$. We can use 4.3.9 to simplify such expressions in the following way.

Define subsets A_n and B_n of D as follows. $A_n = B_n = \phi$ for $n \le 0$ and for $n > 0$, $A_n = \{a_{n,i}\}$ while B_n is defined recursively by

$$B_n = A_n \cup \bigcup_{|J|>0} \{v_J w_J(B_{n-\|J\|})\}.$$

4.3.11. LEMMA. *With notation as above,* $\sum_{n,i}^F a_{n,i} = \sum_{n>0}^F w_\phi(B_n)$.

PROOF. We will show by induction on m that the statement is true in dimensions $< 2(p^m-1)$. Our inductive hypothesis is

$$\sum^F a_{n,i} = \sum_{0<n<m}^F w_\phi(B_n) +_F \sum_{\substack{n<m\\ \|J\|+n\ge m}} v_J w_J(B_n) +_F \sum_{n\ge m}^F a_{n,i},$$

which is trivial for $m=1$. The set of formal summands of dimension $2(p^m-1)$ on the right is B_m. By 4.3.9 the formal sum of these terms is $\sum^F v_J w_J(B_m)$, so we get

$$\begin{aligned}
\sum^F a_{n,i} &= \sum_{0<n<m}^F w_\phi(B_n) +_F \sum_J^F v_J w_J(B_m) +_F \sum_{\substack{n<m\\ \|J\|+n>m}}^F v_J w_J(B_n) +_F \sum_{n>m}^F a_{n,i} \\
&= \sum_{0<n\le m}^F B_n +_F \sum_{\substack{n\le m\\ \|J\|+n>m}}^F v_J w_J(B_n) +_F \sum_{n>m}^F a_{n,i}.
\end{aligned}$$

which completes the inductive step and the proof. □

Recall now the coproduct in $BP_*(BP)$ given by 4.1.18(e), i.e.,

$$\sum_{i\ge 0} \log(\Delta(t_i)) = \sum_{i,j\ge 0} \log(t_i \otimes t_j^{p^i}),$$

which can be rewritten as

$$\sum_{i\geq 0}^{F} \Delta(t_i) = \sum_{i,j\geq 0}^{F} t_i \otimes t_j^{p^i} \tag{4.3.12}$$

To apply 4.3.11, let $M_n = \{t_i \otimes t_{n-i}^{p^i} \mid 0 \leq i \leq n\}$ (M here stands for Milnor since these terms are essentially Milnor's coproduct 3.1.1) and let

$$\Delta_n = M_n \cup \bigcup_{|J|>0} \{v_J w_J(\Delta_{n-\|J\|})\}.$$

Then we get from 4.3.11 and 4.3.12

4.3.13. THEOREM. *With notation as above,*

$$\Delta(t_n) = w_\phi(\Delta_n) \in BP_*(BP) \otimes_{BP_*} BP_*(BP). \qquad \square$$

For future reference we make

4.3.14. DEFINITION. *In $BP_*(BP) \otimes_{BP_*} BP_*(BP)$ let $b_{i,j} = w_{(j+1)}(\Delta_i)$.* $\square$

For example,

$$b_{1,j} = -\frac{1}{p - p^{(p^{j+1})}} \sum_{0<i<p^{j+1}} \binom{p^{j+1}}{i} t_1^i \otimes t_1^{p^{j+1}-i}.$$

This $b_{i,j}$ can be regarded as an element of degree 2 in the cobar complex (A1.2.11) $C(BP_*)$. It will figure in subsequent calculations and we will give a formula for its coboundary (4.3.22) below.

If we reduce modulo I_n, 4.3.13 simplifies as follows.

4.3.15. COROLLARY. *In $BP_*(BP) \otimes_{BP_*} BP_*(BP)/I_n$ for $k \leq 2n$*

$$\Delta(t_k) = \sum_{0\leq i\leq k} t_i \otimes t_{k-i}^{p^i} + \sum_{0\leq j\leq k-n-1} v_{n+j} b_{k-n-j,n+j-1}. \qquad \square$$

Now we will simplify the right unit formula 4.3.1. First we need a lemma.

4.3.16. LEMMA. *In $BP_*(BP)$,*

$$\sum_{i,|I|\geq 0}^{F} [(-1)^{|I|}](t_I t_i^{p^{\|I\|}}) = \sum_{i,|I|\geq 0}^{F} [(-1)^{|I|}](t_i (t_I)^{p^i})$$

(*It can be shown that for $p > 2$, $[-1](x) = -x$ for any p-typical formal group law. $[n](x)$ is defined in* A2.1.19.)

PROOF. In the first expression, for each $I = (i_1, i_2, \ldots, i_n)$ with $n > 0$, the expression t_I appears twice: once as $t_I t_0$ and once as $t_{I'}(t_{i_n})^{p^{\|I'\|}}$ where $I' = (i_1, \ldots, i_{n-1})$. These two terms have opposite formal sign and hence cancel, leaving 1 as the value of the first expression. The argument for the second expression is similar. $\square$

Now we need to use the conjugate formal group law $c(F)$ over $BP_*(BP)$, defined by the homomorphism $\eta_R\colon BP_* \to BP_*(BP)$. Its logarithm is

$$\log_{c(F)}(x) = \sum_{i\geq 0} \eta_R(\lambda_i) x^{p^i} = \sum_{i,j\geq 0} \lambda_i t_j^{p^i} x^{p^{i+j}}.$$

An analog of 4.3.9 holds for $c(F)$ with v_I replaced by $\eta_R(v_I)$.

The last equation in the proof of A2.2.5 reads

$$\sum \lambda_i v_j^{p^i} t_k^{p^{i+j}} = \sum \lambda_i t_j^{p^i} \eta_R(v_k)^{p^{i+j}} = \sum \eta_R(\lambda_i)\eta_R(v_j)^{p^i}$$

while 4.3.16 gives

$$\sum \lambda_i = \sum (-1)^{|K|} \lambda_i t_j^{p^i} t_K^{p^{i+j}}.$$

Combining these and reindexing gives

$$\sum (-1)^{|J|} \eta_R(\lambda_i)(t_J(v_k t_l^{p^k})^{p^{\|J\|}})^{p^i} = \sum \eta_R(\lambda_i)\eta_R(v_j)^{p^i},$$

which is equivalent to

$$\sum_{i\geq 0}^{c(F)} \eta_R(v_i) = \sum_{|I|,j,k\geq 0}^{c(F)} [(-1)^{|I|}]_{c(F)} (t_I(v_j t_k^{p^j})^{p^{\|I\|}}). \tag{4.3.17}$$

We now define finite subsets of $BP_*(BP)$ for $n > 0$

$$N_n = \bigcup_{\|I\|+i+j=n} \left\{ (-1)^{|I|} t^I (v_i t_j^{p^i})^{p^{\|I\|}} \right\}$$
$$R_n = N_n \cup \bigcup_{\substack{\|J\|=i \\ 0<i<n}} \{\eta_R(v_J) w_J(R_{n-i})\}.$$

Then we get

4.3.18. Theorem. *In $BP_*(BP)$, we have $\eta_R(v_n) = w_\phi(R_n)$.* □

4.3.19. Definition. *In $BP_*(BP)$, $c_{i,J} = w_J(R_i)$. For $J = (j)$ this will be written as $c_{i,j}$.*

Again we can simplify further by reducing modulo I_n.

4.3.20. Corollary. *In $BP_*(BP)/I_n$ for $0 < k \leq 2n$,*

$$\sum_{0\leq i\leq k} v_{n+i} t_{k-i}^{p^{n+i}} - \eta_R(v_{n+k-i})^{p^i} t_i = \sum_{0\leq j\leq k-n-1} v_{n+j} c_{k-j,n+j}.$$

(Note that the right-hand side vanishes if $k \leq n$.) □

4.3.21. COROLLARY. *In* $BP_*(BP)/I_n$,

$$\begin{aligned}
\eta_R(v_{n+1}) &= v_{n+1} + v_n t_1^{p^n} - v_n^p t_1 \\
&\qquad\qquad \textit{for } n \geq 1; \\
\eta_R(v_{n+2}) &= v_{n+2} + v_{n+1} t_1^{p^{n+1}} + v_n t_2^{p^n} - v_{n+1}^p t_1 - v_n^{p^2} t_2 \\
&\quad + v_n^{p^2} t_1^{1+p} - v_n^p t_1^{1+p^{n+1}} \\
&\qquad\qquad \textit{for } n \geq 2; \\
\eta_R(v_{n+3}) &= v_{n+3} + v_{n+2} t_1^{p^{n+2}} + v_{n+1} t_2^{p^{n+1}} + v_n t_3^{p^n} - v_{n+2}^p t_1 \\
&\quad - v_{n+1}^{p^2} t_2 - v_n^{p^3} t_3 - v_{n+1}^p t_1^{1+p^{n+2}} - v_n^p t_1 t_2^{p^{n+1}} \\
&\quad - v_n^{p^2} t_2 t_1^{p^{n+2}} + v_{n+1}^{p^2} t_1^{1+p} + v_n^{p^3} t_1 t_2^p + v_n^{p^3} t_2 t_1^{p^2} \\
&\quad + v_n^{p^2} t_1^{1+p+p^{n+2}} - v_n^{p^3} t_1^{1+p+p^2} \\
&\qquad\qquad \textit{for } n \geq 3; \\
\eta_R(v_3) &= v_3 + v_2 t_1^{p^2} + v_1 t_2^p - v_2^p t_1 - v_1^{p^2} t_2 - v_1^p t_1^{1+p^2} \\
&\quad + v_1^{p^2} t_1^{1+p} + v_1 w_1(v_2, v_1 t_1^p, -v_1^p t_1) \\
&\qquad\qquad \textit{for } n = 1,\ p > 2\ (\textit{add } v_1^5 t_1^2 \textit{ for } p = 2) \\
\textit{and} \qquad \eta_R(v_5) &= v_5 + v_4 t_1^{p^4} + v_3 t_2^{p^3} + v_2 t_3^{p^2} - v_4^p t_1 - v_3^{p^2} t_2 - v_2^{p^3} t_3 \\
&\quad - v_3^p t_1^{1+p^4} - v_2^p t_1 t_2^{p^3} - v_2^{p^2} t_1^{p^4} t_2 + v_3^{p^2} t_1^{1+p} + v_2^{p^3} t_1 t_2^p \\
&\quad + v_2^{p^3} t_1^{p^2} t_2 + v_2^{p^2} t_1^{1+p+p^4} - v_2^{p^3} t_1^{1+p+p^2} \\
&\quad + v^2 w_1(v_3, v_2 t_1^{p^2}, -v_2^p t_1^p)^p \\
&\qquad\qquad \textit{for } n = 2,\ p > 2\ (\textit{add } v_2^9 t_1^4 \textit{ for } p = 2). \qquad \square
\end{aligned}$$

Now we will calculate the coboundaries of $b_{i,j}$ (4.3.14) and $c_{i,j}$ (4.3.19) in the cobar complex $C(BP_*/I_n)$ (A1.2.11).

4.3.22. THEOREM. *In* $C(BP_*/I_n)$ *for* $0 < i \leq n$ *and* $0 \leq j$

(a) $d(b_{i,j}) = \sum_{0<k<i} b_{k,j} \otimes t_{i-k}^{p^{k+j+1}} - t_k^{p^{1+j}} \otimes b_{i-k,k+j}$ *and* (b) $d(c_{n+i,j+1}) = \sum_{0\leq k<i} v_{n+k}^{p^{1+j}} b_{i-k,n+k+j} - v_{n+k}^{p^{i+1+j-k}} b_{i-k,j}$.

PROOF. (a) It suffices to assume $i = n$. Recall that in $C(BP_*/I_n)$, $d(t_i) = t_i \otimes 1 + 1 \otimes t_i - \Delta(t_i)$ and $d(v_{n+i}) = n_R(v_{n+i}) - v_{n+i}$. $\Delta(t_{2n}) - 1 \otimes t_{2n} - t_{2n} \otimes 1$, given by 4.3.13, is a coboundary and hence a cocycle. Calculating its coboundary term by term using 4.3.13 and 4.3.17 will give the desired formula for $d(b_{n,n-1})$ and the result will follow. The details are straightforward and left to the reader.

For (b) we assume $i = n$ if $i + n$ is even and $i = n - 1$ if $i + n$ is odd. Then we use the fact that $d(v_{2n+i})$ is a cocycle to get the desired formula, as in the proof of (a). $\square$

Now we will construct an increasing filtration on the Hopf algebroid $BP_*(BP)/I_n$. We will use it in Section 6.3.

To do this we first define integers $d_{n,i}$ by

$$d_{n,i} = \begin{cases} 0 & \text{for } i \leq 0 \\ \max(i, pd_{n,i-n}) & \text{for } i > 0. \end{cases}$$

We then set $\deg t_i^{p^j} = \deg v_{n+i}^{p^j} = d_{n,i}$ for $i,\ j \geq 0$. The subgroups $F_r \subset BP_*(BP)/I_n$ are defined to be the smallest possible subgroups satisfying the above conditions.

The associated graded algebra $E_0BP_*(BP)/I_n$ is defined by $E_0^iBP_*(BP)/I_n = F_i/F_{i-1}$. Its structure is given by

4.3.23. PROPOSITION.

$$E_0BP_*(BP)/I_n = T(t_{i,j}, v_{n+i,j}\colon i > 0,\ j \geq 0),$$

where $t_{i,j}$ *and* $v_{n+1,j}$ *are elements corresponding to* $t_i^{p^j}$ *and* $v_{n+1}^{p^j}$*, respectively,* $T(x) = R[x]/(x^p)$ *and* $R = \mathbf{Z}(p)[v_n]$. □

4.3.24. THEOREM. *With the above filtration,* $BP_*(BP)/I_n$ *is a filtered Hopf algebroid, and* $E_0BP_*(BP)/I_n$ *is a Hopf algebroid.*

PROOF. For a set of elements X in $B_*(BP)/I_n$ or $BP_*(BP)\otimes_{BP_*}BP_*(BP)/I_n$, let deg X be the smallest integer i such that $X \subset F_i$. It suffices to show then that $\deg \Delta_i = \deg R_{n+i} = d_{n,i}$. We do this by induction on i, the assertion being obvious for $i = 1$.

First note that

$$d_{n,a+b} \geq d_{n,a} + d_{n,b} \tag{4.3.25}$$

and

$$d_{n,a+bn} \geq p^b d_{n,a}. \tag{4.3.26}$$

It follows from 4.3.25 that $\deg M_i = degN_{n+i} = d_{n,i}$. It remains then to show that for $\|J\| < i$

$$\deg(v_J w_J(\Delta_{i-\|J\|})) \leq d_{n,i} \tag{4.3.27}$$

and

$$\deg(v_J w_J(R_{n+i-\|J\|})) \leq d_{n,i}. \tag{4.3.28}$$

Since

$$\deg w_J(X) \leq p^{|J|}\deg X, \tag{4.3.29}$$

both 4.3.27 and 4.3.28 reduce to showing

$$d_{n,i} \geq \deg v_J + p^{|J|}d_{n,i-\|J\|}. \tag{4.3.30}$$

Now if $v_J \not\equiv 0 \bmod I_n$ we can write

$$J = (n + j_1', m + j_2', \ldots, n + j_l')$$

with $j_t' \geq 0$, so

$$|J| = l, \quad \|J\| = \ln + \sum_{t=1}^{l} j_t', \quad \text{and} \quad \deg v_J = \sum_{t=1}^{l} d_{n,j_t'}.$$

If we set $k = \|J\| - n|J|$, then 4.3.25 implies

$$d_{n,k} \geq \deg v_J. \tag{4.3.31}$$

However, by 4.3.25 and 4.3.26

$$\begin{aligned} d_{n,i} &\geq d_{n,k} + d_{n,i-\|J\|+n|J|} \\ &\geq d_{n,k} + p^{|J|}d_{n,i-\|J\|} \end{aligned}$$

so 4.3.20 follows from 4.3.31. □

We now turn to the Hopf algebroid structure of $E_0BP_*(BP)/I_n$. Let $\overline{M}_i$, $\bar{\Delta}_i$, $\overline{N}_{n+i}$, and $\overline{R}_{n+i}$ denote the associated graded analogs of M_i, Δ_i, N_{n+i}, and R_{n+i}, respectively, with trivial elements deleted. (An element in one of the latter sets will correspond to a trivial element if its degree is less than $d_{n,i}$.) All we have to do is describe these subsets. Let $\bar{t}_I$, $\bar{v}_I$, and $\overline{w}_I(x)$ denote the associated graded elements corresponding to t_I, v_I, and $w_I(x)$, respectively.

4.3.32. LEMMA.

$$\overline{M}_i = \begin{cases} \bigcup\limits_{0\le j\le i} \{t_{j,0}\otimes t_{i-j,j}\} & \textit{for } i \le m \\ \{t_{i,0}\otimes 1, 1\otimes t_{i,0}\} & \textit{for } i > m \end{cases}$$

$$\overline{N}_{n+i} = \begin{cases} \bigcup\limits_{\|I\|+j+k=i} \{(-1)^{|I|}\bar{t}_I v_{j,\|I\|} t_{k,j+\|I\|}\} & \textit{for } i \le m \\ \{v_{n+i,0}, v_n t_{i,n}, -v_n^{p^i} t_{i,0}\} & \textit{for } i > m \end{cases}$$

where $m = pn/(p-1)$.

PROOF. This follows from the fact that equality holds in 4.3.25 if $a+b\le m$. $\square$

4.3.33. LEMMA.

$$\bar{\Delta}_i = \begin{cases} \overline{M}_i & \textit{for } i < m \\ \overline{M}_i \cup \{v_n \overline{w_1^{p^{n-1}}(M_{i-n})}\} & \textit{for } i = m \\ \overline{M}_i \cup \bigcup\limits_{\substack{\|J\|=n|J| \\ 0<\|J\|<i \\ i-\|J\|\ge m-n}} \{v_J \overline{w_J(\Delta_{i-n|J|})}\} & \textit{for } i > m \end{cases}$$

$$\overline{R}_{n+i} = \begin{cases} \overline{N}_{n+i} & \textit{for } i < m \\ \overline{N}_{n+i} \cup \{v_n \overline{w_1^{p^{n-1}}(R_{i-n})}\} & \textit{for } i = m \\ \overline{N}_{n+i} \cup \bigcup\limits_{\substack{\|J\|=n|J| \\ 0<\|J\|<i \\ i-\|J\|\ge m-n}} \overline{\{v_J w_J(R_{i-n|J|})\}} & \textit{for } i > m \end{cases}$$

[*Note that the case $i = m$ occurs only if $(p-1)|n$, and that the only J's we need to consider for $i > m$ are those of the form $(n, n, \ldots, n)$.*]

PROOF. We use the observation made in the proof of 4.3.32 along with the fact that equality holds in 4.3.26 if $a \ge m = n$.

Now both $\overline{R}_{n+i}$, and $\bar{\Delta}$, will consist only of the terms associated with those J for which equality holds in 4.3.30. For $i > m$ this can occur only if $\deg v_J = 0$, i.e., if $J = (n, n, \ldots, n)$; the condition $i - \|J\| \ge m - n$ is necessary to ensure that $d_{n,i} = p^{|J|} d_{n,i-n|J|}$. For $i \le m$ we still need $i - \|J\| \ge m - n$. Since $\|J\| \ge n$ in all nontrivial terms, the only possibility is $J = (n)$ when $i = m$. $\square$

Now let $\Delta_{i,j}$ and $R_{n+i,j}$ be the subsets obtained from Δ_i and R_{n+i}, respectively, by raising each element to the p^jth power. The corresponding subsets $\bar{\Delta}_{i,j}$ and $\overline{R}_{n+i,j}$ of the appropriate associated graded objects are related to $\bar{\Delta}_i$ and $\overline{R}_{n+i}$ in an obvious way. Note that

$$\begin{aligned} \overline{w_J(\Delta_i)} &= \overline{w_{|J|}(\Delta_{i,\|J\|-|J|})} \\ &= w_{|J|}(\bar{\Delta}_{i,\|J\|-|J|}). \end{aligned}$$

4.3.34. THEOREM. *With $\bar{\Delta}_{i,j}$ and $\overline{R}_{n+i,j}$ as above, the Hopf algebroid structure of $E_0BP_*(BP)/I_n$, is given by*

$$\Delta(t_{i,j}) = w_0(\bar{\Delta}_{i,j})$$
$$\eta_R(v_{n+i,j}) = w_0(\overline{R}_{n+i,j}). \qquad \square$$

None of the $t_{i,j}$ for $i > 1$ are primitive, so we could not get a Hopf algebroid with deg $t_{i,j} < d_{n,i}$ once we have set deg $t_{1,j} = 1$.

Note finally that the structure of $E_0BP_*(BP)/I_n$ depends in a very essential way on the prime p.

Theorem 4.3.34 implies that $E_0BP_*(BP)/I_n$ is cocommutative for $n = 1$ and $p > 2$. For any n and p we can use this filtration to construct a spectral sequence as in A1.3.9. The cocommutativity in the case above permits a complete, explicit determination of the E_2-term, and hence a very promising beginning for a computation of $\mathrm{Ext}_{BP_*(BP)}(BP_*, BP/I_1)$. However, after investigating this method thoroughly we found the E_2-term to be inconveniently large and devised more efficient strategies for computing Ext, which will be described in Chapter 7. Conceivably the approach at hand could be more useful if one used a machine to do the bookkeeping. We leave the details to the interested reader.

PROOF OF 4.3.8. We will prove (a) and (b) simultaneously by induction on $m = |K|$. If $K' = (1 + k_1, k_2, \dots, k_m)$ then it follows from (b) that

$$w_{K'} \equiv w_K^p \qquad \mathrm{mod}\ (p).$$

Let $K'' = (k_1 + k_2, k_3, \dots, k_m)$ and $K''' = (k_2, k_3, \dots, k_m)$. Then by the inductive hypothesis $w_{K''}$ and $w_{K'''}$ exist with

$$w_{K''} \equiv w_{K'''}^{a_1} \qquad \mathrm{mod}\ (p),$$

where $a_1 = p^{k_1}$. Since $\|K\| = \|K''\|$ we have

$$\sum_{IJ=K} \frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}} = \sum_{IJ=K''} \frac{\Pi(K'')}{\Pi(I)} w_J^{p^{\|I\|}}.$$

Expanding both sides partly we get

$$\Pi(K)w_K + \frac{\Pi(K)}{\Pi(k_1)} w_{K'''}^{a_1} + \sum_{\substack{|I|\geq 2\\ IJ=K}} \frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}}$$
$$= \Pi(K'')w_{K''} + \sum_{\substack{|I|\geq 1\\ IJ=K''}} \frac{\Pi(K'')}{\Pi(I)} w_J^{p^{\|I\|}}.$$

Note that the same $w_J^{p^{\|I\|}}$ occur on both sides, and one can use the definition of $\Pi(k)$ to show that they have the same coefficients so the sums cancel. The remaining terms give

$$\Pi(K)\left(w_K + \frac{w_{K'''}^{a_1}}{\Pi(k_1)}\right) = \Pi(K'')w_{K''}.$$

Since $\Pi(k_1) \equiv p \mod (p^2)$ and $w_{K''} = w_{k'''}^{a_1} \mod (p)$, we get an integral expression when we solve for w_K.

This completes the proof of (a).

For (b) we have

$$\sum x_t^{p^{\|K\|}} = \Pi(K)w_K + \sum_{\substack{IJ=K\\|I|>0}} \frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}}.$$

Since $\Pi(K) \equiv p^{|K|} \mod (p^{1+|K|})$ and $\Pi(I) \equiv p^{|I|} \mod (p^{1+|I|})$, we get $\Pi(K)/\Pi(I) \equiv p^{|J|} \mod (p^{1+|J|})$. By definition

$$\begin{aligned}\frac{\Pi(K)}{\Pi(I)} &= \Pi(\|K\|)\Pi(\|K\| - j_{|J|}) \cdots \Pi(\|I\| + j_1)\\ &= (p - p^{p^{\|K\|}})(p - p^{p^{\|K\|-j_{|J|}}}) \cdots (p - p^{p^{\|I\|+j_1}})\\ &\equiv p^{|J|} \qquad \mod (p^{|J|-1+p^{\|I\|+j_1}})\\ &\equiv p^{|J|} \qquad \mod (p^{|K|+1}) \quad \text{since}\end{aligned}$$

$$\begin{aligned}|J| - 1 + p^{\|I\|+j_1} &\geq |J| - 1 + \|I\| + 2\\ &\geq |K| + 1.\end{aligned}$$

By the inductive hypothesis

$$w_J \equiv w_{|J|}^{p^{\|J\|-|J|}} \mod (p)$$

so $w_J^{p^{\|I\|}} \equiv w_{|J|}^{p^{\|K\|-|J|}} \mod (p^{1+\|I\|})$. Combining these two statements gives

$$\frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}} \equiv w_{|J|}^{\|K\|-|J|} \mod (p^{1+|K|}).$$

Hence the defining equation for w_K becomes

$$\sum x_t^{p^{\|K\|}} \equiv p^{|K|}w_K + \sum_{\substack{IJ=K\\|I|>0}} p^{|J|} w_{|J|}^{p^{\|K\|-|J|}} \mod (p^{1+|K|}).$$

Let $n = \|K\| - |K|$. Substituting $x_t^{p^n}$ for x_t in 4.3.5 gives

$$\sum x_t^{p^{\|K\|}} = p^{\|K\|} w_{|K|}(x_t^{p^n}) + \sum_{0\leq j<|K|} p^j w_j^{p^{|K|-j}}(x_t^{p^n}).$$

Since $w_j(x_t^{p^n}) \equiv w_j^{p^n} \mod (p)$,

$$w_j^{p^{|K|-j}}(x_t^{p^n}) \equiv w_j^{p^{n+|K|-j}} \mod (p^{1+|K|-j}),$$

so we get

$$\sum x_t^{p^{\|K\|}} \equiv p^{|K|} w_{|K|}(x_t^{p^n}) + \sum_{0\leq j<|K|} p^j w_j^{p^{\|K\|-j}} \mod (p^{1+|K|}).$$

Comparing this with the defining equation above gives

$$w_K \equiv w_{|K|}(x_t^{p^n}) \equiv w_{|K|}^{p^{\|K\|-|K|}} \mod (p)$$

as claimed. $\square$

4. Beginning Calculations with the Adams–Novikov Spectral Sequence

In this section we introduce the main object of interest in this book, the Adams–Novikov spectral sequence, i.e., the BP_*-Adams spectral sequence (2.2.4). There is a different BP_*-theory and hence a different Adams–Novikov spectral sequence for each prime p. One could consider the MU_*-Adams spectral sequence (as Novikov [**?**] did originally) and capture all primes at once, but there is no apparent advantage in doing so. Stable homotopy theory is a very local (in the arithmetic sense) subject. Even though the structure formulas for $BP_*(BP)$ are more complicated than those of $MU_*(MU)$ (both are given in Section 1) the former are easier to work with once one gets used to them. (Admittedly this adjustment has been difficult. We hope this book, in particular the results of Section 3, will make it easier.)

The Adams–Novikov spectral sequence was first constructed by Novikov [**?**] and the first systematic calculations at the primes 2 and 3 were done by Zahler [**?**]. In this section we will calculate the E_2-term for $t-s \leq 25$ at $p=2$ and for $t-s \leq (p^2+p)q$ for $p>2$, where $q=2p-2$. In each case we will compute all the differentials and extensions and thereby find $\pi(S^0)$ through the indicated range. At $p=2$ this will be done by purely algebraic methods based on a comparison of the Adams–Novikov spectral sequence and Adams spectral sequence E_2-terms. At odd primes we will see that the Adams spectral sequence E_2-term sheds no light on the Adams–Novikov spectral sequence and one must compute differentials by other means. Fortunately, there is only one differential in this range and it is given by Toda [**?, ?**]. The more extensive calculations of later chapters will show that in a much larger range all nontrivial differentials follow formally from the first one.

In Section 2.2 we developed the machinery necessary to set up the Adams–Novikov spectral sequence and we have

4.4.1. Adams–Novikov spectral sequence Theorem (Novikov [**?**]). *For any spectrum X there is a natural spectral sequence $E_*^{**}(X)$ with $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$ such that*

(a) $E_2 = \mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(X))$ *and*

(b) *if X is connective and p-local then E_∞^{**} is the bigraded group associated with the following filtration of $\pi_*(X)$: a map $f\colon S^n \to X$ has filtration $\geq s$ if it can be factored with s maps each of which becomes trivial after smashing the target with BP.* □

The fact that $BP_*(BP)$, unlike the Steenrod algebra, is concentrated in dimensions divisible by $q=2p-2$ has the following consequence.

4.4.2. Proposition: Sparseness. Suppose $BP_*(X)$ is concentrated in dimensions divisible by $q=2p-2$ (e.g., $X=S^0$). Then in the Adams–Novikov spectral sequence for X, $E_r^{s,t}=0$ for all r and s except when t is divisible by q. Consequently d_r is nontrivial only if $r \equiv 1 \mod (q)$ and $E_{mq+2}^{**} = E_{mq+q+1}^{**}$ for all $m \geq 0$. □

For $p=2$ this leads to the "checkerboard phenomenon": $E_r^{s,t}=0$ if $t-s$ and s do not have the same parity.

To compare the Adams spectral sequence and Adams–Novikov spectral sequence we will construct two trigraded spectral sequences converging to the Adams spectral sequence and Adams–Novikov spectral sequence E_2-terms. The former is a

Cartan–Eilenberg spectral sequence (A1.3.15) for a certain Hopf algebra extension involving the Steenrod algebra, while the latter arises from a filtration of $BP_*(BP)$ (A1.3.9). The point is that up to reindexing these two spectral sequences have the same E_2-term. Moreover, at odd primes (but not at $p = 2$) the former spectral sequence collapses, which means that the Adams spectral sequence E_2-term when suitably reindexed is a trigraded E_2-term of a spectral sequence converging to the Adams–Novikov spectral sequence E_2-term. It is reasonable to expect there to be a close relation between differentials in the trigraded filtration spectral sequence, which Miller [**?**] calls the "algebraic Novikov spectral sequence," and the differentials in the Adams spectral sequence. Miller [**?**] has shown that many Adams d_2's can be accounted for in this way. At any rate this indicates that at odd primes the Adams spectral sequence E_2-term has less information than the Adams–Novikov spectral sequence E_2-term.

To be more specific, recall (3.1.1) that the dual Steenrod algebra A_* as an algebra is

$$A_* = \begin{cases} P(\xi_1, \xi_2, \dots) & \text{with} \quad \dim \xi_i = 2^i - 1 \text{ for } p = 2 \\ E(\tau_0, \tau_1, \dots) \otimes P(\xi_1, \xi_2, \dots) & \text{with} \quad \dim \tau_i = 2p^i - 1 \quad \text{and} \\ & \qquad \dim \xi_i = 2p^i - 2 \quad \text{for } p > 2. \end{cases}$$

Let $P_* \subset A_*$ be $P(\xi_1^2, \xi_2^2, \dots)$ for $p = 2$ and $P(\xi_1, \xi_2, \dots)$ for $p > 2$, and let $E_* = A_* \otimes_{P_*} \mathbf{Z}/(p)$, i.e. $E_* = E(\xi_1, \xi_2, \dots)$ for $p = 2$ and $E_* = E(\tau_0, \tau_1, \dots)$ for $p > 2$. Then we have

4.4.3. Theorem. *With notation as above* (a)

$$\operatorname{Ext}_{E_*}(\mathbf{Z}/(p), \mathbf{Z}/(p)) = P(a_0, a_1, \dots)$$

with $a_i \in \operatorname{Ext}^{1,2p^i-1}$ *represented in the cobar complex* (A1.2.11) *by* $[\xi_i]$ *for* $p = 2$ *and* $[\tau_i]$ *for* $p > 2$,

(b) $P_* \to A_* \to E_*$ *is an extension of Hopf algebras* (A1.1.15) *and there is a Cartan–Eilenberg spectral sequence* (A1.3.15) *converging to* $\operatorname{Ext}_{A_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ *with*

$$E_2^{s_1,s_2,t} = \operatorname{Ext}_{P_*}^{s_1}(\mathbf{Z}/(p), \operatorname{Ext}_{E_*}^{s_2,t}(\mathbf{Z}/(p), \mathbf{Z}(p)))$$

and

$$d_r \colon E_r^{s_1,s_2,t} \to E_r^{s_1+r,s_2-r+1,t},$$

(c) *the* P_**-coaction on* $\operatorname{Ext}_{E_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ *is given by*

$$\psi(a_n) = \begin{cases} \sum_i \xi_{n-i}^{2^{i+1}} \otimes a_i & \text{for } p = 2 \\ \sum_i \xi_{n-i}^{p^i} \otimes a_i & \text{for } p > 2, \quad \text{and} \end{cases}$$

(*d) for* $p > 2$ *the Cartan–Eilenberg spectral sequence collapses from* E_2 *with no nontrivial extensions.*

Proof. Everything is straightforward but (d). We can give A_* a second grading based on the number of τ_i's which are preserved by both the product and the coproduct (they do not preserve it at $p = 2$). This translates to a grading of Ext by the number of a_i's which must be respected by the differentials, so the spectral sequence collapses. $\square$

For the algebraic Novikov spectral sequence, let $I = (p, v_1, v_2, \dots) \subset BP_*$. We filter $BP_*(BP)$ by powers of I and study the resulting spectral sequence (A1.3.9).

4.4.4. Algebraic Novikov SS Theorem (Novikov [**?**], Miller [**?**]). There is a spectral sequence converging to $\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*)$ with

$$E_1^{s,m,t} = \mathrm{Ext}_{P_*}^{s,t}(\mathbf{Z}/(p), I^m/I^{m+1})$$

and $d_r\colon E_r^{s,m,t} \to E_r^{s+1,r+m,t}$. The E_1^{***} of this spectral sequence coincides with the E_2^{***} of 4.4.3.

Proof. A1.3.9 gives a spectral sequence with

$$E_1 = \mathrm{Ext}_{E_0BP_*(BP)}(E_0BP_*, E_0BP_*).$$

Now we have $BP_*(BP)/I = E_0BP_*(BP) \otimes_{E_0BP_*} \mathbf{Z}/(p) = P_*$. We apply the change-of-rings isomorphism A1.3.12 to the Hopf algebroid map $(E_0BP_*, E_0BP_*(BP)) \to (\mathbf{Z}/(p), P_*)$ and get

$$\begin{aligned}
&\mathrm{Ext}_{P_*}(\mathbf{Z}/(p), E_0BP_*)\\
&\qquad = \mathrm{Ext}_{E_0BP_*(BP)}(E_0BP_*, (E_0BP_*(BP) \otimes_{E_0BP_*} \mathbf{Z}/(p)) \,\square_{P_*}\, E_0BP_*)\\
&\qquad = \mathrm{Ext}_{E_0BP_*(BP)}(E_0BP_*, P_* \,\square_{P_*}\, E_0BP_*)\\
&\qquad = \mathrm{Ext}_{E_0BP_*(BP)}(E_0BP_*, E_0BP_*).
\end{aligned}$$

The second statement follows from the fact that $E_0BP_* = \mathrm{Ext}_{E_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$. $\square$

In order to use this spectral sequence we need to know its E_1-term. For $p > 2$, 4.4.3(d) implies that it is the cohomology of the Steenrod algebra, i.e., the classical Adams E_2-term suitably reindexed. This has been calculated in various ranges by May [**?**], and Liulevicius [**?**], but we will compute it here from scratch. Theorem 4.4.3(d) fails for $p = 2$ so we need another method, outlined in Miller [**?**] and used extensively by Aubry [**?**].

We start with $\mathrm{Ext}_{P_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$. For $p = 2$ we have $\mathrm{Ext}_{A_*}^{s,t}(\mathbf{Z}/(2), \mathbf{Z}/(2)) = \mathrm{Ext}_{P_*}^{s,2t}(\mathbf{Z}/(2), \mathbf{Z}(2))$, so the latter is known if we know the former through half the range of dimensions being considered. For $p > 2$ we will make the necessary calculation below.

Then we compute $\mathrm{Ext}_{P_*}(\mathbf{Z}/(p), E_0BP_*/I_n)$, by downward induction on n. To start the induction, observe that through any given finite range of dimensions $BP_*/I_n \simeq \mathbf{Z}/(p)$ for large enough n. For the inductive step we use the short exact sequence

$$0 \to \Sigma^{\dim v_n} E_0BP_*/I_n \to E_0BP_*/I_n \to E_0BP_*/I_{n+1} \to 0,$$

which leads to a Bockstein spectral sequence of the form

$$P(a_n) \otimes \mathrm{Ext}_{P_*}(\mathbf{Z}/(p), E_0BP_*/I_{n+1}) \Rightarrow \mathrm{Ext}_{P_*}(\mathbf{Z}/(p), E_0BP_*/I_n). \tag{4.4.5}$$

The method we will use in this section differs only slightly from the above. We will compute the groups $\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*/I_n)$ by downward induction on n; these will be abbreviated by $\mathrm{Ext}(BP_*/I_n)$. To start the induction we note that $\mathrm{Ext}^{s,t}(BP_*/I_n) = \mathrm{Ext}_{P_*}^{s,t}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ for $t < 2(p^n - 1)$. For the inductive step we analyze the long exact sequence of Ext groups induced by the short exact sequence

$$0 \to \Sigma^{\dim v_n} BP_*/I_n \to BP_*/I_n \to BP_*/I_{n+1} \to 0, \tag{4.4.6}$$

either directly or via a Bockstein spectral sequence similar to 4.4.5. The long exact sequence and Bockstein spectral sequence are related as follows. The connecting homomorphism in the former has the form

$$\delta_n\colon \operatorname{Ext}^s(BP_*/I_{n+1}) \to \operatorname{Ext}^{s+1}(\Sigma^{2p^n-2}BP_*/I_n).$$

The target is a module over $\operatorname{Ext}^0(BP_*/I_n)$ which is $\mathbf{Z}/(p)[v_n]$ for $n>0$ and $\mathbf{Z}_{(p)}$ for $n=0$ by 4.3.2. Assume for simplicity that $n>0$. For each $x\in\operatorname{Ext}(BP_*/I_{n+1})$ there is a maximal k such that $\delta_n(x)=v_n^k y$, i.e., such that $y\in\operatorname{Ext}(BP_*/I_n)$ is not divisible by v_n. (This y is not unique but is only determined modulo elements annihilated by v_n^k.) Let $\bar{y}\in\operatorname{Ext}(BP_*/I_{n+1})$ denote the image of y under the reduction map $BP_*/I_n\to BP_*/I_{n+1}$ Then in the Bockstein spectral sequence there is a differential $d_{1+k}(x)=a_n^{1+k}\bar{y}$.

Now we will start the process by computing $\operatorname{Ext}_{P_*}^{s,t}(\mathbf{Z}/(p),\mathbf{Z}/(p))$ for $p>2$ and $t<(p^2+p+1)q$. In this range we have $P_*=P(\xi_1,\xi_2)$. We will apply the Cartan–Eilenberg spectral sequence (A1.3.15) to the Hopf algebra extension

$$P(\xi_1)\to P(\xi_1,\xi_2)\to P(\xi_2). \tag{4.4.7}$$

The E_2-term is $\operatorname{Ext}_{P(\xi_1)}(\mathbf{Z}/(p),\operatorname{Ext}_{P(\xi_2)}(\mathbf{Z}/(p),\mathbf{Z}/(p)))$. The extension is cocentral (A1.1.15) so we have

$$E_2=\operatorname{Ext}_{P(\xi_1)}(\mathbf{Z}/(p),\mathbf{Z}/(p))\otimes\operatorname{Ext}_{P(\xi_2)}(\mathbf{Z}/(p),\mathbf{Z}(p)).$$

By a routine calculation this is in our range of dimensions

$$E(h_{10},h_{11},h_{12},h_{20},h_{21})\otimes P(b_{10},b_{11},b_{20})$$

with

$$h_{i,j}\in\operatorname{Ext}_{P(\xi_i)}^{1,2p^j(p^i-1)}\quad\text{and}\quad b_{i,j}\in\operatorname{Ext}_{P(\xi_i)}^{2,2p^{j+1}(p^i-1)}.$$

The differentials are (up to sign) $d_2(h_{2,j})=h_{1,j}h_{1,j+1}$ and $d_3(b_{20})=h_{12}b_{10}-h_{11}b_{11}$ [compare 4.3.22(a)]. The result is

4.4.8. THEOREM. *For $p>2$ and $t<(p^2+p+1)q$, $\operatorname{Ext}_{P_*}^{s,t}(\mathbf{Z}/(p),\mathbf{Z}/(p))$ is a free module over $P(b_{10})$ on the following 10 generators: 1, h_{10}, h_{11}, $g_0=\langle h_{11},h_{10},h_{10}\rangle$, $k_0=\langle h_{11},h_{11},h_{10}\rangle$, $h_{10}k_0=\pm h_{11}g_0$, h_{12}, $h_{10}h_{12}$, b_{11}, and $h_{10}b_{11}$. There is a multiplicative relation $h_{11}b_{11}=h_{12}b_{10}$ and (for $p=3$) $h_{11}k_0=\pm h_{10}b_{11}$.* □

The extra relation for $p=3$ follows easily from A1.4.6. For $p>3$ there is a corresponding Massey product relation $\langle k_0,h_{11},\dots,h_{11}\rangle=h_{10}b_{11}$ up to a nonzero scalar, where there are $p-2$ factors h_{11}.

The alert reader may observe that the restriction $t<(p^2+p+1)q$ is too severe to give us $\operatorname{Ext}^{s,t}$ for $t-s<(p^2+p)q$ because there are elements in this range with $s>q$, e.g., b_{10}^p. However, one sees easily that in a larger range all elements with $s>q$ are divisible by b_{10} and this division gets us back into the range $t<(p^2+p+1)q$. One could make this more precise, derive some vanishing lines, and prove the following result.

4.4.9. THEOREM. *Let $p>2$.*
(a) $\operatorname{Ext}_{P_*}^{s,t}(\mathbf{Z}/(p),\mathbf{Z}/(p))=0$ *for $t-s<f(s)$ where*

$$f(s)=\begin{cases}(p^2-p-1)s & \text{for } s \text{ even}\\ 2p-3+(p^2-p-1)(s-1) & \text{for } s \text{ odd.}\end{cases}$$

(b) *Let* $R_* = P_*/(\xi_1, \xi_2)$. *Then* $\mathrm{Ext}^{s,t}_{R_*}(\mathbf{Z}/(p), \mathbf{Z}(p)) = 0$ *for* $t - s < g(s)$ *where*

$$g(s) = \begin{cases} (p^4 - p - 1)s & \text{for } s \text{ even} \\ 2p^3 - 3 + (p^4 - p - 1)(s - 1) & \text{for } s \text{ odd.} \end{cases}$$

(c) *The map* $P(\xi_1, \xi_2) \to P_*$ *induces an epimorphism in* $\mathrm{Ext}^{s,t}$ *for* $(t-s) < h(s)$ *and an isomorphism for* $(t - s) < h(s - 1) - 1$, *where*

$$\begin{aligned} h(s) &= 2p^3 - 3 + f(s - 1) \\ &= \begin{cases} 2p^3 - 3 + (p^2 - p - 1)(s - 1) & \text{for } s \text{ odd} \\ 2p^3 + 2p - 6 + (p^2 - p - 1)(s - 2) & \text{for } s \text{ even.} \end{cases} \end{aligned} \qquad \square$$

This result is far more than we need, and we leave the details to the interested reader.

Now we start feeding in the generators v_n inductively. In our range 4.4.8 gives us $\mathrm{Ext}(BP_*/I_3)$. Each of the specified generators is easily seen to come from a cocycle in the cobar complex $C(BP_*/I_2)$ so we have

$$\mathrm{Ext}(BP_*/I_2) = \mathrm{Ext}(BP_*/I_3) \otimes P(v_2),$$

i.e., the Bockstein spectral sequence collapses in our range.

The passage to $\mathrm{Ext}(BP_*/I_1)$ is far more complicated. The following formulas in $C(BP_*/I_1)$ are relevant.

(4.4.10) $$\text{(a)} \quad d(v_2) = v_1 t_1^p - v_1^p t_1$$

and

$$\text{(b)} \quad d(t_2) = -t_1|t_1^p - v_1 b_{10}.$$

These follow immediately from 4.3.20 and 4.3.15. From 4.4.10(a) we get

(4.4.11) $$\text{(a)} \quad \delta_1(v_2^i) \equiv i v_2^{i-1} h_{11} \mod (v_1)$$

and

$$\text{(b)} \quad \delta_1(v_2^p) \equiv v_1^{p-1} h_{12} \mod (v_1^p).$$

Next we look at elements in $\mathrm{Ext}^1(BP_*/I_2)$. Clearly, h_{10}, h_{11}, and h_{12} are in $\ker \delta_i$ as are $v_2^i h_{11}$ for $i < p - 1$ by the above calculation. This leaves $v_2^i h_{10}$ for $1 \le i \le p - 1$ and $v_2^{p-1} h_{11}$. For the former 4.4.10 gives

$$\begin{aligned} d(v_2^i t_1 + i v_1 v_2^{i-1}(t_1^{1+p} - t_2)) &\equiv i v_1^2 v_2^{i-1} b_{10} \\ &\quad + \binom{i}{2} v_1^2 v_2^{i-1}(t_1^{2p}|t_1 - 2t_1^p|t_2 + 2t_1^p|t_1^{1+p}) \mod (v_1^3). \end{aligned}$$

The expression in the second term is a multiple of k_0, so we have

(4.4.12) $$\delta(v_2^i h_{10}) \equiv i v_1 v_2^{i-1} b_{10} \pm \binom{i}{2} v_1 v_2^{i-2} k_0 \mod (v_1^2).$$

To deal with $v_2^{p-1} h_{11}$ we use 4.4.10(a) to show

$$d\left(\sum_{0<i<p} \frac{1}{p}\binom{p}{i} v_2^{p-i} v_1^{i-1} t_1^{pi}\right) \equiv \sum_{0<i<p} \frac{1}{p}\binom{p}{i} v_1^{p-1} t_1^{pi}|t_1^{p^2-pi} \mod (v_1^p)$$

so

$$\delta_1(v_2^{p-1}h_{11}) = \pm v_1^{p-2}b_{11}. \tag{4.4.13}$$

This is a special case of 4.3.22(b).

Now we move on to the elements in $\mathrm{Ext}^2(BP_*/I_2)$. They are $h_{10}h_{12}$, b_{11}, $v_2^ib_{10}$, $v_2^ig_0$, and $v_2^ik_0$ for suitable i. The first two are clearly in $\ker\delta_1$. Equation 4.4.12 eliminates the need to consider $v_2^ib_{10}$ for $i < p-1$, so that leaves $v_2^{p-1}b_{10}$, $v_2^ig_0$, and $v_2^ik_0$. Routine calculation with 4.4.10 gives

$$\text{(a)}\quad \delta_1(v_2^ig_0) \equiv \pm(v_2^ih_{10}b_{10} \pm iv_2^{i-1}h_{10}k_0) \mod (v_1^2)$$

and

$$\text{(b)}\quad \delta_1(v_2^ik_0) \equiv \pm v_2^ih_{11}b_{10} \mod (v_1^2).$$

We have to handle $v_2^{p-1}b_{10}$ more indirectly.

4.4.14. LEMMA. *$\delta_1(v_2^{p-1}b_{10} \pm \frac{1}{2}v_2^{p-2}k_0) = cv_1^{p-3}h_{10}b_{11}$ for some nonzero $c \in \mathbf{Z}/(p)$.*

PROOF. By 4.4.13, $v_1^{p-1}b_{11} = 0$ in $\mathrm{Ext}(BP_*/I_1)$, so $v_1^{p-1}h_{10}b_{11} = 0$ and $v_1^ih_{10}b_{11} = \delta_1(x)$ for some $i < p-1$ and some $x \in \mathrm{Ext}^2(BP_*/I_2)$. The only remaining x is the indicated one. □

From 4.4.14 we get $\delta_1(v_2^{p-1}h_{10}b_{10} \pm \frac{1}{2}v_2^{p-2}h_{10}k_0) \equiv 0 \mod (v_1^{p-2})$. All other elements in $\mathrm{Ext}^s(BP_*/I_2)$ for $s \geq 3$ are divisible by h_{10} or b_{10} and they can all be accounted for in such a way that the above element, which we denote by $\bar{\phi}$, must be in $\ker\delta_1$. Hence δ_1 is completely determined in our range.

Equivalently, we have computed all of the differentials in the Bockstein spectral sequence. However, there are some multiplicative extensions which still need to be worked out.

4.4.15. THEOREM. *For $p > 2$, $\mathrm{Ext}(BP_*/I_1) = P(v_1) \otimes E(h_{10}) \oplus M$, where M is a free module over $P(b_{10})$ on the following generators:*

$$\beta_i = \delta_1(v_2^i),\quad h_{10}\beta_i,\quad \bar\beta_i = v_1^{-1}\delta_1(v_2^ih_{10})\quad (\textit{e.g., } \bar\beta_1 = \pm b_{10}),$$
$$\textit{and}\quad h_{10}\bar\beta_i \quad \textit{for } 1 \leq i \leq p-1;$$
$$\beta_{p/i} = v_1^{1-i}\delta_i(v_2^p) \quad \textit{and} \quad h_{10}\beta_{p/i} \quad \textit{for } 1 \leq i \leq p;$$
$$\bar\beta_{p/i} = v_1^{2-i}\delta_1(v_2^{p-1}h_{11}) \quad \textit{for } 2 \leq i \leq p;$$
$$h_{10}\bar\beta_{p/i} \quad \textit{for } 3 \leq i \leq p;\quad \bar\phi \quad \textit{and} \quad \beta_1\bar\beta_{p/p}.$$

Here δ_1 is the connecting homomorphism for the short exact sequence

$$0 \to \Sigma^q BP_*/I_1 \xrightarrow{v_1} BP_*/I_1 \to BP_*/I_2 \to 0.$$

Moreover,

$$h_{10}\beta_i = v_1\bar\beta_i,\quad v_1\beta_i = 0,\quad v_1^p\beta_{p/p} = 0,$$
$$v_1^{p-1}\bar\beta_{p/p} = 0,\quad \textit{and}\quad v_1^{p-2}h_{10}\bar\beta_{p/p} = 0.$$

(This description of the multiplicative structure is not complete.)

PROOF. The additive structure of this Ext follows from the above calculations. The relations follow from the way the elements are defined. □

FIGURE 4.4.16. $\mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*, BP_*/I_1)$ for $p = 5$ and $t - s \leq 240$.

FIGURE 4.4.16 illustrates this result for $p = 5$. Horizontal lines indicate multiplication by v_1, and an arrow pointing to the right indicates that the element is free over $P(v_1)$. A diagonal line which increases s and t/q by one indicates multiplication by h_{10} and one which increases t/q by 4 indicates the Massey product operation $\langle -, h_{10}, h_{10}, h_{10}, h_{10}\rangle$. Thus two successive diagonal lines indicate multiplication by $b_{10} = \pm\langle h_{10}, h_{10}, h_{10}, h_{10}, h_{10}\rangle$. The broken line on the right indicates the limit of our calculation.

Now we have to consider the long exact sequence or Bockstein spectral sequence associated with

$$0 \to BP_* \xrightarrow{p} BP_* \to BP_*/I_1 \to 0.$$

First we compute $\delta_0(v_1^i)$. Since $d(v_1) = pt_1$ in $C(BP_*)$ we have $d(v_1^i) \equiv ipv_1^{i-1}t_1$ mod (ip^2), so

$$\delta_0(v_1^i) \equiv iv_1^{i-1}h_{10} \qquad \text{mod } (ip^2). \tag{4.4.17}$$

Moving on to $\mathrm{Ext}^1(BP_*/I_1)$ we need to compute δ_0 on β_i and $\beta_{p/i}$. The former can be handled most easily as follows. $\delta_0(\bar{\beta}_i) = 0$ because there is no element in the appropriate grading in Ext^3. δ_0 is a derivation mod (p) so $\delta_0(v_1\bar{\beta}_i) = h_{10}\bar{\beta}_i$. Since $v_1\bar{\beta}_i = h_{10}\beta_i$ we have $h_{10}\bar{\beta}_i = \delta_0(h_{10}\beta_i) = h_{10}\delta_0(\beta_i)$ so

$$\delta_0(\beta_i) = \bar{\beta}_i. \tag{4.4.18}$$

Now $\beta_{p/p} = h_{12} - v_1^{p^2-p}h_{11}$ and $v_1^{p^2-p}h_{11}$ is cohomologous to $v_1^{p^2-1}h_{10}$, which by 4.4.16 is in $\ker\delta_0$. Hence

$$\delta_0(\beta_{p/p}) = \delta_0(h_{12}) = b_{11} = \pm\bar{\beta}_{p/p}. \tag{4.4.19}$$

It follows that

$$\delta_0(\beta_{p/p-i}) = \delta_0(v_1^i\beta_{p/p}) = iv_1^{i-1}h_{10}\beta_{p/p} \pm v_1^i\bar{\beta}_{p/p}.$$

This accounts for all elements in sight but $\delta_0(h_{10\beta_{p/1}})$ which vanishes mod (p). We will show that it is a unit multiple of $p\Phi$ below in 5.1.24.

Putting all this together gives

4.4.20. THEOREM. *For $p > 2$ and $t - s \le (p^2 + p)q$, $\mathrm{Ext}(BP_*)$ is as follows. $\mathrm{Ext}^0 = \mathbf{Z}_{(p)}$ concentrated in dimension zero. $\mathrm{Ext}^{1,qi} = \mathbf{Z}_{(p)}/(pi)$ generated by $\bar{\alpha}_i = i^{-1}\delta_0(v_1^i)$, where $\alpha_1 = h_{10}$. For $s \ge 2$ Ext^s generated by all b_{10}^jx, where x is one of the following: $\beta_i = \delta_0(\beta_i)$ (where $\beta_1 = \pm b_0$) and $\alpha_1\beta_i$ for $1 \le i \le p-1$; $\beta_{p/p-i} = \delta_0(\beta_{p/p-i})$ for $0 \le i \le p-1$; $\alpha_1\beta_{p/p-i}$ for $0 \le i \le p-3$; and $\phi = p^{-1}\delta_0(h_{10}\beta_{p/1})$ which has order p^2. ϕ is a unit multiple of $\langle\beta_{p/2}, \alpha_1, \alpha_1\rangle$ and $p\phi$ is a unit multiple $\alpha_1\beta_{p/1}$. Here $\beta_{i/j}$ denotes the image under δ_0 of the corresponding element in $\mathrm{Ext}(BP_*/I_1)$.* □

For $p = 5$ this is illustrated in FIG. 4.4.21, with notation similar to that of Fig. 4.4.16. It also shows differentials (long arrows originating at $\beta_{5/5}$ and $\beta_1\beta_{5/5}$), which we discuss now. By sparseness (4.4.2) $E_2 = E_{2p-1}$ and $d_{2p-1}\colon E_{2p-1}^{s,t} \to E_{2p-1}^{s+2p-1,t-2p+2}$. It is clear that in our range of dimensions $E_{2p} = E_\infty$ because any higher (than d_{2p-1}) differential would have a target whose filtration (the s-coordinate) would be too high. Naively, the first possible differential is $d_{2p-1}(\alpha_{p^2-1}) = c\beta_i^p$. However, d_{2p-1} respects multiplication by α_1 and $\alpha_1\alpha_{p^2-1}$ so $c\alpha_1\beta_1^p = 0$ and $c = 0$. Alternatively one can show (see 5.3.7) that each element in Ext^1 is a permanent cycle.

FIGURE 4.4.21. The Adams–Novikov spectral sequence for $p = 5$, $t - s \leq 240$, and $s \geq 2$.

4.4.22. THEOREM (Toda [?, ?]). *$d_{2p-1}(\beta_{p/p}) = a\alpha_1\beta_1^p$ for some nonzero $a \in \mathbf{Z}/(p)$.* □

Toda shows that any $x \in \pi_*(S)$ of order p must satisfy $\alpha_1 x^p = 0$. For $x = \beta_1$ this shows $\alpha_1\beta_1^p = 0$ in homotopy. Since it is nonzero in E_2 it must be killed by a differential and our calculation shows that $\beta_{p/p}$ is the only possible source for it. We do not know how to compute the coefficient a, but its value seems to be of little consequence.

Theorem 4.4.22 implies that $d_{2p-1}(\beta_1\beta_{p/p}) = \alpha_1\beta_1^{p+1}$. Inspection of 4.4.20 or 4.4.21 shows that there are no other nontrivial differentials.

Notice that the element $\alpha_1\beta_{p/p}$ survives to E_∞ even though $\beta_{p/p}$ does not. Hence the corresponding homotopy element, usually denoted by ε', is indecomposable. It follows easily from the definition of Massey products (A1.4.1) that $\langle\alpha_1, \alpha_1, \beta_1^p\rangle$ is defined in E_{2p}, has trivial indeterminacy, and contains a unit multiple of $\alpha_1\beta_{p/p}$. It follows from 7.5.4 that ε' is the corresponding Toda bracket. Using A1.4.6 we have

$$\langle\alpha_1, \ldots, \alpha_1, \varepsilon'\rangle = \langle\alpha_1, \ldots, \alpha_1\rangle\beta_1^p = \beta_1^{p+1}$$

with $p-2$ α_1's on the left and p α_1's on the right.

Looking ahead we can see this phenomenon generalize as follows. For $1 \le i \le p-1$ we have $d_{2p-1}(\beta_{p/p}^i) = ia\alpha_1\beta_{p/p}^{pi-1}$. For $i \le p-2$ this leads to $\langle\alpha_1, \ldots, \alpha_1\beta_1^{ip}\rangle$ [with $(i+1)$ α_1's] being a unit multiple of $\varepsilon^{(i)} = \alpha_1\beta_{p/p}^i$, and $\langle\alpha_1, \alpha_1, \ldots, \alpha_1\varepsilon^{(i)}\rangle$ [with $(p-i-1)$ α_1's] is a unit multiple of β_1^{1+ip}. In particular, $\alpha_1\varepsilon^{(p-2)}$ is a unit multiple of $\beta_1^{1+(p-2)p}$. Since $\alpha_1\beta_1^p = 0$ (4.4.22), $\beta_1^{p^2-p+1} = 0$ since it is a unit multiple of $\alpha_1\beta_1^p\varepsilon^{(p-2)}$. However, in the E_2-term all powers of β_1 are nonzero (Section 6.4), so $\beta_1^{p^2-p+1}$ must be killed by a differential, more precisely by $d_{(p-1)q+1}(\alpha_1\beta_{p/p}^{p-1})$.

Now we will make an analogous calculation for $p = 2$. The first three steps are shown FIG. 4.4.23. In (a) we have $\mathrm{Ext}_{P_*}(\mathbf{Z}/(2), \mathbf{Z}(2))$, which is $\mathrm{Ext}(BP_*/I_4)$ for $t - s \le 29$. Since differentials in the Bockstein spectral sequences and the Adams–Novikov spectral sequence all lower $t - s$ by 1, we lose a dimension with each spectral sequence. In (a) we give elements the same names they have in $\mathrm{Ext}_{A_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$. Hence we have $c_0 = \langle h_{11}, h_{12}^2, h_{11}\rangle$ and $Px = \langle x, h_{10}^4, h_{12}$. Diagonal lines indicate multiplication by h_{10}, h_{11}, and h_{12}. The arrow pointing up and to the right indicates that all powers of h_{10} are nontrivial.

The Bockstein spectral sequence for $\mathrm{Ext}(BP_*/I_3)$ collapses and the result is shown in Fig. 4.4.23(b). The next Bockstein spectral sequence has some differentials. Recall that δ_2 is the connecting homomorphism for the short exact sequence

$$0 \to \Sigma^6 BP_*/I_2 \xrightarrow{v_2} BP_*/I_2 \to BP_*/I_3 \to 0.$$

Since $\eta_R(v_3) \equiv v_3 + v_2t_1^4 + v_2^2t_1 \mod I_2$ by 4.3.1 we have

$$\text{(4.4.24)} \qquad \begin{aligned} &\text{(a)} \quad \delta_2(v_3h_{10}^i) = (h_{12} + v_2h_{10})h_{10}^i && \text{for } i \le 2,\\ &\text{(b)} \quad \delta_2(v_3h_{10}^i) = v_2h_{10}^{i+1} && \text{for } i \ge 3,\\ &\text{(c)} \quad \delta_2(v_3h_{12}^i) = h_{12}^{i+1} && \text{for } i = 1, 2,\\ &\text{(d)} \quad \delta_2(v_2^2) = v_2h_{13} + v_2^3h_{11}. \end{aligned}$$

This accounts for all the nontrivial values of δ_2. In $\mathrm{Ext}(BP_*/I_2)$ we denote $\delta_2(v_3^i)$ by γ_i and $v_2^{-1}\delta_2(v_3^2)$ by $\gamma_{2/2}$. The elements $v_3h_{1,1}$, $v_3h_{1,0}h_{1,2} \in \mathrm{Ext}(BP_*/I_3)$

FIGURE 4.4.23. (a) $\mathrm{Ext}(BP_*/I_4)$ for $p = 2$ and $t - s < 29$. (b) $\mathrm{Ext}(BP_*/I_3)$ for $t - s \leq 28$. (c) $\mathrm{Ext}(BP_*/I_2)$ for $t - s \leq 27$.

FIGURE 4.4.32. $\mathrm{Ext}(BP_*/I_1)$ for $p=2$ and $t-s \le 26$

are in $\ker \delta_2$ and hence lift back to $\mathrm{Ext}(BP_*/I_2)$, where we denote them by ζ_2 and $x_{2,2}$, respectively. They are represented in $C(BP_*/I_2)$ by

$$\text{(4.4.25)} \qquad \text{(a)} \quad \zeta_2 = v_3 t_1^2 + v_2(t_2^2 + t_1^6) + v_2^2 t_2 \quad \text{and}$$

$$\text{(b)} \quad x_{22} = v_3 t_1|t_1^4 + v_2(t_2|t_2^2 + t_1|t_1^2 t_2^2 + t_2|t_1^6 + t_3|t_1^2 + t_1^5|t_1^4) + v_2^2(t_2|t_2 + t_1|t_1^2 t_2 + t_1 t_2|t_1^2).$$

Now we pass to $\mathrm{Ext}(BP_*/I_1)$. To compute δ_1 on $\mathrm{Ext}^0(BP_*/I_2)$ we have $\eta_R(v_2) \equiv v_2 + v_1 t_1^2 + v_1^2 t_1 \mod I_1$, so

$$\text{(4.4.26)} \qquad \begin{aligned} \delta_1(v_2) &\equiv h_{11} && \mod (v_1),\\ \delta_1(v_2^2) &\equiv v_1 h_{12} \equiv v_1(\gamma_1 + v_2 h_{10}) && \mod (v_1^2),\\ \delta_1(v_2^3) &\equiv v_2^2 h_{11} && \mod (v_1),\\ \delta_1(v_2^4) &\equiv v_1^3 h_{13} \equiv v_1^3(\gamma_{2/2} + v_2^2 h_{11}) && \mod (v_1^4),\\ \delta_1(v_2^5) &\equiv v_2^4 h_{11} && \mod (v_1), \end{aligned}$$

This means that in $\mathrm{Ext}^1(BP_*/I_2)$ it suffices to compute δ_1 on $v_2 h_{10}, \zeta_2$, $v_2^3 h_{10}$, γ_2, and $v_2\zeta_2$. We find $\delta_1(v_2 h_{10})$ and the element pulls back to

$$\text{(4.4.27)} \qquad x_7 = v_2 t_1 + v_1(t_2 + t_1^3).$$

For ζ_2 we compute in $C(BP_*/I_1)$ and get

$$d(\zeta_2 + v_1 t_1^2 t_2^2) \equiv v_1(t_1^4|t_1^4 + v_2^2 t_1|t_1) \mod (v_1^2)$$

so

$$\text{(4.4.28)} \qquad \delta_1(\zeta_2) \equiv \gamma_1^2 \mod (v_1)$$

For $v_2^3 h_{10}$ we compute

$$d(v_2^3 t_1 + v_1 v_2^2(t_2 + t_1^3) + v_1^2 v_3 t_1) \equiv v_1^2 v_2^2 t_1|t_1 \mod (v_1^3)$$

so

$$\text{(4.4.29)} \qquad \delta_1(v_2^3 h_{10}) \equiv v_1 v_2^2 h_{10}^2 \mod (v_1^2).$$

Similar calculations give

$$\text{(4.4.30)} \qquad \delta_1(\gamma_2) \equiv h_{11}\gamma_{2/2} \mod (v_1^2)$$

and

$$\delta_1(v_2\zeta_2) \equiv h_{11}\zeta_2 + v_2^3 h_{10} \mod (v_1^2)$$

In $\mathrm{Ext}^2\, BP_*/I_2)$ it suffices to compute δ_1 on x_{22}. We will show

$$\text{(4.4.31)} \qquad \delta_1(x_{22}) = c_0$$

using Massey products. Since x_{22} projects to $v_3 h_{10} h_{12}$ we have $x_{22} \in \langle v_2, \gamma_1^2, h_{10}\rangle$, so $\delta_1(x_{22}) \in \langle \delta_1(v_2), \gamma_1^2, h_{10}\rangle$ by A1.4.11. This is $\langle h_{11}, \gamma_1^2, h_{10}\rangle$, which is easily seen to be c_0.

This completes our calculation of δ_1. The resulting value of $\mathrm{Ext}(BP_*/I_1)$ is shown in FIG. 4.4.32. The elements 1 and x_7 are free over $P(v_1, h_{10})$. As usual we denote $v_1^{1-j}\delta_1(v_2^i)$ by $\beta_{i/j}$. x_7 is defined by 4.4.27. η_1 and η_2 (not to be confused with the η_j of Mahowald [**?**]) denote $\delta_1(\zeta_2)$ and $\delta_1(v_2\zeta_2)$.

We must comment on some of the relations indicated in 4.4.32.

4.4.33. LEMMA. *In* $\mathrm{Ext}(BP_*/I_1)$ *for* $p = 2$ *the following relations hold.*
(a) $h_{10}\beta_3 = v_1\beta_{2/2}^2$
(b) $\beta_{2/2}^3 = \beta_1^2\beta_{4/4} + h_{10}^2\beta_{4/2}$
(c) $h_{10}^3 x_7 \beta_{4/4} = v_1 P\beta_1$.

PROOF.
(a) $\beta_{2/2} = h_{12} \mod (v_1)$ so $v_1\beta_{2/2}^2 = \delta_1(v_2^2 h_{12})$ while $h_{10}\beta_3 = \delta_1(v_2^3 h_{10})$. Since $\eta_R(v_2v_3) \equiv v_2v_3 + v_2^2t_1^4 + v_2^3t_1 \mod I_2$, we have $v_2^2h_{12} = v_2^3h_{10}$ in $\mathrm{Ext}(BP_*/I_2)$.
(b) $\beta_{2/2} = h_{12} + v_1^2h_{11} = h_{12} + v_1^3h_{10}$ so

$$\begin{aligned}\beta_{2/2}^3 &\equiv h_{12}^3 \equiv h_{11}^2h_{13} \equiv \beta_{4/4}h_{11}^2\\ &\equiv \beta_{4/4}(\beta_1^2 + v_1^2h_{10}^2) \equiv \beta_{4/4}\beta_1^2 + \beta_{4/2}h_{10}^2 \mod (v_1^3).\end{aligned}$$

$$\begin{aligned}\text{(c)}\quad v_1P\beta_1 &= v_1\langle \beta_1, h_{10}^4, \beta_{4/4}\rangle\\ &= \langle v_1, \beta_1, h_{10}^4\rangle\beta_{4/4} \quad \text{by A1.4.6}\\ &= \langle v_1, \beta_1, h_{10}\rangle h_{10}^3\beta_{4/4} \quad \text{by A1.4.6.}\end{aligned}$$

The last Massey product is easily seen to contain x_7. □

Now we pass to $\mathrm{Ext}(BP_*)$ by computing δ_0, beginning with $\mathrm{Ext}^0(BP_*/I_1) = P(v_1)$. By direct calculation we have

$$\begin{aligned}\delta_0(v_1^{2i+1}) &\equiv v_1^{2i}h_{10} \mod (2)\\ \delta_0(v_1^2) &= 2\beta_1\end{aligned} \tag{4.4.34}$$

To handle larger even powers of v_1, consider the formal expression $u = v_1^2 - 4v_1^{-1}v_2$. Using the formula (in terms of Hazewinkel's generators A2.2.1)

$$\eta_R(v_2) = v_2 - 5v_1t_1^2 - 3v_1^2t_1 + 2t_2 - 4t_1^3,$$

we find that $d(u) = 8v_1^{-2}x_7$ in $C(v_1^{-1}BP_*/(2^4))$. It follows that

$$d(u^2 - 2^4v_1^{-2}v_2^2) \equiv 2^4(x_7 + \beta_{2/2}) \mod (2^5)$$

and for $i > 2$

$$d(u^i) \equiv 8iv_1^{2i-4}x_7 \mod (16i)$$

so

$$\delta_0(v_1^4) \equiv 2^3(x_7 + \beta_{2/2}) \mod (2^4) \tag{4.4.35}$$

and

$$\delta_0(v_i^{2i}) \equiv 4iv_1^{2i-4}x_7 \mod (8i) \quad \text{for } i \geq 3.$$

Combining this with

$$\delta_0(v_1^{2i+1}h_0^j) = v_1^{2i}h_0^{j+1} \tag{4.4.36}$$

and

$$\delta_0(v_1^{2i+1}h_0^jx_7) = v_1^{2i}h_0^{j+1}x_7$$

accounts for all elements of the form $v_1^ih_0^jx_7^\varepsilon$ for $i, j \geq 0$ and $\varepsilon = 0, 1$ we have

4.4.37. THEOREM. *For $p=2$ $\mathrm{Ext}^1(BP_*)$ is generated by $\bar\alpha_i$ for $i \geq 1$ where*

$$\bar\alpha_i = \begin{cases} \delta_0(v_1^i) & \text{for } i \text{ odd} \\ \frac{1}{2}\delta_0(v_1^2) & \text{for } i=2 \\ (1/2i)\delta_0(v_1^i) & \text{for even } i \geq 4. \end{cases}$$

In particular $\bar\alpha_1 = h_{10}$. Moreover $\bar\alpha_1^j\bar\alpha_i \neq 0$ for all $j>0$ and $i \neq 2$. □

Moving on to $\mathrm{Ext}^1(BP_*/I_1)$ we still need to compute δ_0 on h_{12}, v_1h_{12}, β_3, and $v_1^jh_{13}$ for $0 \leq j \leq 3$. An easy calculation gives

$$\begin{aligned} \delta_0(h_{12}) &\equiv h_{11}^2 && \mod (2), \\ \delta_0(v_1h_{12}) &\equiv h_{10}h_{12} && \mod (2), \\ \delta_0(h_{13}) &\equiv h_{12}^2 && \mod (2), \\ \delta_0(v_1h_{13}) &\equiv v_1h_{12}^2 + h_{10}h_{13} && \mod (2), \\ \delta_0(v_1^2h_{13}) &\equiv 2(h_{11}+v_1h_{10})h_{13} && \mod (4), \end{aligned} \tag{4.4.38}$$

and

$$\delta_0(v_1^3h_{13}) \equiv v_1^2h_{10}h_{13} \qquad \mod (2).$$

For β_3 we have

$$\delta_0(\beta_3) = \beta_{2/2}^2 + \eta_1. \tag{4.4.39}$$

The proof is deferred until the next chapter (5.1.25).

In $\mathrm{Ext}^2(BP_*/I_1)$ all the calculations are straightforward except η_2 and $x_7\beta_{4/j}$. The former gives

$$\delta_0(\eta_2) = c_0, \tag{4.4.40}$$

which we defer to 5.1.25. For the latter we have

$$\delta_0(x_7\beta_{4/4}) \equiv x_7h_{12}^2 \mod (2).$$

Computing in $C(BP_*/I_2)$ we get

$$d(t_1^2|t_3 + t_2|t_2^2 + t_1^3|t_2^2 + t_1^2t_2|t_1^4) = t_1|t_1^4|t_1^4 + v_2t_1^2|t_1^2|t_1^2$$

so $x_7h_{12}^2 \equiv \beta_3\beta_1^2 \mod (v_1^2)$ and

$$\delta_0(x_7\beta_{4/4}) \equiv \beta_1^2\beta_3 + ch_{10}^2\beta_{4/2} \mod (2) \tag{4.4.41}$$

for $c = 0$ or 1. Note that

$$\delta_0(h_{10}\beta_4) = h_{10}^2\beta_{4/2}.$$

We also get from 4.4.37

$$\delta_0(x_7\beta_{4/3}) \equiv ch_{10}^2\beta_4 + h_{10}x_7\beta_{4/4} \mod (2). \tag{4.4.42}$$

$\delta_0(x_7\beta_{4/2})$ must be a multiple of $h_{10}x_7\beta_{4/3}$ but the latter is not in $\ker\delta_0$ so

$$\delta_0(x_7\beta_{4/2}) = 0. \tag{4.4.43}$$

Of the remaining calculations of δ_0 all are easy but $\beta_1^2\beta_{4/4}$ and $h_{10}^3\beta_4 = \beta_1^3\beta_{4/4}$. It is clear that $\delta_0(\beta_1^2\beta_{4/4})$ and $\delta_0(\beta_1^3\beta_{4/4})$ are multiples of elements which reduce to h_{10}^3 and $P\beta_1$, respectively. Since $\beta_1^2\beta_{2/2}^2 = 0$ and $\beta_1\beta h_{10}^2\beta_{4/3} = 0$ we have

FIGURE 4.4.45. $\mathrm{Ext}(BP_*)$ for $p = 2$, $t - s \le 25$.

$\delta_0(\beta_1^2\beta_{4/4}) \equiv 0 \mod (2)$ and $\delta_0(\beta_1^2\beta_{4/4}) \equiv 0 \mod (4)$. Thus the simplest possible result is

$$\frac{1}{2}\delta_0(\beta_1^2\beta_{4/4}) \equiv h_{10}^3\beta_{4/3} \mod (2),$$
$$\frac{1}{4}\delta_0(\beta_1^3\beta_{4/4}) \equiv P\beta_1 \mod (2). \tag{4.4.44}$$

We will see below that larger values of the corresponding Ext groups would lead to a contradiction.

The resulting value of $\mathrm{Ext}(BP_*)$ is shown in FIG. 4.4.45. Here squares denote elements of order greater than 2. The order of the elements in Ext^1 is given in 4.4.37. The generators of $\mathrm{Ext}^{2,20}$ and $\mathrm{Ext}^{4,24}$ have order 4 while that of $\mathrm{Ext}^{5,28}$ has order 8.

We compute differentials and group extensions in the Adams–Novikov spectral sequence for $p = 2$ by comparing it with the Adams spectral sequence. The E_2-term of the latter as computed by Tangora [?] is shown in FIG. 4.4.46. This procedure will determine all differentials and extensions in the Adams spectral sequence in this range as well.

FIGURE 4.4.46. $\mathrm{Ext}_{A_*}(\mathbf{Z}/2, \mathbf{Z}/2)$ for $t - s \le 25$.

The Adams element h_1 corresponds to the Novikov $\bar{\alpha}_1$. Since $h_1^4 = 0$, α_1^4 must be killed by a differential, and it must be $d_3(\bar{\alpha}_3)$. It can be shown that the periodicity operator P in the Adams spectral sequence (see 3.4.6) corresponds to multiplication by v_1^4, so $P^i h_1$ corresponds to $\bar{\alpha}_{4i+1}$, so $d_3(\bar{\alpha}_{4i+3}) = \bar{\alpha}_1^3\bar{\alpha}_{4i+1}$. The relation $h_0^2h_2 = h_1^3$ gives a group extension in the Adams–Novikov spectral sequence, $2\alpha_{4i+2} = \bar{\alpha}_1^2\bar{\alpha}_{4i+1}$ in homotopy. The element $P^i h_2$ for $i > 0$ corresponds to $2\bar{\alpha}_{4i+1}$. This element is not divisible by 2 in the Adams spectral sequence so we deduce $d_3(\bar{\alpha}_{4i+2}) = \alpha_1^3\bar{\alpha}_{4i}$ for $i > 0$. Summing up we have

4.4.47. THEOREM. *The elements in* $\mathrm{Ext}(BP_*)$ *for* $p = 2$ *listed in* 4.4.37 *behave in the Adams–Novikov spectral sequence as follows.* $d_3(\bar{\alpha}_{4i+3}) = \bar{\alpha}_1^3\bar{\alpha}_{4i+3}$ *for* $i \ge 0$ *and* $d_3(\bar{\alpha}_{4i+2}) = \bar{\alpha}_1^3\bar{\alpha}_{4i}$ *for* $i \ge 1$. *Moreover the homotopy element corresponding to* $\alpha_{4i+2} = r\bar{\alpha}_{4i+2}$ *does not have order* 2; *twice it is* $\bar{\alpha}_1^2\bar{\alpha}_{4i}$ *for* $i \ge 1$ *and* $\bar{\alpha}_1^3$ *for* $i = 0$. □

As it happens, there are no other Adams–Novikov spectral sequence differentials in this range, although there are some nontrivial extensions.

These elements in the Adams–Novikov spectral sequence E_∞-term correspond to Adams elements near the vanishing line. The towers in dimensions congruent to $7 \mod (8)$ correspond to the groups generated by $\bar{\alpha}_{4i}$. Thus the order of $\bar{\alpha}_{4i}$ determines how many elements in the tower survive to the Adams E_∞-term. For example, the tower in dimension 15 generated by h_4 has 8 elements. $\bar{\alpha}_8$ has order 2^5 so only the top elements can survive. From this we deduce $d_3(h_0^i h_4) = h_0^i d_0$ for $i = 1, 2$ and either $d_3(h_4) = d_0$ or $d_2(h_4) = h_0h_3^2$. To determine which of these two occurs we consult the Adams–Novikov spectral sequence and see that β_3 and $\beta_{4/4}$

must be permanent so $\pi_{14}^s = \mathbf{Z}/(2) \oplus \mathbf{Z}/(2)$. If $d_3(h_4) = d_0$ the Adams spectral sequence would give $\pi_{14}^s = \mathbf{Z}/(4)$, so we must have $d_2(h_4) = h_0 h_3^2$.

One can also show that $P^i c_0$ corresponds to $\alpha_1 \bar{\alpha}_{4i+4}$ for $i > 1$ and this leads to a nontrivial multiplicative extension in the Adams spectral sequence. For example, the homotopy element corresponding to Pc_0 is α_1 times the one corresponding to $h_0^3 h_4$.

The correspondence between Adams–Novikov spectral sequence and Adams spectral sequence permanent cycles is shown in the following table.

TABLE 4.4.48. Correspondence between Adams–Novikov spectral sequence and Adams spectral sequence permanent cycles for $p = 2$, $14 \leq t - s \leq 24$

Adams–Novikov element	*Adams element*	*Adams–Novikov element*	*Adams element*
$\beta_{4/4}$	h_3^2	β_4	g
β_3	d_0	$\frac{1}{2}\bar{\alpha}_2^2 \beta_{4/4}$	$h_0 g$
$\beta_{4/3}$	$h_1 h_4$	$\frac{1}{4}\alpha_1^3 \beta_4$	$h_2 g$
$\frac{1}{2}\beta_{4/2}$	$h_2 h_4$	$\beta_{4/4}\bar{\alpha}_4$	$h_4 c_0$
η_2	c_1		

4.4.49. COROLLARY. *The Adams–Novikov spectral sequence has nontrivial group extensions in dimensions* 18 *and* 20 *and the homotopy product* $\beta_4 \bar{\alpha}_2$ *is detected in filtration* 4. □

4.4.50. COROLLARY. *For* $14 \leq t - s \leq 24$ *the following differentials occur in the Adams spectral sequence for* $p = 2$.

$$d_2(h_4) = h_0 h_3^2, \quad d_3(h_0 h_4) = h_0 d_0, \quad d_2(e_0) = h_1^2 d_0,$$

$$d_2(f_0) = h_0^2 e_0, \quad d_2(i) = h_0 P d_0, \quad \textit{and} \quad d_2(Pe_0) = h_1^2 P d.$$

There are nontrivial multiplicative extensions as follows:

$$h_1 \cdot h_0^3 h_4 = Pc_0, \quad h_1 \cdot h_1 g = Pd_0, \quad \textit{and} \quad h_0 \cdot h_2^2 e_0 = h_1 P d_0 = h_2 \cdot h_2^2 d_0. \quad \square$$