ON THE NON-EXISTENCE OF ELEMENTS OF KERVAIRE INVARIANT ONE

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ABSTRACT. We show that the Kervaire invariant one elements $\theta_j \in \pi_{2^{j+2}-2}S^0$ exist only for $j \leq 6$. By Browder's Theorem, this means that smooth framed manifolds of Kervaire invariant one exist only in dimensions 2, 6, 14, 30, 62, and possibly 126. Except for dimension 126 this resolves a longstanding problem in algebraic topology.

CONTENTS

1. INTRODUCTION

The existence of smooth framed manifolds of Kervaire invariant one is one of the oldest unresolved issues in differential and algebraic topology. The question originated in the work of Pontryagin in the 1930's. It took a definitive form in the paper [17] of Kervaire in which he constructed a combinatorial 10-manifold with no smooth structure, and in the work of Kervaire-Milnor [18] on h-cobordism classes of manifolds homeomorphic to the sphere. The question was connected to homotopy theory by Browder in his fundamental paper [3] where he showed that smooth framed manifolds of Kervaire invariant one exist only in dimension of the form $(2^{j+1}-2)$, and that a manifold exists in that dimension if and only if the class

$$h_j^2 \in \mathrm{Ext}_A^{2,2^{j+1}}(\mathbb{Z}/2, \mathbb{Z}/2)$$

in the E_2-term of the classical Adams spectral represents an element

$$\theta_j \in \pi_{2^{j+1}-2}S^0$$

in the stable homotopy groups of spheres. The classes θ_j for $j \leq 5$ were shown to exist by Barratt-Mahowald, and by Barratt-Jones-Mahowald (see [2]). Many open issues in algebraic and differential topology depend on knowing whether or not the Kervaire invariant one elements θ_j exist for $j \geq 6$.

The purpose of this paper is to prove the following theorem

Theorem 1.1. *For $j \geq 7$ the class $h_j^2 \in \text{Ext}_A^{2,2^{j+1}}(\mathbb{Z}/2, \mathbb{Z}/2)$ does not represent an element of the stable homotopy groups of spheres. In other words, the Kervaire invariant elements θ_j do not exist for $j \geq 7$.*

Smooth framed manifolds of Kervaire invariant one therefore exist only in dimensions 2, 6, 14, 30, 62, and possibly 126. At the time of writing, our methods still leave open the existence of θ_6.

1.1. **Outline of the argument.** Our proof builds on the the strategy used by the third author in [26] and on the homotopy theoretic refinement developed by the second author and Haynes Miller (see [28]). We consider the situation

$$
\begin{array}{ccc}
\text{Adams-Novikov} & \Longrightarrow & \text{An easier} \\
\text{spectral sequence} & & \text{spectral sequence} \\
\Big\Downarrow & & \\
\text{Adams spectral} & & \\
\text{sequence} & &
\end{array}
$$

While the Adams and Adams-Novikov spectral sequences compute the group $\pi_k S^0$, our "easier" spectral sequence will compute the homotopy groups of some other spectrum Ω. We write it simply as

$$E_2^{s,t} \implies \pi_{t-s}\Omega.$$

We will find that this "easier" spectral sequence has the properties

(1) The groups $E_2^{s,t}$ are zero for $s < 0$, and for $s = 0$ and t odd.

(2) Any element b_j in the Adams-Novikov E_2-term whose image in the Adams E_2-term is h_j^2 has non-zero image in the "easier" spectral sequence.

(3) The group $\pi_i\Omega$ is zero for $-4 < i < 0$.

(4) The groups $\pi_*\Omega$ are periodic with period $2^8 = 256$.

It follows from the last two items that $\pi_{2^{j+1}-2}\Omega = 0$ for $j \geq 7$.

Note that these properties prevent the existence of θ_j for $j \geq 7$. Indeed, suppose that $\theta_j : S^{2^{j+1}-2} \to S^0$ is a map represented by h_j^2 in the Adams spectral sequence. Then θ_j has Adams filtration 0, 1 or 2 in the Adams-Novikov spectral sequence, since the Adams filtration can only increase under a map. Since both

$$\text{Ext}_{MU_*MU}^{0,2^{j+1}-2} \quad \text{and} \quad \text{Ext}_{MU_*MU}^{1,2^{j+1}-1}$$

are zero, the class θ_j must be represented in Adams filtration 2 by some element b_j which is a permanent cycle. By the second property above the element b_j has a non-trivial image in the "easy spectral sequence." Since for $j \geq 7$, $\pi_{2^{j+1}-2}\Omega = 0$ the image b_j cannot survive the easy spectral sequence. Since $E_2^{s,t} = 0$ for $s = 0$ and t odd, and for $s < 0$ (hence for $s \leq 0$ and $(t-s)$ odd), the image b_j cannot be the

target of a differential, and it is therefore the source of a non-trivial differential. But this means that b_j is the source of a non-trivial differential in the Adams-Novikov spectral sequence, contradicting the fact that it represents the map θ_j.

Our main results are properties (2), (3), and (4). We call them the *Detection Theorem* (Theorem 12.2), the *Gap Theorem* (Theorem 9.5) and the *Periodicity Theorem* (Corollary 10.16) respectively. Combined, as just described, they give a proof of Theorem 1.1.

1.2. **The easier spectral sequence.** Let C_n denote the cyclic group of order n. The "easier" spectral sequence is the spectral sequence for the homotopy fixed points of the group $G = C_8$ acting on a certain spectrum $\widetilde{\Omega}$, which we now describe.

Start with the C_2-spectrum $MU_{\mathbb{R}}$ of *real bordism* ([19, 9, 16]). This is the equivariant E_∞ Thom spectrum whose underlying spectrum is MU, with C_2-action given by complex conjugation. Write $MU^{((G))}$ for the C_8-equivariant spectrum whose underlying spectrum is

$$MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}$$

and on which the generator γ of C_8 acts by

$$(a, b, c, d) \mapsto (\bar{d}, a, b, c).$$

The C_8-spectrum $MU^{((G))}$ is an equivariant E_∞ ring-spectrum.

Associated to any virtual representation V of G is an equivariant sphere spectrum S^V. When V is an actual representation, S^V is the equivariant suspension spectrum of the one point compactification of V. The definition is extended to more general V by requiring an equivalence $S^V \wedge S^W \approx S^{V \oplus W}$. It is customary to denote the group of equivariant homotopy classes of maps from S^V to a G-spectrum X by $\pi_V^G(X)$. We're interested specifically in the case $V = m\rho_G$, where ρ_G is the real regular representation of $G = C_8$. The non-equivariant spectrum underlying $S^{m\rho_G}$ is the sphere S^{8m}.

We will specify in Corollary 10.16 an element $D \in \pi_{\ell\rho_G}^G MU^{((G))}$, and define $\widetilde{\Omega}$ to be

$$\widetilde{\Omega} = D^{-1}MU^{((G))} = \varinjlim S^{-m\ell\rho_G} \wedge MU^{((G))}.$$

There is some flexibility in the choice of D, but it needs to be chosen in order that the periodicity theorem holds, and in order that the map from the fixed point spectrum of $\widetilde{\Omega}$ to the homotopy fixed point spectrum is a weak equivalence. That such an D can be chosen with these properties is a relatively easy fact, albeit mildly technical. Because of this we leave a more specific discussion to the main body of the paper.

The spectrum $\widetilde{\Omega}$ is an equivariant E_∞ ring spectrum. We define $\Omega = \widetilde{\Omega}^{hG}$ to be its homotopy fixed point spectrum. The "easy" spectral sequence is the spectral sequence

$$E_2^{s,t} = H^s(G; \pi_t\widetilde{\Omega}) \implies \pi_{t-s}\Omega.$$

Since the non-equivariant spectrum $i_0^*\widetilde{\Omega}$ underlying $\widetilde{\Omega}$ is complex orientable, there is a map

$$\text{Ext}_{MU_*MU}^{s,t}(MU_*, MU_*\Omega) \implies \pi_{t-s}\Omega$$

$$\Big\downarrow \qquad\qquad\qquad \Big\|$$

$$H^s(G; \pi_t\widetilde{\Omega}) \qquad \implies \qquad \pi_{t-s}\Omega$$

from the Adams-Novikov spectral sequence for Ω to the homotopy fixed point spectral sequence. This map is actually an isomorphism of spectral sequences, though we do not need this fact. The inclusion of the unit $S^0 \to \Omega$ gives a map from the Adams-Novikov spectral sequence for the sphere, to the Adams-Novikov spectral sequence for Ω. Composing with the map described above gives the map from the Adams-Novikov spectral sequence to the "easy" spectral sequence.

The proof of the Detection Theorem is given in §12. It is a matter of pure algebra and the argument is independent of the rest of the paper. We have put it at the end, because it makes explicit reference to the element D.

1.3. **The slice filtration and the Gap Theorem.** While the Detection Theorem and the proof of Theorem 1.1 involve the homotopy fixed point spectral sequence for Ω, the Gap Theorem and the Periodicity Theorem result from studying $\widetilde{\Omega}$ as an honest equivariant spectrum. What permits the mixing of the two approaches is the following result, which is part of Theorem 11.3.

Theorem 1.2 (Homotopy Fixed Point Theorem). *The map from the fixed point spectrum of $\widetilde{\Omega}$ to the homotopy fixed point spectrum of $\widetilde{\Omega}$ is a weak equivalence.*

In particular, for all n, the map

$$\pi_n^G \widetilde{\Omega} \to \pi_n \widetilde{\Omega}^{hG} = \pi_n \Omega$$

is an isomorphism, where the symbol $\pi_n^G \widetilde{\Omega}$ denotes the group of equivariant homotopy classes of maps from S^n (with the trivial action) to $\widetilde{\Omega}$.

We will study the equivariant homotopy type of $\widetilde{\Omega}$ using an analogue of the Postnikov tower. We call this tower the *slice tower*. Versions of it have appeared in work of Dan Dugger [7], Hopkins-Morel (unpublished), Voevodsky[31, 32, 33], and Hu-Kriz [16].

For a subgroup $K \subseteq G$, let ρ_K denote its regular representation and write

$$\widehat{S}(m\rho_K) = G_+ \underset{K}{\wedge} S^{m\rho_K} \qquad m \in \mathbb{Z}.$$

Definition 1.3. The set of *slice cells* is

$$\mathcal{A} = \{\widehat{S}(m\rho_K), \Sigma^{-1}\widehat{S}(m\rho_K) \mid m \in \mathbb{Z}, K \subseteq G\}.$$

Definition 1.4. A slice cell $\widehat{S}$ is *free* if it is of the form $G_+ \wedge S^m$ for some m. An *isotropic* slice cell is one which is not free.

We define the *dimension* of a cell $\widehat{S} \in \mathcal{A}$ by

$$\dim \widehat{S}(m\rho_K) = m|K|$$

$$\dim \Sigma^{-1}\widehat{S}(m\rho_K) = m|K| - 1.$$

Finally the *slice section* $P^n X$ is constructed by attaching cones on slice cells $\widehat{S}$ with $\dim \widehat{S} > n$ to kill all maps $\widehat{S} \to X$ with $\dim \widehat{S} > n$. There is a natural map

$$P^n X \to P^{n-1} X$$

The *n-slice of X* is defined to be its homotopy fiber $P_n^n X$.

In this way a tower $\{P^n X\}$, $n \in Z$ is associated to each equivariant spectrum X. The homotopy colimit $\underset{n}{\text{holim}} \, P^n X$ is contractible, and $\underset{n}{\text{holim}} \, P^n X$ is just X. The *slice spectral sequence* for X is the spectral sequence of the slice tower, relating $\pi_* P_n^n X$ to $\pi_* X$.

The key technical result of the whole paper is the following

Theorem 1.5 (The Slice Theorem). *The G-spectrum $P_n^n MU^{((G))}$ is contractible if n is odd. If n is even then $P_n^n MU^{((G))}$ is weakly equivalent to $H\underline{\mathbb{Z}} \wedge W$, where $H\underline{\mathbb{Z}}$ is the Eilenberg-Mac Lane spectrum associated to the constant Mackey functor $\underline{\mathbb{Z}}$, and W is a wedge of isotropic slice cells of dimension n.*

Just to be clear, the expression *weak equivalence* refers to the notion of weak equivalence in the usual model category structure for equivariant stable homotopy theory (and not, for example to an equivariant map which is a weak homotopy equivalence of underlying, non-equivariant spectra). Between fibrant-cofibrant objects it corresponds to the notion of an equivariant homotopy equivalence. Similarly, the term *contractible* means weakly equivalent to the terminal object.

The Gap Theorem depends on the following result.

Lemma 1.6 (The Cell Lemma). *Let $G = C_{2^n}$ for some n. If $\widehat{S}$ is a slice cell of even dimension which is not free, then the groups $\pi_k H\underline{\mathbb{Z}} \wedge \widehat{S}$ are zero for $-4 < k < 0$.*

This is an easy explicit computation. It reduces to showing that for $m \geq 0$ and $-4 < i < 0$

$$H_i(S^{m\rho_K}; \underline{\mathbb{Z}}) = 0$$
$$H^{-i}(S^{m\rho_K}; \underline{\mathbb{Z}}) = 0$$

where $K \subseteq G$ is a non-trivial subgroup, and the (co)homology group are equivariant cohomology groups with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$. The first assertion is trivial, while the second is a simple explicit computation, and makes essential use of the fact that K is not the trivial group.

Since the restriction of ρ_G to K is isomorphic to $(|G/K|)\,\rho_K$ there is an equivalence

$$S^{m\rho_G} \wedge G_+ \wedge_K S^{n\rho_K} = G_+ \wedge_K S^{(n+m|G/K|)\rho_K}.$$

It follows that if $\widehat{S}$ is a slice cell of dimension d, then for any m, $S^{m\rho_G} \wedge \widehat{S}$ is a slice cell of dimension $d + m|G|$. Moreover, if $\widehat{S}$ is not free, then neither is $S^{m\rho_G} \wedge \widehat{S}$. The Cell Lemma and the Slice Theorem then imply that for any m, the group

$$\pi_i S^{m\rho_G} \wedge MU^{((G))}$$

is zero for $-4 < i < 0$. Since

$$\pi_i \widetilde{\Omega} = \varinjlim \pi_i S^{-m\ell\rho_G} MU^{((G))}$$

this implies that

$$\pi_i \widetilde{\Omega} = 0$$

for $-4 < i < 0$, which is the Gap Theorem.

The Periodicity Theorem is proved with a small amount of computation in the $RO(G)$-graded slice spectral sequence for $\widetilde{\Omega}$. It makes use of the fact that $\widetilde{\Omega}$ is an equivariant E_∞-ring spectrum. Using the nilpotence machinery of [4, 14] instead of explicit computation, it can be shown that the groups $\pi_* \Omega$ are periodic with *some* period which a power of 2. This would be enough to show that only finitely many of the θ_j can exist. Some computation is necessary to get the actual period stated in the Periodicity Theorem.

All of the results are fairly easy consequences of the Slice Theorem, which in turn reduces to a single computational fact: that the quotient of $MU^{((G))}$ by the analogue of the "Lazard ring" is the Eilenberg-Mac Lane spectrum $H\mathbb{Z}$ associated to the constant Mackey functor $\underline{\mathbb{Z}}$. We call this the *Reduction Theorem* and it appears as Theorem 4.55. It is proved for $G = C_2$ in Hu-Kriz [16], and its analogue in motivic homotopy theory is the main result of the (unpublished) work of second author and Morel mentioned earlier, where it is used to identify the Voevodsky slices of MGL. It would be very interesting to find a proof of Theorem 4.55 along the lines of Quillen's argument in [25].

1.4. **Summary of the contents.** We now turn to a more detailed summary of the contents of this paper. In §2 we recall the basics of equivariant stable homotopy theory, and establish many conventions and explains some simple computations. Our main new construction, introduced in §2.3.3 is the multiplicative *norm functor*. We merely state our main results about the norm, deferring the details of the proofs to an appendix which will appear in a future version of this document.

Section 3 introduces the slice filtration and establishes many of its basic properties, including the strong convergence of the slice spectral sequence (Theorem 3.38), and an important result concerning the distribution of groups in the E_2-term of the slice spectral sequence (Corollary 3.40). The material of these first sections makes no restriction on the group G.

From §4 and forward we restrict attention to the case in which G is cyclic of order a power of 2, and we localize all spectra at the prime 2. The spectra $MU^{((G))}$ are introduced and some of the basic properties are established. The groundwork is laid for the proof of the Slice Theorem. The Reduction Theorem (Theorem 4.55) is stated in §4.4. The Reduction Theorem is the backbone of Slice Theorem, and in some sense, is the only part that is not "formal."

In §5 the results on $MU^{((G))}$ are used to derive more refined information on the slice tower. Two important methods for determining slices are introduced, one for even slices in §5.2.2 and one for odd slices in §5.2.3. The notions of a *perfect spectra* and *isotropic spectra* are introduced in §5.2.1.

The proof of the Slice Theorem is by induction on $|G|$. In §6, a refined version of the Slice Theorem (Theorem 6.1) is stated, and some auxiliary results used in the induction loop are established.

The proof of the Reduction Theorem is in §7.

The Slice Theorem is proved in §8, the Gap Theorem in §9. Section10 contains the Periodicity Theorem. The Homotopy Fixed Point Theorem is proved in §11, and the Detection Theorem in §12.

A future version of this document will contain an appendix on equivariant stable homotopy theory, providing more detailed proofs of some of the claims in §2.

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of influence to that paper, and to thank the authors for writing it. We were also helped by the thesis of Dan Dugger (which appears as [7]). The second author would like to thank Dan Dugger, Marc Levine, Jacob Lurie, and Fabien Morel for several useful conversations. Early drafts of this manuscript were read by Mark Hovey and Tyler Lawson, and the authors would like to express their gratitude for their many detailed comments. Finally, the authors would also like to thank Mark Mahowald for a lifetime of mathematical ideas, and for many helpful discussions in the early stages of this project.

2. Some equivariant stable homotopy theory

The purposes of this paper require a good multiplicative theory of equivariant spectra, and in particular a good symmetric monoidal category of modules over an equivariant E_∞ ring spectrum. To meet these demands, we will use as foundations the theory of equivariant orthogonal spectra [22, 21]. In this section we state the main homotopy theoretic results we need. The proofs, constructions, and other foundational details are explained in the Appendix, which will appear in a later version of this paper. The reader is referred to the survey of Greenlees and May [10] for an overview of equivariant stable homotopy theory, and for further references.

The results described here in the main body of the paper are homotopy theoretic in nature, and it will be convenient to use some abbreviated terminology.

Definition 2.1. Suppose that $F, G : \mathcal{C} \to \mathcal{D}$ are two continuous functors between topological model categories. A continuous transformation $T : F \to G$ is a *weak equivalence* if for every fibrant-cofibrant object $X \in \mathcal{C}$, the map $T(X) : FX \to GX$ is a weak equivalence.

A continuous functor of topological model categories automatically preserves weak equivalences of fibrant-cofibrant objects, so induces a functor of homotopy categories. The above definition could be rephrased as saying that a natural transformation is a weak equivalence if and only if it induces an equivalence of the associated derived homotopy functors.

2.1. *G*-**spaces.** Let G be a finite group, and write $\mathcal{T}_0^G$ for the category of topological spaces equipped with an action of G, and G-spaces of non-equivariant maps, with G acting by conjugation. Let $\mathcal{T}^G$ be the category with the same objects, but with equivariant maps. Thus $\mathcal{T}^G(X, Y)$ is the space of G-fixed points in $\mathcal{T}_0^G(X, Y)$.

The homotopy set (group, for $n > 0$) $\pi_n^H(X)$ of a pointed G-space is defined for $H \subseteq G$ and $n \geq 0$ to be the set of H-equivariant homotopy classes of maps

$$S^n \to X.$$

This is the same as the ordinary homotopy set (group) $\pi_n(X^H)$ of the space of H fixed-points in X.

A *weak equivalence* in $\mathcal{T}^G$ is a map inducing an isomorphism on equivariant homotopy groups π_n^H for all $H \subset G$ and all $n \geq 0$. A *fibration* is a map $X \to Y$ which for every $H \subset G$ is a Serre fibration on fixed points $X^H \to Y^H$. These two classes of maps make $\mathcal{T}^G$ into a topological model category. The cofibrations are the retracts of the *cellular maps* constructed by attaching equivariant cells of the form

$$G/H \times S^{n-1} \to G/H \times D^n.$$

For $X, Y \in \mathcal{T}_0^G$ we'll use the notation $[X, Y]^H$ for the set of homotopy classes of H-equivariant maps from X to Y. As is customary we'll write

$$\mathrm{ho}\,\mathcal{T}^G(X, Y)$$

for the set of maps from X to Y in the homotopy category of the model category $\mathcal{T}^G$. When X is cofibrant and Y is fibrant there is a canonical isomorphism

$$\mathrm{ho}\,\mathcal{T}^G(X, Y) = [X, Y]^G.$$

An important role is played by the equivariant spheres S^V arising as the one point compactification of real orthogonal representations V of G. When V is the trivial representation of dimension n, S^V is just the n-sphere S^n with the trivial G-action. We combine these two notations and write

$$S^{n+V} = S^{\mathbb{R}^n \oplus V}.$$

Associated to S^V is the equivariant homotopy group

$$\pi_V^H X$$

defined to the be set of homotopy classes of H-equivariant maps

$$S^V \to X.$$

2.2. Equivariant stable homotopy theory.

2.2.1. *G-spectra.* The category $\mathcal{S}^G$ of G-spectra can be characterized as the target of the universal functor from $\mathcal{T}^G$ to a stable topological model category in which all of the operators $S^V \wedge (-)$ are (weakly) invertible. (Similarly, the category $\mathcal{S}$ of spectra can be characterized as the target of the universal functor from the category of topological spaces $\mathcal{T}$ to a stable topological model category in which all of the operators $S^n \wedge (-)$ are (weakly) invertible.) The universal functor

$$\mathcal{T}^G \to \mathcal{S}^G$$

will be written

$$T \mapsto S^0 \wedge T,$$

or sometimes

$$T \mapsto \Sigma^\infty T,$$

and the symbol S^{-V} will stand for an object equipped with a weak equivalence

$$S^{-V} \wedge S^V \to S^0.$$

The category $\mathcal{S}^G$ is enriched over pointed spaces. The symbol $X \wedge T$ denotes (as above) the "tensor" product of a G-spectrum X and a G-space T, and the symbol

$$\mathcal{S}^G(X, Y)$$

denotes the space of equivariant maps from X to Y. There is a related category $\mathcal{S}_0^G$ consisting of equivariant spectra and non-equivariant maps. It is enriched over G-spaces, and one has

$$\mathcal{S}^G(X, Y) = \mathcal{S}_0^G(X, Y)^G.$$

2.2.2. *Equivariant stable homotopy groups.* There are several notions of homotopy groups relevant to equivariant stable homotopy. For an integer $k \in \mathbb{Z}$ and a subgroup $H \subset G$ let

$$\pi_k^H(X) = \varinjlim \pi_{k+V}^H X_V$$

denote the group of H-equivariant homotopy classes of maps $S^k \to X$. As H varies, the abelian groups $\pi_k^H X$ fit together to define a *Mackey functor* (see §2.5.1 below) which we'll denote $\underline{\pi}_k X$. The extreme values $H = G$ and $H = \{e\}$ are especially important. We offer no special notation for $\pi_k^G(X)$, but will use $\pi_k^u X$ as a shorthand for the *underlying* homotopy group $\pi_k^{\{e\}}(X)$.

2.2.3. *Homotopy theory of equivariant spectra.* The category $\mathcal{S}^G$ is a topological model category in which a weak equivalence is a map $X \to Y$ inducing an isomorphism $\pi_n^H(-)$ for all $H \subseteq G$ and all $n \in \mathbb{Z}$. As is customary, the expression

$$\mathrm{ho}\, \mathcal{S}^G(X, Y)$$

denotes the set of maps between X and Y in the homotopy category of $\mathcal{S}^G$. When X is cofibrant and Y is fibrant this can be identified with the set $\pi_0 \mathcal{S}^G(X, Y)$. We will also use the more customary notation

$$[X, Y]^G$$

for $\mathrm{ho}\, \mathcal{S}^G(X, Y)$.

The cofibrations are the retracts of the *cellular maps* constructed by attaching equivariant cells of the form

$$S^{-m} \wedge (G/H \times S^{n-1})_+ \to S^{-m} \wedge (G/H \times D^n)_+.$$

Up to weak equivalence, every G-spectrum X can be functorially presented as the homotopy colimit of a sequence

$$(2.2) \qquad \cdots \to S^{-V_n} \wedge X_{V_n} \to S^{-V_{n+1}} \wedge X_{V_{n+1}} \to \cdots,$$

in which each X_{V_n} is a G-space and $\{V_n\}$ is any increasing sequence of representations eventually containing every finite dimensional representation of G. We will often just abbreviate this important *canonical presentation* as

$$(2.3) \qquad X = \varinjlim_V S^{-V} \wedge X_V.$$

It will be convenient to have a name for key property of the sequence $\{V_n\}$.

Definition 2.4. An increasing sequence $V_n \subset V_{n+1} \subset \ldots$ of finite dimensional representations of G is *exhausting* if any finite dimensional representation V of G admits an equivariant embedding in some V_n.

Clearly any two exhausting sequences are cofinal in each other, so any two presentations (2.3) can be nested into each other.

2.2.4. *Change of group.* For a subgroup $H \subset G$, there is a restriction functor

$$i^* = i_H^* : \mathcal{S}^G \to \mathcal{S}^H.$$

It is derived from an analogous functor $i_H^* : \mathcal{T}^G \to \mathcal{T}^H$ formed by simply restricting the G action to H. To describe the functor of spectra, choose an exhausting

sequence $\{W\}$ of G-representations, and let $\{i^*W\}$ be the same sequence regarded as an exhausting sequence of H-representations. For $X = \varinjlim_V S^{-V} \wedge X_V$ one has

$$i_H^* X = \varinjlim_{i^*V} S^{-i^*V} \wedge i^* X_V.$$

In case $H = \{e\}$ is the trivial group, we'll write the restriction functor as

$$i_0^* : \mathcal{S}^G \to \mathcal{S}.$$

The functor i_H^* has both a left and right adjoint

$$G_+ \underset{H}{\wedge} (-) \quad \text{and} \quad F_H(G_+, -).$$

If

$$Y = \varinjlim_{i^*W} S^{-i^*W} \wedge Y_{i^*W}$$

is an H-spectrum, then

$$G_+ \underset{H}{\wedge} (X) = \varinjlim_W S^{-W} \wedge G_+ \underset{H}{\wedge} Y_{i^*W}$$

$$F_H(G_+, Y) = \varinjlim_W S^{-W} \wedge F_H(G_+, Y_W),$$

where $F_H(G_+, Y_W)$ is the space of H-equivariant maps from G to Y_W. The Wirthmüller isomorphism ([34], [10, Theorem 4.10]) gives an equivariant weak equivalence

$$G_+ \underset{H}{\wedge} (-) \to F_H(G_+, -).$$

Because of this we will make little mention of the functor $F_H(G_+, X)$.

2.3. Multiplicative properties.

2.3.1. *Smash product.* The category $\mathcal{S}^G$ is a symmetric monoidal category under the smash product operation $X \wedge Y$. The tensor unit is the sphere spectrum S^0. The smash product commutes with enriched colimits in each variable. One has

$$S^{-V} \wedge S^{-W} = S^{-V \oplus W}$$

and in terms of the canonical presentation

$$X = \operatorname*{holim}_V S^{-V} \wedge X_V$$

$$Y = \operatorname*{holim}_W S^{-W} \wedge Y_W$$

one has

$$X \wedge Y = \operatorname*{holim}_{V,W} S^{-V \oplus W} \wedge X_V \wedge Y_W.$$

The smash product makes $\mathcal{S}^G$ into a *symmetric monoidal model category* in the sense of Hovey [15, Definition 4.2.6] and Schwede-Shipley [29, Definition 3.1]. This means that the analogue of Quillen's axiom SM7 holds (called the *pushout-product axiom* [29]), and for any cofibrant X, the map

$$\tilde{S}^0 \wedge X \to X$$

is a weak equivalence, where $\tilde{S}^0 \to S^0$ is a cofibrant approximation (called the *unit axiom* [29]).

The axioms of a symmetric monoidal category imply that one can make unambiguous sense of

$$\bigwedge_{j \in J} X_j \tag{2.5}$$

for any finite set J. The symmetric monoidal structure on $\mathcal{S}^G$ is actually G-monoidal in the sense that one can make sense of (2.5) when J is a finite G-set. In case all the $X_j = X$ one defines

$$\bigwedge_{j \in J} X = X^{(J)} = J_+ \underset{\Sigma_n}{\wedge} X^{(n)},$$

where $n = |J|$ and the group G is acting diagonally. It's important that this expression is not the homotopy quotient by the action of Σ_n, but actually the coequalizer of

$$J_+ \wedge (\Sigma_n)_+ \wedge X^{(n)} \rightrightarrows J_+ \wedge X^{(n)}.$$

More general expressions like (2.5) are formed by breaking J into orbits, using the construction just described, and smashing the results together.

2.3.2. *Commutative and associative algebras.*

Definition 2.6. A *commutative algebra* is a unital commutative monoid in $\mathcal{S}^G$ with respect to the smash product operation. An *associative algebra* is a unital associative monoid with respect to the smash product.

Commutative algebras will also be referred to as *equivariant commutative algebras*, E_∞-*algebras* and *equivariant E_∞-algebras*. Similarly, the terms *equivariant associative algebra*. A_∞-*algebra* and *equivariant A_∞ algebra* will be used for associative algebra.

There is a weaker "up to homotopy notion" that sometimes comes up.

Definition 2.7. A *homotopy associative algebra* is an associative algebra in ho $\mathcal{S}^G$. A *homotopy commutative algebra* is a commutative algebra in ho $\mathcal{S}^G$.

We will sometimes use the terms *equivariant homotopy associative (commutative) algebra, homotopy associative (commutative) algebra in $\mathcal{S}^G$*, or *equivariant homotopy associative (commutative) algebra*.

The category of commutative algebras in $\mathcal{S}^G$ will be denoted $\mathcal{E}_\infty^G$. It is tensored and cotensored over $\mathcal{T}^G$ and is a topological model category. The tensor product of an equivariant E_∞-ring spectrum R and a G-space T will be denoted

$$R \otimes T$$

to distinguish it from the smash product. By definition

$$\mathcal{E}_\infty^G(R \otimes T, E) = \mathcal{T}^G(T, \mathcal{E}_\infty^G(R, E)).$$

This "tensor product" with a G-space is related to the G-monoidal structure described in §2.3.1. Indeed, for a G-set J, the spectrum underlying $R \otimes J$ is just $R^{(J)}$.

The forgetful functor and its left adjoint, the free commutative algebra functor form a Quillen morphism

$$\mathcal{S}^G \leftrightarrows \mathcal{E}_\infty^G.$$

Modules over an equivariant E_∞ ring are defined in the evident way using the smash product. The category of left modules over R will be denoted $\mathcal{M}_R$. It is a symmetric monoidal model category under the operation

$$M \underset{R}{\wedge} N$$

where M is regarded as a right R-module via

$$M \wedge R \xrightarrow{\text{flip}} R \wedge M \to M.$$

A map of R-modules is a weak equivalence if and only if the underlying map of spectra is a weak equivalence.

The "free module" and "forgetful" functors

$$X \mapsto R \wedge X : \mathcal{S}^G \leftrightarrows \mathcal{M}_R : M \mapsto M$$

are adjoint and form a Quillen morphism.

2.3.3. *Norm induction.* In addition to the additive transfer, there is a multiplicative or *norm* transfer from $\mathcal{S}^H$ to $\mathcal{S}^G$ whenever $H \subset G$. The norm sends an H-spectrum X to the iterated smash product

$$\bigwedge_{i \in G/H} X_i,$$

where $X_i = (H_i)_+ \underset{H}{\wedge} X$, and $H_i \subset G$ is the right H-coset corresponding to i. This notion appeared first in group cohomology in the work of Evens [8], and is often referred to as the "Evens transfer" or the "norm transfer." The analogue in stable homotopy theory appears in the Greenlees-May [11].

We start with the construction for G-spaces. Suppose that $H \subset G$ is an inclusion of finite groups, and $X \in \mathcal{T}^H$. The norm of X is defined to be

$$N(X) = N_H^G(X) = \text{Map}_H(G, X)/\{f \mid f(g) = * \text{ for some } g \in G\}$$
$$= \bigwedge_{j \in H \backslash G} X^j,$$

where $X^j = \hom_H(H_j, X)$, and $H_j \subset G$ is the left H-coset corresponding to j. A choice of coset representative identifies X^i with X. This makes the G-action transparent, but for later purposes it will be more convenient to write

(2.8) $$N(X) = \bigwedge_{i \in G/H} X_i$$

with $X_i = H_i \times_H X$ and H_i the right coset corresponding to i. The correspondence between these two expressions is given by the map $g \mapsto g^{-1}$, which interchanges left and right cosets, and the left and right actions.

As described in the appendix, the norm functor is extended to a functor of spectra

$$N = N_H^G : \mathcal{S}^H \to \mathcal{S}^G.$$

by defining

$$NS^{-V} = S^{-\text{ind}_H^G V}$$

where $\operatorname{ind}_H^G V = \hom_H(G, V) \approx \rho_G \otimes_H V$ is the induced representation (and ρ_G is the real regular representation of V). As is evident from the canonical presentation, the analogue of (2.8) still holds

$$(2.9) \qquad i_0^* N(X) = \bigwedge_{i \in G/H} X_i$$

with

$$X_i = (H_i)_+ \bigwedge_H X.$$

The functor N is *weakly symmetric monoidal* in the sense that it is lax monoidal, and the map

$$N(X) \wedge N(Y) \to N(X \wedge Y)$$

is a weak equivalence whenever X and Y are cofibrant (this terminology is borrowed from Schwede-Shipley [29]). Among other things, this gives a weak equivalence

$$(2.10) \qquad NS^V \equiv S^{\operatorname{ind} V}$$

for any virtual representation V.

In terms of the canonical presentation

$$X = \operatorname*{holim}_{\overrightarrow{V}} S^{-V} \wedge X_V$$

one has

$$NX \approx \operatorname*{holim}_{\overrightarrow{V}} S^{-\operatorname{ind}_H^G V} \wedge N(X_V).$$

The norm distributes over wedges in much the same way as the iterated smash product. We will need an explicit formula for the norm of a wedge in a fairly special situation. In the proposition below, we consider the case of an abelian group G a subgroup $H \subset G$ and a wedge

$$X = \bigvee_{j \in J} X_j$$

of H-equivariant spectra. Associated to this is the left G-set

$$K = \hom(G/H, J),$$

and for each $\phi \in K$ the stabilizer group $H_\phi \subset G$ of ϕ. Thus H_ϕ consists of the elements $g \in G$ such that for all $x \in G/H$

$$\phi(gx) = \phi(x),$$

and the map ϕ factors through

$$\bar{\phi} : G/H_\phi \to J.$$

Set

$$X_{\bar{\phi}} = \bigwedge_{t \in G/H_\phi} X_{\bar{\phi}(t)},$$

and

$$X_\phi = N_H^{H_\phi} X_{\bar{\phi}}.$$

Informally, one can think of X_ϕ as

$$\bigwedge_{t \in G/H} X_{\phi(t)}.$$

Proposition 2.11. *Suppose that G is abelian, $H \subset G$ and*

$$X = \bigvee_{j \in J} X_j$$

is a wedge of H-spectra. Then with the notation just described

$$N_H^G X = \bigvee_{\phi \in K} G_+ \underset{H_\phi}{\wedge} X_\phi.$$

Proof: The canonical presentation reduces this to the analogous assertion for spaces, which is straightforward to check. □

Because it is lax monoidal, the functor N take commutative algebras to commutative algebras, and so induces a functor

$$N : \mathcal{E}_\infty^H \to \mathcal{E}_\infty^G.$$

The following result is proved in the Appendix.

Proposition 2.12. *The functor*

$$N : \mathcal{E}_\infty^H \to \mathcal{E}_\infty^G.$$

is left adjoint to the restriction functor i^*. Together they form a Quillen morphism of model categories.

The norm is also related to the G-monoidal structure. The following result is proved in the Appendix.

Proposition 2.13. *There is a natural isomorphism*

$$N i_H^* R \to R \otimes (G/H),$$

under which the counit of the adjunction is identified with the map

$$R \otimes (G/H) \to R \otimes (pt)$$

given by the unique G-map $G \to pt$.

A useful consequence Proposition 2.13 is that the group $Z(H)/H$ of G-automorphisms of G/H acts naturally on $N i_H^* R$. The result below is used in the main computational assertion of Proposition 4.51.

Corollary 2.14. *For $\gamma \in Z(H)/H$ the following diagram commutes*

$$N i_H^* R \xrightarrow{\ \gamma\ } N i_H^* R$$
$$\searrow \qquad \swarrow$$
$$R$$

Proof: Immediate from Proposition 2.13. □

2.3.4. *Other uses of the norm.* There are several important construction derived from the norm functor which also go by the name of "the norm."

Suppose that R is a G-equivariant E_∞ ring spectrum, and X is an H-spectrum for a subgroup $H \subset G$. Write

$$R_H^0(X) = [X, i_H^* R]^H.$$

There is a norm map

$$N : R_H^0(X) \to R_G^0(NX)$$

defined to be the composite

$$NX \to Ni_H^* R \to R$$

in which the second map is the counit of the restriction-norm adjunction. This is the *norm map on equivariant spectrum cohomology*, and is the form in which the norm is described in Greenlees-May [11].

When V is a representation of H and $X = S^V$ the above gives a map

$$N = N_H^G : \pi_V^H R \to \pi_{\mathrm{ind}\, V}^G R$$

where

$$\mathrm{ind}\, V = \hom_H(G, V)$$

is the induced representation.

Now suppose that X is a pointed G-space. There is a norm map

$$N : R_H^0(X) \to R_G^0(X)$$

sending

$$x \in R_H^0(X) = [S^0 \wedge X, i_H^* R]^H$$

to the composite

$$S^0 \wedge X \to S^0 \wedge NX \approx N(S^0 \wedge X) \to Ni_H^* R \to R,$$

in which

$$X \to NX = \hom_H(G, X)$$

is the map adjoint to the action map

$$G \underset{H}{\times} X \to X.$$

One can combine these construction to define the *norm on $RO(G)$-graded cohomology*

$$N : R_H^V(X) \to R_G^{\mathrm{ind}\, V}(X)$$

sending

$$S^0 \wedge X \xrightarrow{a} S^V \wedge i_H^* R$$

to the composite

$$S^0 \wedge X \to S^0 \wedge NX \xrightarrow{Na} S^{\mathrm{ind}\, V} \wedge Ni_H^* R \to S^{\mathrm{ind}\, V} \wedge R.$$

2.3.5. *The relative norm.* We'll also need a relative version of the norm functor N for modules over an equivariant E_∞ ring spectrum. Suppose that R is a G-equivariant E_∞ ring spectrum, and let $\mathcal{M}_R$ denote the category of left R-modules. The category $\mathcal{M}_R$ is a symmetric monoidal category under the operation $M_1 \underset{R}{\wedge} M_2$. We define

$$N = N^R : \mathcal{M}_{i_H^* R} \to \mathcal{M}_R$$

by

$$N^R M = R \underset{N i_H^* R}{\wedge} N M,$$

in which the E_∞ map

$$N i_H^* R \to R$$

is the counit of the adjunction described in Proposition 2.12.

Clearly N^R is a weakly symmetric monoidal functor and commutes with directed homotopy colimits.

2.3.6. *Equivariant polynomial algebras.* In this section we construct a class associative algebras which are in some sense equivariant polynomial extension of other rings. A word of warning, though. These ring spectra are not necessarily E_∞ rings, and are not free commutative algebras.

For a representation V of H let

$$S^0[S^V] = \bigvee_{k \geq 0} S^{kV}$$

be the free (H-equivariant) associative algebra generated by S^V, and

$$\bar{x} \in \pi_V^H S^0[S^V]$$

the homotopy class of the generating inclusion. The spectrum $S^0[S^V]$ is not a commutative algebra, though the $RO(H)$-equivariant homotopy groups make it appear so:

$$\pi_\star^H S^0[S^V] = \pi_\star^H S^0[\bar{x}].$$

It will be convenient to use the notation

$$S^0[\bar{x}] = S^0[S^V].$$

Using the norm we can then form the G-equivariant "polynomial" algebra

$$S^0[G \cdot S^V] = S^0[G \cdot \bar{x}] = N_H^G S^0[S^V].$$

By smashing examples like these together we can make associative algebras that "look like" equivariant polynomial algebras over S^0, in which the group G is allowed to act on the polynomial generators. More explicitly, suppose we are given a sequence of subgroups $H_i \subset G$ and for each i a virtual representation V_i of H_i. For each i form

$$S^0[G \cdot \bar{x}_i]$$

as described above, smash the first m together to make

$$S^0[G \cdot \bar{x}_1, \ldots, G \cdot x_m],$$

and then pass to the homotopy colimit to construct the G-equivariant associative algebra

$$S^0[G \cdot \bar{x}_1, G \cdot x_2, \ldots],$$

which we think of as an equivariant polynomial algebra over S^0.

All of this can be done relative to an equivariant commutative ring R by defining

$$R[G \cdot \bar{x}_1, G \cdot x_2, \dots]$$

to be

$$R \wedge S^0[G \cdot \bar{x}_1, G \cdot x_2, \dots].$$

2.3.7. *Weakly commutative algebras.*

Definition 2.15. An associative algebra A is *weakly commutative* if it is an associative algebra retract of a commutative algebra. More precisely, A is weakly commutative if there is an equivariant E_∞ ring spectrum R and equivariant associative algebra maps

$$A \to R \to A$$

whose composite is equal, in the homotopy category of associative algebras, to the identity map of A.

The identity map makes a commutative algebra weakly commutative. The following is straightforward.

Proposition 2.16. *If A is a weakly commutative H-spectrum then $N_H^G A$ is a weakly commutative G-spectrum. A (possibly infinite) smash product of weakly commutative ring spectra is weakly commutative.* $\square$

Definition 2.17. Suppose that

$$f_i : B_i \to A, \quad i = 1, \dots m$$

are algebra maps to a weakly commutative algebra $A \to R \to A$. The *smash product* of the f_i is the algebra map

$$\bigwedge^m f_i : \bigwedge^m B_i \to \bigwedge^m A \to \bigwedge^m R \to R \to A.$$

If B is an H-equivariant associative algebra, and $f : B \to i_H^* A$ is an algebra map, the *norm of f* is the G-equivariant algebra map

$$N_H^G B \to A$$

given by

$$NB \to N i_H^* A \to N i_H^* R \to R \to A.$$

Remark 2.18. Composition with an algebra map $g : A \to C$ of weakly commutative algebras does not necessarily preserve the smash product or norm. On the other hand, since the map of equivariant spectra underlying

$$\bigwedge^m f_i A$$

can be computed without reference to R, the smash product is functorial as a map of underlying equivariant spectra.

It's easy to map a polynomial algebra to a weakly commutative algebra. Suppose that A is a G-equivariant weakly commutative algebra, and we're given a sequence

$$\bar{x}_i \in \pi_{V_i}^{H_i} A, \qquad i = 1, 2, \dots$$

A choice of representative

$$S^{V_i} \to A$$

of $\bar{x}_i$ determines an associative algebra map

$$S^0[\bar{x}_i] \to A.$$

Applying the norm gives a G-equivariant associative algebra map

$$S^0[G \cdot \bar{x}_i] \to A.$$

By smashing these together we can make a sequence of equivariant algebra maps

$$S^0[G \cdot \bar{x}_1, \ldots, G \cdot x_m] \to A.$$

Passing to the homotopy colimit gives an equivariant algebra map

$$S^0[G \cdot \bar{x}_1, G \cdot x_2, \ldots] \to A$$

representing the sequence $\bar{x}_i$.

2.3.8. *Quotient rings.* One important construction in ordinary stable homotopy theory is the formation of the quotient of an E_∞ ring spectrum R by the ideal generated by a regular sequence $\{x_1, x_2, \ldots\} \subset \pi_* R$. This is done by inductively forming the cofibration sequence of R-modules

$$\Sigma^{|x_n|} R/(x_1, \ldots, x_{n-1}) \to R/(x_1, \ldots, x_{n-1}) \to R/(x_1, \ldots, x_n)$$

and passing to the colimit in the end. One can also write

$$R/(x_1, \ldots, x_n) = R/(x_1) \underset{R}{\wedge} \cdots \underset{R}{\wedge} R/(x_n).$$

The situation is slightly trickier in equivariant stable homotopy theory, where the group G might be permuting the elements x_i, and preventing the inductive approach described above. One can get around this difficulty using the functor N^R as we now describe. We are indebted to Jacob Lurie for suggesting that the norm functor might be used in this way.

Fix $H \subseteq G$ and a set $\{g_i\}$ of coset representatives. Suppose that R is a G-equivariant E_∞ ring spectrum, and to ease the notation a little write

$$R_H = i_H^* R$$
$$R_0 = i_0^* R.$$

Given a sequence of elements $x_i \in \pi_{V_i}^H R_H$ form the equivariant R_H-module

$$R_H/(x_i) = \underrightarrow{\mathrm{holim}}\, R_H/(x_1, \ldots, x_n)$$

as in the non-equivariant case, using the cofibration sequences

$$S^{V_i} \wedge R_H/(x_1, \ldots, x_{n-1}) \to R_H/(x_1, \ldots, x_{n-1}) \to R_H/(x_1, \ldots, x_n).$$

We define $R/(G\,x_j)$ by

$$R/(G \cdot x_j) = N^R(R_H/(x_j)).$$

Even though the construction does not depend on the choice of coset representatives $\{g_i\}$ it is useful to employ the notation

$$R/(G \cdot x_j) = R/(g_i x_j)$$

as a reminder of the factors being smashed together (Lemma 2.23 below).

There is another very useful description of $R/(G\,x_j)$. For each j choose an H-equivariant map

$$S^{V_j} \to i_H^* R$$

representing x_j, and extend it to a map of associative algebras

$$S[x_j] \to i_H^* R$$

where

$$S[x_j] = \bigvee_{k \geq 0} S^{kV_j}$$

is the free H-equivariant associative algebra generated by S^{V_j}. Now apply the norm to get a G-equivariant algebra map

$$S[G \cdot x_j] \to R$$

with $S[G \cdot x_j] = N_H^G S[x_j]$. Similarly, use the constant map

$$S^{V_j} \to *$$

to define an algebra map

$$S[G \cdot x_j] \to S^0.$$

Smashing these together over j, multiplying, and passing to the colimit gives the following diagram of G-equivariant associative algebras

(2.19) $$R \leftarrow S[G \cdot x_1, G \cdot x_2, \dots] \to S^0$$

Lemma 2.20. *The R-module $R/(g_i, x_j)$ is isomorphic to*

$$R \underset{T}{\wedge} S^0 \approx R \underset{R \wedge T}{\wedge} R,$$

where

$$T = S[G \cdot x_1, G \cdot x_2 \dots].$$

Remark 2.21. The smash products in the above lemma are the "derived" smash products, and are given by the evident two-sided bar constructions. Before forming them the map $T \to S^0$ needs to be replaced by a cofibration.

Proof: Write

$$R_H = i_H^* R$$
$$T_H = S[x_1, x_2, \dots].$$

Because the norm is symmetric monoidal it suffices to prove that

$$R_H/(x_1, \dots) \approx R_H \underset{T_H}{\wedge} S^0.$$

By induction and passage to colimits, this in turn reduces to the assertion that for an R_H-module M,

$$M \underset{R_H}{\wedge} R_H/(x) \approx M \underset{S[x]}{\wedge} S^0,$$

for a single $x \in \pi_V^H R$. One easily checks that both sides are given as the cofiber of the R_H-module map

$$\Sigma^V M \xrightarrow{x} M$$

given by multiplication by x. $\square$

We will also make use of the properties of quotient rings described in the next two lemmas.

Lemma 2.22. *There are natural weak equivalences of equivariant R-modules*

$$R/(g_i x_1, \ldots, g_i x_n) \approx R/(g_i x_1) \underset{R}{\wedge} \cdots \underset{R}{\wedge} R/(g_i x_n)$$

$$R/(g_i x_j) \approx \varinjlim_n R/(g_i x_1) \underset{R}{\wedge} \cdots \underset{R}{\wedge} R/(g_i x_n)$$

Proof: The second assertion follows easily from the fact that N^R commutes with the formation of directed colimits. The first is formal consequence of the fact that it is weakly symmetric monoidal, and is proved by induction on n. To keep the notation under control let (x_k) stand for the sequence $(x_1, \ldots, x_{n-1})$. For the inductive step write

$$R_H/(x_k, x_n) = R_H \underset{R_H \wedge R_H}{\wedge} (R_H/(x_k) \wedge R_H/(x_n))$$

and note that

$$
\begin{aligned}
R/(G \cdot x_k, G \cdot x_n) &= N^R R_H/(x_k, x_n) \\
&= R \underset{N R_H}{\wedge} N(R_H/(x_k, x_n)) \\
&= R \underset{N R_H}{\wedge} N(R_H \underset{(R_H \wedge R_H)}{\wedge} (R_H/(x_k) \wedge R_H/(x_n))) \\
&= R \underset{(N R_H \wedge N R_H)}{\wedge} (N R_H/(x_k) \wedge N R_H/(x_n)) \\
&= R \underset{(R \wedge R)}{\wedge} (R \wedge R) \underset{(N R_H \wedge N R_H)}{\wedge} (N R_H/(x_k) \wedge N R_H/(x_n)) \\
&= R \underset{(R \wedge R)}{\wedge} (R/(G \cdot x_k) \wedge R/(G \cdot x_n)) \\
&= R/(G \cdot x_k) \underset{R}{\wedge} R/(G \cdot x_n).
\end{aligned}
$$

$\square$

Lemma 2.23. *The R_0-module $i_0^*(R/(G \cdot x_j))$ is naturally weakly equivalent to the quotient $(R_0)/(g_i x_j)$.*

Proof: By Lemma 2.22 it suffices to consider the case of a single x_j. In that case $R_H/(x_j)$ sits in a cofibration sequence

$$(2.24) \qquad\qquad S^{V_j} \wedge R_H \xrightarrow{x_j} R_H \to R_H/(x_j).$$

By the decomposition (2.9), the factors being smashed together over R_0 to form $i_0^* R/(G \cdot x_j)$ are

$$R_0 \underset{R_i}{\wedge} M_i,$$

where

$$R_i = H_i \underset{H}{\otimes} i_0^* R_H \approx (H_i)_+ \underset{H}{\wedge} R_0$$

and M_i is gotten from $i_0^* R_H/(x_j)$ by applying $(H_i)_+ \underset{H}{\wedge} (-)$ to (2.24). The choice of coset representative $g_i \in H_i$ identifies

$$R_i \to R_0$$

with

$$g : R_0 \to R_0.$$

The result then follows easily. $\square$

We end this section with a remark that plays an important role in the proof of the Reduction Theorem in §7.

Remark 2.25. It follows from the fact that N^R is symmetric monoidal that if $R_H/(x_j)$ is an R_H-algebra. Then the R-module $R/(g_i x_j)$ is naturally an R-algebra.

2.4. **Fixed points and geometric fixed points.**

2.4.1. *Fixed point spectra.* The *fixed point spectrum* of a G-spectrum X is defined to be the spectrum of G fixed points in the underlying, non-equivariant spectrum $i_0^* X$. In other words it is given by

$$X \mapsto i_0^* X^G.$$

The functor of fixed points is right adjoint the functor sending $S^{-V} \wedge X_V \in \mathcal{S}$ to $S^{-V} \wedge X_V \in \mathcal{S}^G$, where in the latter expression, V is regarded as a representation of G with trivial G-action and X_V is regarded as a space with trivial G-action. The fixed point functor doesn't always have the properties one might expect. For example it does not generally commute with smash products, or with the formation of suspension spectra.

2.4.2. *The geometric fixed point spectrum.* There is a related *geometric fixed point functor*, Φ^G which has much more convenient properties. In terms of the canonical presentation

$$X \approx \underset{\overrightarrow{V}}{\mathrm{holim}}\, S^{-V} \wedge X_V$$

one has

$$(2.26) \qquad \Phi^G X \approx \underset{\overrightarrow{V}}{\mathrm{holim}}\, S^{-V_0} X_V^G,$$

where $V_0 \subset V$ is the space of G-invariant vectors, and X_V^G is the space of fixed points.

The functor Φ^G has the following properties

    i) it is weakly monoidal and commutes with directed homotopy colimits;
    ii) it commutes with the formation of suspension spectra of cofibrant objects.

In fact these properties determine Φ^G up to weak equivalence. To see this note that for a representation V of G one has

$$S^0 = \Phi^G S^0 \approx \Phi^G(S^V \wedge S^{-V}) \approx \Phi^G S^V \wedge \Phi^G S^{-V} \approx S^{V_0} \wedge \Phi^G S^{-V},$$

where $V_0 \subset V$ is the part on which G acts trivially. Thus the properties above imply

    iii) $\Phi^G S^{-V} = S^{-V_0}$.

Threading Φ^G through the canonical presentation then leads to the formula (2.26).

Because Φ^G is weakly monoidal, it determines a functor

$$\Phi^G : \mathcal{E}_\infty^G \to \mathcal{E}_\infty.$$

2.4.3. *Isotropy separation.* These properties often make it easy to compute $\Phi^G X$, but that's only half of the story. What makes the notion so fundamental is the fact that there is a completely different approach. Let $\mathcal{F}$ denote the family of proper subgroups of G, and $E\mathcal{F}$ the "classifying space" for $\mathcal{F}$, characterized by the property that the space of fixed points $E\mathcal{F}^G$ is empty, while for any proper $H \subset G$, $E\mathcal{F}^H$ is contractible. The space $E\mathcal{F}$ can be constructed as the join of infinitely many copies of G/H with H ranging through the proper subgroups of G. Let $\tilde{E}\mathcal{F}$ be the mapping cone of $E\mathcal{F} \to \mathrm{pt}$, with the cone point taken as base point. Smashing with a G spectrum X gives the *isotropy separation sequence*

(2.27) $$E\mathcal{F}_+ \wedge X \to X \to \tilde{E}\mathcal{F} \wedge X.$$

The remarkable thing about $\tilde{E}\mathcal{F} \wedge X$ is that there is an equivalence

$$\Phi^G X \approx i_0^*(\tilde{E}\mathcal{F} \wedge X)^G.$$

So one can harness the fixed point spectrum of X by determining the geometric fixed points $\Phi^G X$ and the fixed points in $E\mathcal{F}_+ \wedge X$. But $E\mathcal{F}_+ \wedge X$ is built entirely from G-cells of the form $G/H_+ \wedge S^n$ with H a proper subgroup of G, and so one can often get at its fixed points by induction.

When $G = C_{2^n}$, the space $E\mathcal{F}$ is the space EC_2 with G acting through the epimorphism $G \to C_2$. Because of this we'll use the notation $EC_{2+} \wedge X$ and $\tilde{E}C_2 \wedge X$ instead of $E\mathcal{F}_+ \wedge X$ and $\tilde{E}\mathcal{F} \wedge X$.

Remark 2.28. The isotropy separation sequence often leads to the situation of needing to show that a map $X \to Y$ of G-spectra induces a weak equivalence

$$\tilde{E}\mathcal{F} \wedge X \to \tilde{E}\mathcal{F} \wedge Y.$$

Since for every proper $H \subset G$, $\pi_*^H \tilde{E}\mathcal{F} \wedge X = \pi_*^H \tilde{E}\mathcal{F} \wedge Y = 0$, this is equivalent to showing that the map of geometric fixed point spectra $\Phi^G X \to \Phi^G Y$ is a weak equivalence.

Remark 2.29. Since for every proper $H \subset G$, $\pi_*^H \tilde{E}\mathcal{F} \wedge X = 0$, it is also true that

$$[T, \tilde{E}\mathcal{F} \wedge X]_*^G = 0$$

for every G-CW spectrum T built entirely from G-cells of the form $G_+ \underset{H}{\wedge} D^n$ with H a proper subgroup of G. Similarly, if T is gotten from T_0 by attaching G-cells induced from proper subgroups, then the restriction map

$$[T, \tilde{E}\mathcal{F} \wedge X]_*^G \to [T_0, \tilde{E}\mathcal{F} \wedge X]_*^G$$

is an isomorphism. This holds, for example, if T is the suspension spectrum of a G-CW complex, and $T_0 \subset T$ is the subcomplex of G-fixed points.

Remark 2.30. For a subgroup $H \subset G$ and a G-spectrum X it will be convenient to use the abbreviation

$$\Phi^H X$$

for the more correct $\Phi^H i_H^* X$. This situation comes up in our proof of the "homotopy fixed point" property of Theorem 11.3, where the more compound notation becomes a little unwieldy.

2.4.4. *Multiplicative properties.* The geometric fixed point construction interacts well with the norm. Suppose that $H \subset G$. Note that for a representation V of H and a space X_V with an H-action we have

$$\Phi^G N(S^{-V} \wedge X_V) \approx \Phi^G(S^{-\operatorname{ind} V} \wedge NX_V) \approx S^{-V_0} \wedge (NX_V)^G$$
$$\approx S^{-V_0} \wedge X_V^H \approx \Phi^H(S^{-V} \wedge X_V),$$

where $V_0 \subset V$ is the space of H-fixed vectors. Using the canonical presentation this leads easily to

Proposition 2.31. *There is a natural weak equivalence*
$$\Phi^G NX \approx \Phi^H X.$$

There is also an analogue for the relative norm

Proposition 2.32. *Suppose that R is an equivariant E_∞ ring spectrum. For cofibrant $i_H^* R$-modules there is a natural weak equivalence*
$$\Phi^G(N^R M) \approx \Phi^G R \underset{\Phi^H i^* R}{\wedge} \Phi^H(M).$$

Proposition 2.32 has a simple but very useful consequence.

Lemma 2.33. *The composite functor*
$$\Phi^G N^R : \mathcal{M}_{i^* R} \to \mathcal{M}_{\Phi^G R}$$
is homotopy colimit preserving. In particular, it preserves wedges and cofiber sequences. $\square$

Proposition 2.32 leads to a formula for the geometric fixed point spectrum of a quotient module. As in §2.3.8, suppose that R is a G-equivariant E_∞ ring spectrum, $H \subset G$, and $x \in \pi_V^H R$.

Proposition 2.34. *There is a cofibration sequence of R-modules*
$$\Sigma^{V_0} \Phi^G R \xrightarrow{y} \Phi^G R \to \Phi^G R/(G \cdot x)$$
in which
$$y = \Phi^G Nx = \Phi^H x.$$

Proof: Starting from the cofibration sequence of R_H-modules
$$\Sigma^V R_H \xrightarrow{x} R_H \to R_H/x$$
apply $\Phi^G \circ N^R$ and use Propositions 2.32 and 2.33. $\square$

There is another useful result describing the interaction of the geometric fixed point functor with the norm map in $RO(G)$-graded cohomology described in §2.3.4. Again, suppose that R is a G-equivariant E_∞ ring spectrum, X is a G-space, and V a real representation of H. One can then compose the norm
$$N : R_H^V(X) \to R_G^{\operatorname{ind} V}(X)$$
with the geometric fixed point map
$$\Phi^G : R_G^{\operatorname{ind} V}(X) \to (\Phi^G R)^{V_0}(X^G),$$
where $V_0 \subset V$ is the subspace of H-fixed vectors, and X^G is the space of G-fixed points in X.

Proposition 2.35. *The composite*

$$\Phi^G N : R_H^V(X) \to (\Phi^G R)^{V_0}(X^G)$$

is a ring homomorphism.

Proof: Multiplicativity is a consequence of the fact that both the norm and the geometric fixed point functors are weakly monoidal. Additivity follows from the fact that the composition $\Phi^G \circ N$ preserves wedges (Lemma 2.33). $\square$

2.5. **Preliminary computations in equivariant cohomology.**

2.5.1. *Mackey Functors and cohomology.* In equivariant homotopy theory, the role of "abelian group" is played by the notion of a *Mackey functor*. The following formulation is taken from Greenlees-May [10].

Definition 2.36 (Dress [5])**.** A *Mackey functor* consists of a pair $M = (M_*, M^*)$ of functors on the category of finite G-sets. The two functors have the same object function (denote M) and take disjoint unions to direct sums. The functor M_* is covariant, while M^* is contravariant, and together they take a pullback diagram of finite G-sets

$$
\begin{array}{ccc}
P & \xrightarrow{\delta} & X \\
\gamma \downarrow & & \downarrow \alpha \\
Y & \xrightarrow{\beta} & Z
\end{array}
$$

to a commutative square

$$
\begin{array}{ccc}
M(P) & \xrightarrow{\delta_*} & M(X) \\
\gamma^* \uparrow & & \uparrow \alpha^* \\
M(Y) & \xrightarrow{\beta_*} & M(Z)
\end{array}
$$

where $\alpha^* = M^*(\alpha)$, $\beta_* = M_*(\beta)$, etc.

A Mackey functor can also be defined as a contravariant additive functor from the full subcategory of $\mathcal{S}^G$ consisting of the suspension spectra $\Sigma^\infty B_+$ of finite G-sets B. It is a non-trivial result of tomDieck that these definitions are equivalent. See [10, §5].

The equivariant homotopy groups of a G-spectrum X are naturally part of the Mackey functor defined by

$$\underline{\pi}_n X(B) = \text{ho}\,\mathcal{S}^G(S^n \wedge B_+, X).$$

For $B = G/H$ one has

$$\underline{\pi}_n X(B) = \pi_n^H X.$$

As described above, the symbol $\underline{\pi}_n X$ will refer to this Mackey functor, though we'll make little use of the notation $\underline{\pi}_n X(B)$, using $\pi_n^H X$ instead.

Just as every abelian group can occur as a stable homotopy group, every Mackey functor M can occur as an equivariant stable homotopy group. In fact associated

to each Mackey functor M is an equivariant Eilenberg-Mac Lane spectrum HM, characterized by the property

$$\underline{\pi}_n HM = \begin{cases} M & n = 0 \\ 0 & n \neq 0. \end{cases}$$

The homology and cohomology groups of a G-spectrum X with coefficients in M are defined by

$$H_k^G(X; M) = \pi_k^G HM \wedge X$$
$$H_G^k(X; M) = \mathrm{ho}\, \mathcal{S}^G(X, \Sigma^k HM).$$

For a pointed G-space Y one defines

$$H_n^G(Y; M) = H_n(S^0 \wedge Y; M)$$
$$H_G^n(Y; M) = H^n(S^0 \wedge Y; M).$$

One particularly important example is the "constant" Mackey functor $\underline{\mathbb{Z}}$ represented on the category of G-sets by the abelian group $\mathbb{Z}$ with trivial G-action. By definition, the value of $\underline{\mathbb{Z}}$ on a finite G-set $\mathcal{O}$ is the group of functions

$$\underline{\mathbb{Z}}(\mathcal{O}) = \hom^G(\mathcal{O}, \mathbb{Z}) = \hom(\mathcal{O}/G, \mathbb{Z}).$$

The restriction maps are given by (pre-)composition. For $K \subset H \subset G$, the transfer map associated by $\underline{\mathbb{Z}}$ to the covering

$$G/K \to G/H$$

is the map $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by the index of K in H.

It will also be useful to have the notation

$$H_n^u(X; \mathbb{Z}) \text{ and } H_u^n(X; \mathbb{Z})$$

for the ordinary, non-equivariant homology and cohomology groups of the underlying spectrum $i_0^* X$. Of course there are isomorphisms

$$H_n^u(X; \mathbb{Z}) \approx H_n^G(G_+ \wedge X; \underline{\mathbb{Z}})$$
$$H_u^n(X; \mathbb{Z}) \approx H_G^n(G_+ \wedge X; \underline{\mathbb{Z}}).$$

The Mackey functor homology and cohomology groups of a G-CW spectrum Y can be computed from a chain complex analogous to the complex of cellular chains. Write $Y^{(n)}$ for the n-skeleton of Y so that

$$Y^{(n)}/Y^{(n-1)} \approx B_+ \wedge S^n$$

with B a discrete G-set. Set

$$C_n^{\mathrm{cell}}(Y; M) = \pi_n HM \wedge Y^{(n)}/Y^{(n-1)} = \pi_0 HM \wedge B_+$$
$$C_{\mathrm{cell}}^n(Y; M) = [Y^{(n)}/Y^{(n-1)}, \Sigma^n HM]^G = [\Sigma^\infty B_+, HM]^G.$$

The map

$$Y^{(n)}/Y^{(n-1)} \to \Sigma Y^{(n-1)}/Y^{(n-2)}$$

defines boundary and coboundary maps

$$C_n^{\mathrm{cell}}(Y; M) \to C_{n-1}^{\mathrm{cell}}(Y; M)$$
$$C_{\mathrm{cell}}^{n-1}(Y; M) \to C_{\mathrm{cell}}^n(Y; M).$$

The equivariant homology and cohomology groups of Y with coefficients in M are the homology and cohomology groups of these complexes. By writing the G-set B

as a coproduct of finite G-sets B_α one can express $C_n^{\text{cell}}(Y; M)$ and $C_{\text{cell}}^n(Y; M)$ in terms of the values of the Mackey functor M on the B_α.

2.5.2. *$RO(G)$-graded homotopy groups.* In addition to the Mackey functor homotopy groups $\underline{\pi}_* X$ there are the $RO(G)$ graded homotopy groups $\pi_\star X$ defined by

$$\pi_V^G X = \text{ho}\, \mathcal{S}^G(S^V, X) \qquad V \in RO(G).$$

Here $RO(G)$ is the Grothendieck group of real representations of G. The use of $\star$ for the wildcard symbol in $\pi_\star$ is taken from Hu-Kriz [16]. The $RO(G)$-graded homotopy groups are also part of a Mackey functor $\underline{\pi}_\star(X)$ defined by

$$\underline{\pi}_V X(B) = \text{ho}\, \mathcal{S}^G(S^V \wedge B_+, X).$$

As with $\mathbb{Z}$-graded homotopy groups, we'll use the abbreviation

$$\underline{\pi}_V^H X = \underline{\pi}_V X(G/H).$$

There are a few distinguished elements of $RO(G)$-graded homotopy groups we'll need. For a representation V of G we let

$$a_V \in \pi_{-V}^G S^0$$

be the equivariant map

$$(2.37) \qquad\qquad a_V : S^0 \to S^V$$

corresponding to the inclusion $\{0\} \subset V$. The element a_V is the equivariant Euler class of V. If V contains a trivial representation then $a_V = 0$. For two representations V and W one has

$$a_{V \oplus W} = a_V a_W \in \pi_{-V-W}^G S^0.$$

If V is an *oriented* representation of G of dimension d, there is a unique map

$$(2.38) \qquad\qquad u_V : S^d \to H\underline{\mathbb{Z}} \wedge S^V$$

with the property that $i_0^* u_V$ is the generator of the non-equivariant homology group $H_d^u(S^V; \mathbb{Z})$ corresponding to the orientation of V (see Example 2.41 below). We'll regard u_V as an element of the $RO(G)$-graded group

$$\pi_{d-V}^G H\underline{\mathbb{Z}}$$

If V and W are two oriented representations of G, and $V \oplus W$ is given the direct sum orientation, then

$$u_{V \oplus W} = u_V u_W.$$

Among other things this implies that the class u_V is stable in V in the sense that $u_{V+1} = u_V$.

For any V, the representation $V \oplus V$ has a canonical orientation, and so there's always a class

$$u_{V \oplus V} \in \pi_{2d-2V}^G H\underline{\mathbb{Z}}.$$

When V is oriented this class can be identified, up to sign, with u_V^2.

The classes a_V and u_V behave well with respect to the norm. The following result is a simple consequence of the fact (2.10) that $NS^V = S^{\text{ind}\, V}$.

Lemma 2.39. *Suppose that V is a representation of a subgroup $H \subset G$ of dimension d. Then*

$$Na_V = a_{\mathrm{ind}\,V}$$
$$u_{\mathrm{ind}\,d} \cdot Nu_V = u_{\mathrm{ind}\,V},$$

where $\mathrm{ind}\,V = \mathrm{ind}_H^G V$ is the induced representation and d is the trivial representation. $\square$

It is sometimes useful to think of the second identity above as

$$Nu_V = u_{(\mathrm{ind}\,V - \mathrm{ind}\,d)} = u_{\mathrm{ind}(V-d)},$$

even though the symbol $u_{\mathrm{ind}(V-d)}$ has no defined meaning.

In the case $G = C_{2^n}$ the sign representation of dimension 1 will play an important role. We'll denote this representation as σ, or σ_G if more than one cyclic group is under consideration.

2.6. **Computations with $H\mathbb{Z}$.** We now turn to the special case of the constant Mackey functor $\underline{\mathbb{Z}}$, and some computations which will play an important role later in the paper.

Lemma 2.40. *Suppose that B is a discrete G-set. The Mackey functor $\pi_0 H\underline{\mathbb{Z}} \wedge B_+$ is the functor associating to each finite G-set $\mathcal{O}$ the group of equivariant homomorphisms*

$$\mathcal{O} \to \mathbb{Z}[B]$$

from $\mathcal{O}$ to the free abelian group on B.

Proof: One reduces easily to the case in which B is finite. Using equivariant Spanier-Whitehead duality one finds

$$[\mathcal{O}_+, H\underline{\mathbb{Z}} \wedge B_+]^G = [\mathcal{O}_+ \wedge B_+, H\underline{\mathbb{Z}}]^G = \hom(\mathcal{O} \underset{G}{\times} B, \mathbb{Z}).$$

from which the result follows. $\square$

Suppose that Y is a G-CW complex, with n-skeleton denoted $Y^{(n)}$. Then by definition $Y^{(n)}/Y^{(n-1)} = B_+ \wedge S^n$ for some discrete G-set B. The Mackey functor $\pi_n H\underline{\mathbb{Z}} \wedge Y^{(n)}/Y^{(n-1)}$ is then the one associating to $\mathcal{O}$ the group of equivariant functions

$$\mathcal{O} \to C_n^{\mathrm{cell}} Y$$

and so the Mackey functor chain complex for $H\mathbb{Z} \wedge Y$ is just the usual cellular chain complex for Y associated to a G-equivariant cell decomposition. The equivariant homology group $H_*^G(Y; \underline{\mathbb{Z}})$ are just the homology groups of the complex

$$C_*^{\mathrm{cell}}(Y)^G$$

of G-invariant cellular chains. Similarly the equivariant cohomology groups $H_G^*(Y; \mathbb{Z})$ are given by the cohomology groups of the complex

$$C^*_{\mathrm{cell}}(Y)^G$$

of equivariant cochains. The equivariant homology and cohomology groups depend only on the equivariant chain homotopy type of these complexes.

Example 2.41. Suppose that V is a representation of G of dimension d, and consider the equivariant cellular chain complex

$$C_d^{\text{cell}} S^V \to C_{d-1}^{\text{cell}} S^V \to \cdots \to C_0^{\text{cell}} S^V.$$

The homology groups are those of the sphere S^V, and so in particular the kernel of

$$C_d^{\text{cell}} S^V \to C_{d-1}^{\text{cell}} S^V$$

is isomorphic, as a G-module, to $H_d^u(S^V; \mathbb{Z})$. If V is orientable then the G-action is trivial, and one finds that the restriction map

$$H_d^G(S^V; \mathbb{Z}) \to H_d^u(S^V; \mathbb{Z})$$

is an isomorphism. A choice of orientation gives equivariant isomorphism

$$H_d^u(S^V; \mathbb{Z}) \approx \mathbb{Z}$$

Thus when V is oriented there is unique isomorphism

$$H_d^G(S^V; \mathbb{Z}) \approx \mathbb{Z}$$

extending the non-equivariant isomorphism given by the orientation.

We now turn to some specific computations which will play an important role in this paper. Let G be the group C_{2^n}, and write ρ_G for its real regular representation. We calculate the groups

$$\tilde{H}_*^G(S^{m\rho_G}; \underline{\mathbb{Z}}) \text{ and } \tilde{H}_G^*(S^{m\rho_G}; \underline{\mathbb{Z}})$$

for $m > 0$. In this section we regard $S^{m\rho_G}$ as a G-space, and not a G-spectrum. Let's first start with $n = 1$. In this case the G-fixed point space is S^m and so the complex of equivariant cellular chains starts with a $\mathbb{Z}$ in dimension m. The remaining cells, ranging in dimensions from $m+1$ to $2m$ are free, and so the complex of cellular chains takes the form

$$C_{2m} \to C_{2m-1} \to \cdots \to C_{m+1} \to \mathbb{Z}$$

where each C_i is a free module over G. The homology of the underlying chain complex consists only of the group $\mathbb{Z}$ in dimension $2m$. From this it is a simple matter to check that the complex of equivariant chains on $S^{m\rho_G}$ is equivariantly chain homotopy equivalent to the complex

$$\mathbb{Z}[G] \to \cdots \to \mathbb{Z}[G] \to \mathbb{Z}$$

ranging in dimensions from m to $2m$. The differentials are the usual ones occurring in the minimal resolution of the trivial module $\mathbb{Z}$.

Returning to the case of $G = C_{2^n}$, and for $k \le n$ write G_k for the unique quotient of G of order 2^k. The fixed point space of the $C_2 \subset G$ action on $S^{m\rho_G}$ is a sphere of dimension $m \, 2^{n-1}$, and as a G_{n-1}-space can be identified with $S^{m\rho_{G_{n-1}}}$. The kernel of

$$C_{m \, 2^{n-1}}^{\text{cell}}(S^{m\rho_G}) \to C_{m \, 2^{n-1}-1}^{\text{cell}}(S^{m\rho_G})$$

is therefore $\mathbb{Z}_\pm^{\otimes m}$, a copy of $\mathbb{Z}$ equipped with the action of G on which a generator acts by $(-1)^m$.

The remaining cells of $S^{m\rho_G}$ are free, and range in dimension from $(m \, 2^{n-1} + 1)$ to $m \, 2^n$. The portion of the equivariant cellular chain complex for $S^{m\rho_G}$ in this range of dimension therefore fits into a sequence

$$C_{m \, 2^n} \to \cdots \to C_{m \, 2^{n-1}+1} \to \mathbb{Z}_\pm^{\otimes m}$$

which is exact everywhere except at the leftmost term, where the kernel is $\mathbb{Z}_{\pm}^{\otimes m}$. A simple argument with basic homological algebra shows that this complex is equivariantly chain homotopy equivalent to

$$\mathbb{Z}[G] \to \cdots \to \mathbb{Z}[G] \to \mathbb{Z}_{\pm}^{\otimes m}$$

with the copies if $\mathbb{Z}[G]$ ranging in dimensions from $(m\,2^{n-1}+1)$ to $m\,2^n$.

Putting it all together one finds that the complex of equivariant cellular chains on $S^{m\rho_G}$ is equivariantly chain homotopy equivalent to the composition of the complexes

$$\mathbb{Z}[G_k] \to \cdots \to \mathbb{Z}[G_{2^k}] \to \mathbb{Z}_{\pm}^{\otimes m}$$

with $2 < k \leq n$, and

$$\mathbb{Z}[G_2] \to \cdots \to \mathbb{Z}[G_2] \to \mathbb{Z}.$$

For example, when $G = C_8$ and $m = 1$, the complex of equivariant cellular chains is constructed by composing

$$\mathbb{Z}[G_8] \to \mathbb{Z}[G_8] \to \mathbb{Z}[G_8] \to \mathbb{Z}[G_8] \to \mathbb{Z}_{\pm}$$
$$\mathbb{Z}[G_4] \to \mathbb{Z}[G_4] \to \mathbb{Z}_{\pm}$$
$$\mathbb{Z}[G_2] \to \mathbb{Z}$$

to give

$$\mathbb{Z}[G_8] \to \mathbb{Z}[G_8] \to \mathbb{Z}[G_8] \to \mathbb{Z}[G_8] \to \mathbb{Z}[G_4] \to \mathbb{Z}[G_4] \to \mathbb{Z}[G_2] \to \mathbb{Z},$$

with the $\mathbb{Z}$ on the right in dimension 1. Passing to invariants gives

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z}.$$

The complex of equivariant maps to $\mathbb{Z}$ (for calculating equivariant cohomology) is

$$\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{1} \mathbb{Z}.$$

For $G = C_8$ and $m = 2$ the complex of invariants works out to be

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{8} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{8} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{8} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{8} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{4} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{4} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z},$$

with the rightmost $\mathbb{Z}$ in dimension 2. The complex for computing cohomology is

$$\mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{8} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{8} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{8} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{8} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{4} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{4} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{1} \mathbb{Z}.$$

We will return to this kind of computations in much greater detail in a later paper.

Proposition 2.42. *Let $G = C_{2^n}$. For any G-spectrum X, the $RO(G)$-graded homotopy groups of $\Phi^G X$ are given by*

$$\pi_\star^G \Phi^G X = a_\sigma^{-1} \pi_\star^G X.$$

The homotopy groups of the E_∞ ring spectrum $\Phi^G H\mathbb{Z}$ are given by

$$\pi_\star^G \Phi^G H\mathbb{Z} = \mathbb{Z}/2[b],$$

where $b = u_{2\sigma} a_\sigma^{-2} \in \pi_2 \Phi^G H\mathbb{Z} \subset a_\sigma^{-1} \pi_\star^G H\mathbb{Z}$.

Proof: The space $\tilde{E}C_2$ is $\lim_{n\to\infty} S^{n\sigma}$. This gives the first assertion. The usual equivariant cell decomposition of S^m with the antipodal action gives a complex of equivariant chains on $\lim_{n\to\infty} S^{n\sigma}$

$$\cdots \to \mathbb{Z}[G_2] \to \cdots \to \mathbb{Z}[G_2] \to \mathbb{Z}[G_2] \to \mathbb{Z}.$$

The complex of invariants is

$$\cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z}$$

from which it follows that

$$\pi_k \Phi^G H\mathbb{Z} = \begin{cases} 0 & k < 0 \text{ or odd} \\ \mathbb{Z}/2 & k \geq 0 \text{ and even.} \end{cases}$$

That the non-zero element in π_{2n} is b^n is immediate from the definition. $\qquad\square$

3. The slice filtration

The slice filtration is an equivariant analogue of the Postnikov tower, to which it reduces in the case of the trivial group. In this section we introduce the slice filtration and establish some of its basic properties. We work at the outset with a general finite group G, though our deepest results (in §5) apply only to the case $G = C_{2^n}$. While situation for general G exhibits many remarkable properties, the reader should regard as exploratory the apparatus of definitions at this level of generality.

3.1. **Slice cells.**

3.1.1. *Slice cells and their dimension.* For a subgroup $K \subseteq G$ let ρ_K denote its regular representation, and write

$$\widehat{S}(m\rho_K) = G_+ \wedge_K S^{m\rho_K} \qquad m \in \mathbb{Z}.$$

When G is cyclic, any subgroup is determined by its order, so we can write $\widehat{S}(m\rho_n) = \widehat{S}(m\rho_K)$, where K is the unique subgroup of order n. Strictly speaking we should denote this instead by $\widehat{S}(m, K)$ to avoid any ambiguity when $m = 0$, but we find the notation above more useful.

Definition 3.1. The set of *slice cells* is

$$\mathcal{A} = \{\widehat{S}(m\rho_K), \Sigma^{-1}\widehat{S}(m\rho_K) \mid m \in \mathbb{Z}, K \subseteq G\}.$$

This brings two notions of "cell" into the story. The slice cells, and the more usual equivariant cells of the form $G/H_+ \wedge S^m$, used to manufacture G-CW spectra. We'll always refer the traditional equivariant cells as "G-cells" in order to easily distinguish them from the "slice cells" which are our main focus.

Definition 3.2. A slice cell is *induced* if it is of the form

$$G_+ \wedge_H \widehat{S},$$

where $\widehat{S}$ is a slice cell for H and $H \subset G$ is a proper subgroup. It is *free* if H is the trivial group. A slice cell is *isotropic* if it is not free.

Remark 3.3. An isotropic slice cell is not necessarily an induced slice cell. The cell S^{ρ_G} is an example.

Since

$$[G_+ \underset{H}{\wedge} S, X]^G = [\widehat{S}, i_H^* X]^H$$

$$[X, G_+ \underset{H}{\wedge} S]^G = [i_H^* X, S]^H$$

induction on $|G|$ usually reduces claims about cells to the case of those which are not induced. The slice cells which are not induced are those of the form $S^{m\rho_G}$ and $S^{m\rho_G - 1}$.

The *dimension* of a slice cell is that of its underlying spheres, namely

$$\dim \widehat{S}(m\rho_K) = m|K|$$

$$\dim \Sigma^{-1} \widehat{S}(m\rho_K) = m|K| - 1.$$

Remark 3.4. Not every suspension of a slice cell is a slice cell. Typically, the spectrum $\Sigma^{-2} \widehat{S}(m\rho_K)$ will *not* be a slice cell, and will *not* exhibit the properties of a slice cell of dimension $\dim \widehat{S}(m\rho_K) - 2$.

The following is immediate from the definition.

Proposition 3.5. *Let $H \subset G$ be a subgroup. If $\widehat{S}$ is a G-slice cell of dimension d, then $i_H^* \widehat{S}$ is a wedge of H-slice cells of dimension d. If $\widehat{S}$ is an H-slice cell of dimension d then $G_+ \underset{H}{\wedge} \widehat{S}$ is a G slice cell of dimension d.* $\qquad \square$

3.1.2. *The slice analogue of "connectivity".* Underlying the theory of the Postnikov tower is the notion of "connectivity" and the class of $(n-1)$-connected spectra. In this section we describe the slice analogues of these ideas. There is a simple relationship between "connectivity" and "slice-connectivity" which we will describe in detail in §3.4.

Definition 3.6. A subcategory $\mathcal{C}$ of $\mathcal{S}^G$ is *closed under homotopy colimits* if it is closed under weak equivalences, the formation of arbitrary wedges and mapping cones.

To clarify, closure under the formation of mapping cones means that if $X \to Y \to Z$ is a cofibration sequence and X and Y are in $\mathcal{C}$ then so is Z.

Example 3.7. Let $X \to Y$ be a map of G-spectra. The class of spectra T for which the map of functions spaces $\mathcal{S}^G(T, X) \to \mathcal{S}^G(T, Y)$ is a weak equivalence is closed under homotopy colimits. So is the class of spectra T for which the map of G-spaces

$$\mathcal{S}_0^G(T, X) \to \mathcal{S}_0^G(T, Y)$$

is a weak equivalence.

Example 3.8. Suppose $\mathcal{F}$ is a collection of G-spectra. The class of G-spectra X with the property that for all $Z \in \mathcal{F}$, the G-space $\mathcal{S}_0^G(X, Z)$ is contractible is closed under homotopy colimits.

Example 3.9. The smallest subcategory of $\mathcal{S}^G$ closed under homotopy colimits, and containing the G-cells $G \underset{H}{\wedge} S^m$, $H \subset G$ is the category of $(m-1)$-connected G-spectra.

Lemma 3.10. *Let $\mathcal{C} \subset \mathcal{S}^G$ be a full subcategory which is closed under homotopy colimits. Then following are equivalent*

i) $\mathcal{C}$ contains all slice cells $\widehat{S}$ with $\dim \widehat{S} \geq n$;

ii) $\mathcal{C}$ contains all spectra of the form $\widehat{S} \wedge T$, where $\widehat{S}$ is a slice cell with $\dim \widehat{S} \geq n$ and T is a suspension spectrum.

Proof: The second condition clearly implies the first. For the other direction it suffices to show that $\mathcal{C}$ contains all spectra of the form $G/H_+ \wedge \widehat{S}$ with $\widehat{S}$ a cell of dimension greater than or equal to n. But $G/H_+ \wedge \widehat{S} \approx G_+ \underset{H}{\wedge} \widehat{S}$ is a wedge of G-cells of dimension greater than or equal to n by Proposition 3.5.   □

Definition 3.11. Let
$$\mathcal{S}^G_{\geq n} \subset \mathcal{S}^G$$
be the smallest full subcategory which is stable under homotopy colimits and satisfies the equivalent conditions of Lemma 3.10.

It will be convenient to use the notation
$$Y \geq n$$
to indicate $Y \in \mathcal{S}^G_{\geq n}$. It will also be convenient to write $\mathcal{S}^G_{>n}$ for $\mathcal{S}^G_{\geq n+1}$ and
$$Y > n$$
when $Y \in \mathcal{S}^G_{>n}$.

One might think of the condition $Y \geq n$ as the slice analogue of being $(n-1)$-connected, and this suggests the use of the term "$(n-1)$ slice-connected" for the spectra in $\mathcal{S}^G_{\geq n}$. We have not found ourselves using this terminology, in part because there is no a priori relationship between being in $\mathcal{S}^G_{\geq n}$ and any notion of connectivity. It is only because of the relation between the connectivity and dimension of spheres that these two notions are so closely related in ordinary homotopy theory. Nevertheless, in forming the slice tower we are definitely forcing the spectra in $\mathcal{S}^G_{\geq n}$ to behave as if they are "$(n-1)$ slice-connected."

Proposition 3.12. *For a G-spectrum X, the following are equivalent*

i) *If $\widehat{S}$ is a slice cell with $\dim \widehat{S} \geq d$ then the G-space $\mathcal{S}^G_0(\widehat{S}, X)$ is contractible;*

ii) *If $Y \geq d$ then the G-space $\mathcal{S}^G_0(Y, X)$ is contractible.*

Proof: We prove that the first assertion implies the second. The reverse implication is trivial. Let $\mathcal{C} \subset \mathcal{S}^G$ be the full subcategory consisting of X for which the G-space $\mathcal{S}^G_0(X, Y)$ is equivariantly contractible. The category $\mathcal{C}$ is closed under homotopy colimits and contains all of the slice cells $\widehat{S}$ with $\dim \widehat{S} \geq d$. It therefore contains $\mathcal{S}^G_{\geq d}$.   □

Proposition 3.13. *For a G-spectrum Y, the following are equivalent*

i) $Y \geq 0$

ii) *Y is (-1)-connected: for $i < 0$, the Mackey functor $\underline{\pi}_i Y$ is zero.*

Proof: Since the slice cells $\widehat{S}$ with $\dim \widehat{S} \geq 0$ are all (-1)-connected, the first assertion implies the second. On the other hand since $\widehat{S}(0 \cdot \rho_H) = G/H_+$, the set of slice cells of dimension 0 is exactly the set of equivariant cells of dimension 0. The class of (-1)-connected spectra is the smallest class of spectra which is closed under homotopy colimits and which contain these cells. $\square$

Remark 3.14. For $n \neq 0$, the condition $Y \geq n$ can have unexpected behavior. For instance it is not the case that if $Y > 0$ then $\pi_0 Y = 0$. In Proposition 5.1 we'll see that the fiber F of $S^0 \to H\mathbb{Z}$ has the property that $F > 0$. On the other hand $\pi_0 F$ is the augmentation ideal of the Burnside ring.

We conclude this section with some results of a technical nature, which will be used later.

Lemma 3.15. *Suppose that* $f : X \to Y$ *with* $X, Y \geq n$. *Then* f *is a weak equivalence if an only if for all slice cells* $\widehat{S}$ *with* $\dim \widehat{S} \geq n$, *and all* $t \geq 0$

$$[\Sigma^t \widehat{S}, X]^G \to [\Sigma^t \widehat{S}, Y]^G$$

is an isomorphism.

Proof: The only if part is trivial. For the "if" part, first note that the condition is equivalent to the assertion that the map of function spaces

$$\mathcal{S}^G(\widehat{S}, X) \to \mathcal{S}^G(\widehat{S}, Y)$$

is a weak equivalence. Consider the class $\mathcal{C}$ of G-spectra T for which $\mathcal{S}^G(T, X) \to \mathcal{S}^G(T, Y)$ is a weak equivalence. The class $\mathcal{C}$ is closed under homotopy colimits, and it contains all of the slice cells $\widehat{S}$ with $\dim \widehat{S} \geq n$. By Lemma 3.10 it therefore contains X and Y. $\square$

In fact the factor Σ^t can be dropped from the hypotheses of Lemma 3.15.

Proposition 3.16. *Suppose that* $X, Y \geq n$ *and that* $f : X \to Y$ *is a map of* G-*spectra with the property that*

$$[\widehat{S}, X]^G \to [\widehat{S}, Y]^G$$

is an isomorphism for all slice cells $\widehat{S}$ *with* $\dim \widehat{S} \geq n$. *Then* f *is a weak equivalence.*

Proof: Under the stated assumptions, we'll prove that for all $t \geq 0$,

$$(3.17) \qquad\qquad [S^t \wedge \widehat{S}, X]^G \to [S^t \wedge \widehat{S}, Y]^G$$

is an isomorphism. The result then follows from Lemma 3.15. The case in which G is the trivial group is trivial since for all t, the spectrum $\Sigma^t \widehat{S}$ is a slice cell of dimension $t + \dim \widehat{S} \geq n$. By induction on $|G|$ we may assume the result for all proper $H \subset G$, so that we know that $i_H^* f$ is a weak equivalence, and hence that

$$[T \wedge Z, X]^G \to [T \wedge Z, Y]^G$$

is an isomorphism for all Z and all equivariant CW-spectra T built entirely from G-cells induced from proper subgroups ("induced G-cells"). Since for $t > 0$, the spectrum $S^{t\rho_G}$ is built from S^t by attaching induced G-cells, the quotient $S^{t\rho_G}/S^t$ is such a T. We now prove that (3.17) is an isomorphism for $t \geq 0$ by induction on t. The case $t = 0$ is one of the hypotheses. For the inductive step, assume the result for $t' < t$. If $\dim \widehat{S}$ is odd, then $\Sigma \widehat{S}$ is a slice cell of dimension $\dim \widehat{S} + 1 \geq n$,

$\Sigma^t \widehat{S} = \Sigma^{t-1}(\Sigma \widehat{S})$, and the situation is covered by the induction hypothesis. Suppose then that dim $\widehat{S}$ is even, and consider the diagram of exact sequences below, with T standing in for $S^{t\rho_G}/S^t$:

$$[T \wedge \widehat{S}, X]^G \longrightarrow [S^{t\rho_G} \wedge \widehat{S}, X]^G \longrightarrow [S^t \wedge \widehat{S}, X]^G \longrightarrow [\Sigma^{-1}T \wedge \widehat{S}, X]^G \longrightarrow [\Sigma^{-1}S^{t\rho_G} \wedge \widehat{S}, X]^G$$

$$\Big\downarrow\approx \qquad \Big\downarrow \qquad \Big\downarrow \qquad \Big\downarrow\approx \qquad \Big\downarrow$$

$$[T \wedge \widehat{S}, Y]^G \longrightarrow [S^{t\rho_G} \wedge \widehat{S}, Y]^G \longrightarrow [S^t \wedge \widehat{S}, Y]^G \longrightarrow [\Sigma^{-1}T \wedge \widehat{S}, Y]^G \longrightarrow [\Sigma^{-1}S^{t\rho_G} \wedge \widehat{S}, Y]^G.$$

The five-lemma shows that (3.17) is an isomorphism if and only if

$$(3.18) \qquad\qquad [S^{t\rho_G} \wedge \widehat{S}, X]^G \to [S^{t\rho_G} \wedge \widehat{S}, Y]^G$$

and

$$(3.19) \qquad\qquad [\Sigma^{-1}S^{t\rho_G} \wedge \widehat{S}, X]^G \to [\Sigma^{-1}S^{t\rho_G} \wedge \widehat{S}, Y]^G$$

are. But $S^{t\rho_G} \wedge \widehat{S}$ is a slice cell of dimension dim $\widehat{S} + t|G| \geq n$, and since dim $\widehat{S}$ is even,

$$\Sigma^{-1}S^{t\rho_G} \wedge \widehat{S} = S^{t\rho_G} \wedge \Sigma^{-1}\widehat{S}$$

is a slice cell of dimension dim $\widehat{S} + t|G| - 1 \geq \dim \widehat{S} \geq n$, and so (3.18) and (3.19) are isomorphisms by the induction hypothesis. $\qquad\square$

3.1.3. *Constructing slice cells from G-cells.* Our convergence result for the slice spectral sequence depends on knowing how slice cells are constructed from G-cells.

Lemma 3.20. *Let $\widehat{S} \in \mathcal{A}$ be a slice cell. If* dim $\widehat{S} = n \geq 0$, *then $\widehat{S}$ can be built from G-cells $G/H_+ \wedge S^k$ with $\lfloor n/|G| \rfloor \leq k \leq n$. If* dim $\widehat{S} = n < 0$ *then $\widehat{S}$ can be built from G-cells $G/H_+ \wedge S^k$ with $n \leq k \leq \lfloor n/|G| \rfloor$.*

   Proof: We start with $\widehat{S} = S^{m\rho_G}$, $m \geq 0$. In this case $\widehat{S}$ is the suspension spectrum of a G-CW complex that can be built starting with S^m and attaching cells of dimension up to $m|G|$. So the result is clear in this case. Desuspending, we find that $\widehat{S} = \Sigma^{-1}S^{m\rho_G}$, of dimension $m|G| - 1$, can be built with G-cells ranging in dimension from

$$m - 1 = \left\lfloor \frac{m|G| - 1}{|G|} \right\rfloor$$

to $m|G| - 1 = n$. For $m < 0$, Spanier-Whitehead duality gives an equivariant cell decomposition of $S^{m\rho_G}$ into cells whose dimensions range from $m|G|$ to m and of $\Sigma^{-1}S^{m\rho_G}$ into cells whose dimensions range from $n = m|G| - 1$ to $m - 1 = \lfloor n/|G| \rfloor$. Finally, the case in which $\widehat{S}$ is a cell induced from a subgroup $K \subset G$ is proved by left inducing its K-equivariant cell decomposition. $\qquad\square$

Corollary 3.21. *Let $Y \in \mathcal{S}^G_{\geq n}$. If $n \geq 0$, then Y is weakly equivalent to a G-CW spectrum built from G-cells $G/H_+ \wedge S^m$ with $m \geq \lfloor n/|G| \rfloor$. If $n \leq 0$ then Y is weakly equivalent to a G-CW spectrum built from G-cells $G/H_+ \wedge S^m$ with $m \geq n$.*

   Proof: The class of G-spectra Y weakly to a G-CW spectrum built from G-cells $G/H_+ \wedge S^m$ with $m \geq \lfloor n/|G| \rfloor$ is closed under homotopy colimits. By Lemma 3.21 it contains the slice cells $\widehat{S}$ with dim $\widehat{S} \geq n$. It therefore contains all $Y \in \mathcal{S}^G_{\geq n}$. A similar argument handles the case $n < 0$. $\qquad\square$

3.2. The slice tower. Let $P^n X$ be the Bousfield localization, or Dror nullification of X with respect to the class $\mathcal{S}^G_{>n}$, and $P_{n+1} X$ the $\mathcal{A}$-colocalization, or $\mathcal{A}$-cellularization, of X with respect to $\mathcal{A} = \mathcal{S}^G_{>n}$ ([6, 13]). There is a functorial fibration sequence

$$P_{n+1} X \to X \to P^n X.$$

The functor P_{n+1} is characterized up to a contractible space of choices by the properties

   i) for all X, $P_{n+1} X \in \mathcal{S}^G_{>n}$;
   ii) for all $A \in \mathcal{S}^G_{>n}$ and all X, the map $\mathcal{S}^G(A, P_{n+1} X) \to \mathcal{S}^G(A, X)$ is a weak equivalence of G-spaces.

In other words, $P_{n+1} X \to X$ is the "universal map" from an object of $\mathcal{S}^G_{>n}$ to X. Similarly $X \to P^n X$ is the universal map from X to a G-spectrum Z which is $\mathcal{S}^G_{>n}$-null in the sense that $\mathcal{S}^G(A, Z) \sim *$ for all $A \in \mathcal{S}^G_{>n}$. More specifically

   iii) the spectrum $P^n X$ is $\mathcal{S}^G_{>n}$-null;
   iv) for any $\mathcal{S}^G_{>n}$-null spectrum Z, the map

$$\mathcal{S}^G(P^n X, Z) \to \mathcal{S}^G(X, Z)$$

   is a weak equivalence.

The functor $P^n X$ is the colimit of a sequence of functors

$$W_0 X \to W_1 X \to \cdots .$$

The $W_i X$ are defined inductively starting with $W_0 X = X$, and taking $W_k X$ to be the cofiber of

$$\bigvee_I \widehat{S} \to W_{k-1} X,$$

in which the indexing set I is the set of maps $\widehat{S} \to W_{k-1} X$ with $\widehat{S} > n$ a slice cell.

Since $\mathcal{S}^G_{>n} \subset \mathcal{S}^G_{>n-1}$, there is a natural transformation

$$P^n X \to P^{n-1} X.$$

Definition 3.22. The *slice tower* of X is the tower $\{P^n X\}_{n \in \mathbb{Z}}$. The spectrum $P^n X$ is the nth *slice section* of X.

When considering more than one group, we will write $P^n X = P^n_G X$ and $P_n X = P^G_n X$.

Proposition 3.23. *The functor P^n commutes with restriction to a subgroup. More precisely, there is a canonical equivalence*

$$i^*_H P^n_G X \to P^n_H i^*_H X.$$

Proof: The first assertion of Proposition 3.5 implies that the map

$$i^*_H X \to i^*_H P^n_G X$$

gives an equivalence after applying P^n_H, so it suffices to show

$$i^*_H P^n_G X \to P^n_H i^*_H P^n_G X$$

is an equivalence. Suppose that $\widehat{S}$ is an H-cell with $\dim \widehat{S} > n$. Then

$$[\widehat{S}, i^*_H P^n_G X]_H = [G_+ \underset{H}{\wedge} \widehat{S}, P^n_G X]^G = 0$$

by the second assertion of Proposition 3.5.   □

The next result is straightforward, but useful. It says that if a tower looks like the slice tower, then it is the slice tower. We leave the proof to the reader.

Proposition 3.24. *Suppose that $X \to \{\tilde{P}^n\}$ is a map from X to a tower of fibrations with the properties*

  i) *the map $X \to \varprojlim \tilde{P}^n$ is a weak equivalence;*

  ii) *the spectrum $\varinjlim_n \tilde{P}^n$ is contractible;*

  iii) *for all n, the fiber of the map $\tilde{P}^n \to \tilde{P}^{n-1}$ is an n-slice.*

Then $\tilde{P}^n$ is the slice tower of X.   □

3.3. **Multiplicative properties of the slice tower.** The slice filtration does not quite have the multiplicative properties one might expect. In this section we collect a few results describing how things work. One important result is Corollary 3.35 asserting that the slice sections of a (-1)-connected commutative or associative algebra are ((-1)-connected) commutative or associative algebras. We'll show in §5.3 show that for the group $G = C_{2^n}$ the slice filtration does behave in the expected way for the special class of "perfect" spectra, defined in §5.2.1.

Lemma 3.25. *Smashing with $S^{m\rho_G}$ gives a bijection of the set of cells $\widehat{S}$ with $\dim \widehat{S} = k$ and those with $\dim \widehat{S} = k + m|G|$.*

Proof: Immediate from the definition.   □

Corollary 3.26. *Smashing with $S^{m\rho_G}$ gives an equivalence*
$$\mathcal{S}_{\geq n}^G \to \mathcal{S}_{\geq n+m|G|}^G \, .$$
                                                                                                  □

Corollary 3.27. *The natural maps*
$$S^{m\rho_G} \wedge P_{k+1}X \to P_{k+m|G|+1}\left(S^{m\rho_G} \wedge X\right)$$
$$S^{m\rho_G} \wedge P^k X \to P^{k+m|G|}\left(S^{m\rho_G} \wedge X\right)$$

are weak equivalences.   □

Lemma 3.28. *Let $\widehat{S}$ be a slice cell of dimension d. If $X \geq 0$ then $X \wedge \widehat{S} \geq d$.*

Proof: Let $\mathcal{C}$ denote the full subcategory of $\mathcal{S}^G$ consisting of those X for which $X \wedge \widehat{S} \geq d$. The category $\mathcal{C}$ contains all the cells of the form $G/H_+ \wedge S^0$ by Proposition 3.5. Since $\mathcal{C}$ is closed under homotopy colimits it contains all (-1)-connected spectra, and hence all $X \geq 0$ by Proposition 3.13.   □

Corollary 3.29. *If $X \geq n$, $Y \geq m$, and n is divisible by $|G|$ then $X \wedge Y \geq n + m$.*

Proof: By smashing X with $S^{(-n/|G|)\rho_G}$ and using Corollary 3.27 we may assume $n = 0$. The class of Y for which $X \wedge Y \geq m$ is closed under homotopy colimits. It also contains all cells $\widehat{S}$ with $\widehat{S} \geq m$ by Lemma 3.28. It therefore contains all $Y \geq m$.   □

Definition 3.30. A map $X \to Y$ is a P^n-*equivalence* if $P^n X \to P^n Y$ is an equivalence.

Lemma 3.31. *If the fiber F of $f : X \to Y$ is in $\mathcal{S}^G_{>n}$, then f is a a P^n equivalence.*

Proof: The map $X \to Y$ is a P^n-equivalence if and only if

$$[Y, Z]^G \to [X, Z]^G$$

is an isomorphism for all $\mathcal{S}^G_{>n}$-null spectra Z. If $F > n$ then $\Sigma F > n$ since $\mathcal{S}^G_{>n}$ is closed under homotopy colimits. the result then follows from the exact sequence

$$[\Sigma F, Z]^G \to [Y, Z]^G \to [X, Z]^G \to [F, Z]^G.$$

$\square$

Remark 3.32. The converse of the above result is not true. For instance, $* \to S^0$ is a P^{-1}-equivalence, but the fiber S^{-1} is not in $\mathcal{S}^G_{>-1}$.

Lemma 3.33. i) *If X, Y, and Z are ≥ 0, and $Y \to Z$ is a P^n-equivalence, then $X \wedge Y \to X \wedge Z$ is a P^n-equivalence;*

ii) *For $X_1, \ldots, X_k \in \mathcal{S}^G_{\geq 0}$, the map*

$$X_1 \wedge \cdots \wedge X_k \to P^n X_1 \wedge \cdots \wedge P^n X_k$$

is a P^n-equivalence.

Proof: The first assertion follows from Corollary 3.29 and Lemma 3.31 since it tells us that the smash product of X with the fiber of $Y \to Z$ is in $\mathcal{S}^G_{>n}$. The second is proved by induction on k, the case $k = 1$ being trivial. For the induction step consider

$$X_1 \wedge \cdots \wedge X_{k-1} \wedge X_k \longrightarrow P^n X_1 \wedge \cdots \wedge P^n X_{k-1} \wedge X_k$$
$$\downarrow$$
$$P^n X_1 \wedge \cdots \wedge P^n X_{k-1} \wedge P^n X_k.$$

The first map is a P^n-equivalence by the induction hypothesis and part i). The second map is a P^n-equivalence by part i). $\square$

Remark 3.34. Lemma 3.33 can be described as asserting that the functor

$$P^n : \{(-1)\text{-connected spectra}\} \to \left\{ \mathcal{S}^G_{>n}\text{-null spectra} \right\}$$

is weakly monoidal.

Corollary 3.35. *Let R be a (-1)-connected G-spectrum. If R is a (homotopy) commutative or (homotopy) associative algebra, then so is $P^n R$ for all n.*

3.4. **The slice spectral sequence.** Write $P_n^n X$ for the fiber of the map

$$P^n X \to P^{n-1} X.$$

The slice spectral sequence is the spectral sequence associated to the tower of fibration $\{P^n X\}$, and it takes the form

$$E_2^{s,t} = \pi_{t-s}^G P_t^t X \implies \pi_{t-s}^G X.$$

It can be regarded as a spectral sequence of Mackey functors, or of individual homotopy groups. We have chosen our indexing so that the display of the spectral sequence is in accord with the classical Adams spectral sequence: the $E_r^{s,t}$-term is placed in the plane in position $(t-s, s)$. The situation is depicted in Figure 1. The differential d_r maps $E_r^{s,t}$ to $E_r^{s+r,t+r-1}$, or in terms the display in the plane, the group in position $(t-s, s)$ to the group in position $(t-s-1, s+r)$.

The main point of this section is to establish strong convergence of the slice spectral sequence, and to show that for any X the E_2-term is distributed in the gray region of Figure 1.

We begin with a fundamental technical result.

Proposition 3.36. *Write $g = |G|$.*

  i) *If $n \geq 0$ and $k > n$ then $(G/H)_+ \wedge S^k > n$.*

 ii) *If $m \leq -1$ and $k \geq m$ then $(G/H)_+ \wedge S^k \geq (m+1)g - 1$.*

iii) *If $Y \geq n$ with $n \geq 0$, then $\pi_i Y = 0$ for $i < \lfloor n/g \rfloor$.*

 iv) *If $Y \geq n$ with $n \leq 0$, then $\pi_i Y = 0$ for $i < n$.*

Proof: For the first assertion we'll prove that if $k > n \geq 0$ then for all G-spectra X

$$[(G/H)_+ \wedge S^k, P^n X]^G = 0.$$

We do this by induction on $|G|$, the case of the trivial group being obvious. Since

$$[(G/H)_+ \wedge S^k, P^n X]^G = [S^k, P^n X]^H$$

we're reduced to showing

$$[S^k, P^n X]^G = 0.$$

The induction hypothesis implies that

$$[T, P^n X]^G = 0$$

for any T weakly equivalent to a G-CW spectrum built entirely from induced cells $G/H_+ \wedge S^j$, with $j > n$. Since the trivial part of $k\rho_G$ is $\mathbb{R}^k$, the homotopy fiber of $S^k \to S^{k\rho_G}$ is such a T. Call it T. It then follows from the exact sequence

$$[\Sigma T, P^n X] \to [S^{k\rho_G}, P^n X] \to [S^k, P^n X] \to [T, P^n X]$$

that the restriction map

$$[S^{k\rho_G}, P^n X]^G \xrightarrow{\approx} [S^k, P^n X]^G$$

is an isomorphism. But the group in the left is zero, since the dimension of $S^{k\rho_G}$ is greater than n. The second assertion is trivial for $k \geq 0$ since in that case $G/H_+ \wedge S^k \geq 0$ and $(m+1)g - 1 \leq -1$. The case $k \leq -1$ is handled by writing

$$(G/H)_+ \wedge S^k = \Sigma^{-1}(G/H)_+ \wedge S^{(k+1)\rho_G} \wedge S^{-(k+1)(\rho_G - 1)}.$$

Since $-(k+1) \geq 0$, the spectrum $S^{-(k+1)(\rho_G - 1)}$ is a suspension spectrum and so

$$(G/H)_+ \wedge S^k \geq (k+1)g - 1 \geq (m+1)g - 1.$$

The third and fourth assertions are immediate from Corollary 3.21. □

Remark 3.37. We've stated part ii) of Proposition 3.36 in the form it is most clearly proved. When it comes up, it is needed as the implication that for $n < 0$,

$$k \geq \lfloor (n+1)/g \rfloor \implies G/H_+ \wedge S^k > n.$$

To relate these, write $m = \lfloor (n+1)/g \rfloor$, so that

$$m + 1 > (n+1)/g$$

and by part ii) of Proposition 3.36

$$G/H_+ \wedge S^k \geq (m+1)g - 1 > n.$$

The following is an immediate consequence. Again, we write $g = |G|$.

Theorem 3.38. *Let X be a G-spectrum. The Mackey functor homotopy groups of $P^n X$ satisfy*

$$\underline{\pi}_k P^n X = 0 \ for \ \begin{cases} k > n & if \ n \geq 0 \\ k \geq \lfloor (n+1)/g \rfloor & if \ n < 0 \end{cases}$$

and the map $X \to P^n X$ induces an isomorphism

$$\underline{\pi}_k X \xrightarrow{\approx} \underline{\pi}_k P^n X \ for \ \begin{cases} k < \lfloor (n+1)/g \rfloor & if \ n \geq 0 \\ k < n+1 & if \ n < 0. \end{cases}$$

Thus for any X,

$$\varinjlim_n P^n X$$

is contractible, the map

$$X \to \varprojlim_n P^n X$$

is a weak equivalence, and for each k, the map

$$\{\underline{\pi}_k(X)\} \to \{\underline{\pi}_k P^n X\}$$

from the constant tower to the slice tower of Mackey functors is a pro-isomorphism.
□

Theorem 3.38 gives the strong convergence of the slice spectral sequence, while Proposition 3.40 shows that the E_2-term vanishes outside of a restricted ranges of dimension. The situation is depicted in Figure 1. The homotopy groups of individual slices lie along lines of slope -1, and the groups contributing to $\pi_* P^n X$ lie to the left of a line of slope -1 intersecting the $(t-s)$-axis at $(t-s) = n$. All of the groups outside the gray region are zero. The vanishing in the regions labeled 1-4 correspond to the four parts of Proposition 3.36.

Definition 3.39. The *n-slice* of a spectrum X is $P_n^n X$. A spectrum M is an *n-slice* if $M = P_n^n M$.
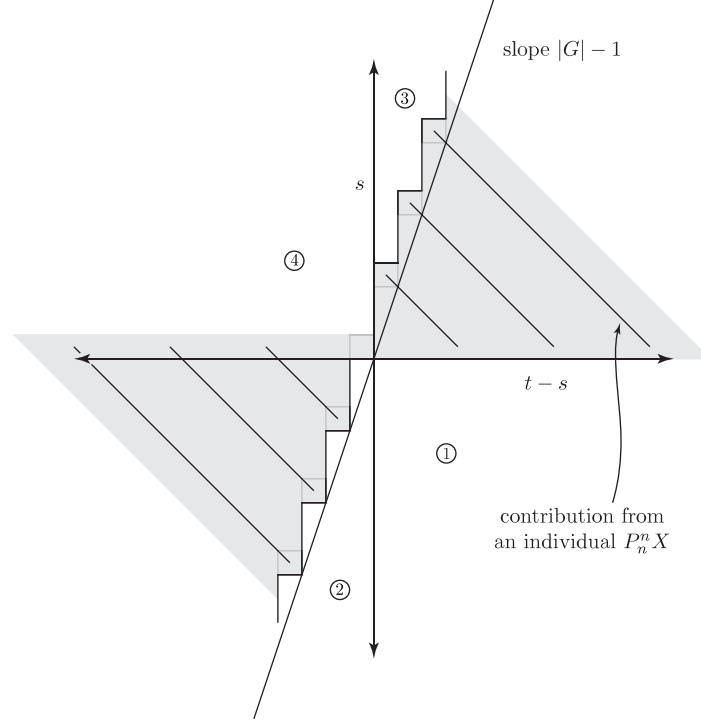
The following is an immediate consequence of Theorem 3.38.

FIGURE 1. The slice spectral sequence

Corollary 3.40. *If M is an n-slice then*

$$\underline{\pi}_k M = 0$$

if $n \geq 0$ and k lies outside of the region $\lfloor n/g \rfloor \leq k \leq n$, or if $n < 0$ and k lies outside of the region $n \leq k < \lfloor (n+1)/g \rfloor$. □

Proposition 3.36 also gives a relationship between the Postnikov tower and the slice tower.

Corollary 3.41. *If X is an $(n-1)$-connected G-spectrum with $n \geq 0$ then $X \geq n$.*

Proof: The assumption on X means it is weakly equivalent to a G-CW spectrum having cells $G/H_+ \wedge S^m$ only in dimensions $m \geq n$. By part i) of Proposition 3.36 these cells are in $\mathcal{S}^G_{\geq n}$. □

We single out some special cases, for convenient reference.

Corollary 3.42. *If M is a 0-slice, then $\underline{\pi}_i M = 0$ for $i \neq 0$.* □

Corollary 3.43. *For any Y, the Eilenberg-Mac Lane spectrum $H\underline{\pi}_0 Y$ and Y have the same 0-slice. More precisely, $P_0 Y$ is the (-1)-connected cover of Y, and the resulting map*

$$P_0 Y \to H\underline{\pi}_0 Y$$

is a P^0-equivalence.

Proof: It follows from Proposition 3.13 that P_0Y is the (-1)-connected cover of Y. The fiber of

$$P_0Y \to H\underline{\pi}_0Y$$

is 0-connected, hence in $\mathcal{S}_{>0}^G$ by Corollary 3.41. □

In spite of Corollary 3.43, it is not necessarily true that $\underline{\pi}_0X$ is isomorphic to $\underline{\pi}_0P_0^0X$. However the situation with odd slices is different. In the slice spectral sequence, only the (-1)-slice can contribute to $\underline{\pi}_{-1}$, and there are no differentials that can enter or leave $\underline{\pi}_{-1}P_{-1}^{-1}$. This gives

Corollary 3.44. *For any* X, $\underline{\pi}_{-1}X = \underline{\pi}_{-1}P_{-1}^{-1}X$. *If* A *is any Mackey functor, then* $\Sigma^{-1}HA$ *is a* (-1)-*slice.* □

Corollary 3.45. *If* $\widehat{S}$ *is a slice cell of odd dimension* d, *then for any* X,

$$[\widehat{S}, X]^G = [\widehat{S}, P_d^dX].$$

Proof: Since the formation of P_d^dX commutes with the functors i_H^*, induction on $|G|$ reduces us to the case when $\widehat{S}$ is not an induced cell. So we may assume $\widehat{S} = S^{m\rho_G - 1}$. Smashing $\widehat{S}$ and X with $S^{-m\rho_G}$, and using Corollary 3.27 reduces to the case $m = 0$. But that is just Corollary 3.44. □

3.5. **The** $RO(g)$**-graded slice spectral sequence.** Often the most useful information about certain homotopy groups is not gotten directly from the slice tower, but from smashing the slice tower with another fixed spectrum. This is especially true when considering an $RO(G)$-graded homotopy group

$$\pi_V^GX = [S^V, X]^G = [S^0, X \wedge S^{-V}].$$

Rather than work with the slice tower for $X \wedge S^{-V}$ we'll work with the slice tower for X.

Definition 3.46. Let V be a virtual representation of G, of virtual dimension d. The *slice spectral sequence for* $\pi_{*+V}X$ is the spectral sequence derived from the tower of fibrations

$$\{S^{-V} \wedge P^{n+d}\}.$$

It has the form

$$E_2^{s,t,V} = \pi_{V+t-s}P_{d+t}^{d+t}X \implies \pi_{V+t-s}X.$$

We've chosen this indexing convention in part to restore some familiar properties of the distribution of groups in the classical Adams spectral sequence. For example, suppose that V is the trivial virtual representation of dimension d. Then the slice spectral sequence for $\pi_{V+*}X$ is gotten from the slice spectral sequence for X by simply shifting the display d units to the left. The vanishing regions shown in Figure 1 do not necessarily hold for the $RO(G)$-graded slice spectral sequence. But by Corollary 3.27 they do hold as stated when $V = m\rho_G$. In case V is trivial the vanishing regions are just shifted along the $(t-s)$-axis.

Our indexing convention amounts roughly to thinking of the t-index as an element of $RO(G)$, which enters only through its dimension when written as an index of a slice. The authors have found the mnemonic "V goes with t" to be a helpful reminder.

4. The complex cobordism spectrum

From here forward we specialize to the case $G = C_{2^n}$, and for convenience localize all spectra at the prime 2. Write

$$g = |G|,$$

and let $\gamma \in G$ be a fixed generator.

We now introduce our equivariant variation on the complex cobordism spectrum by defining

$$MU^{((G))} = N_{C_2}^G MU_{\mathbb{R}},$$

where $MU_{\mathbb{R}}$ is the C_2-equivariant *real bordism* spectrum of Landweber [19] and Fujii [9] (and further studied by Araki [1] and Hu-Kriz [16]). The norm is taken along the unique inclusion $C_2 \subset G$. The spectrum $MU^{((G))}$ is an equivariant E_∞ ring spectrum. For $H \subset G$ the unit of the restriction-norm adjunction (Proposition 2.12) gives a canonical E_∞ map

$$(4.1) \qquad\qquad MU^{((H))} \to i_H^* MU^{((G))}.$$

It will also be convenient to have the shorthand notation

$$i_1^* = i_{C_2}^*$$

for the restriction map $\mathcal{S}^G \to \mathcal{S}^{C_2}$ induced by the unique inclusion $C_2 \subset G$. The C_2-equivariant E_∞ ring spectrum $i_1^* MU^{((G))}$ can be written as

$$(4.2) \qquad\qquad i_1^* MU^{((G))} = \bigwedge_{j=0}^{g/2-1} \gamma^j MU_{\mathbb{R}}.$$

For general purposes the notation $MU^{((G))}$ works quite well, but it can get a bit cumbersome when referring to a specific group. Therefore we'll also use the alternate notation

$$MU^{((n-1))}$$

for $MU^{((G))}$ when $G = C_{2^n}$. The exponent is the 2-log of the number of factors of $MU_{\mathbb{R}}$ being smashed together. For example the case most important to our main result is $G = C_8$ which we'll denote $MU^{((4))}$ since

$$i_{C_2}^* MU^{((C_8))} \approx MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}.$$

4.1. Real bordism, real orientations and formal groups.
In this section we begin with the case $G = C_2$ and a review of the work work of Araki [1] and H-Kriz [16] on real bordism.

4.1.1. *The formal group.* Consider $\mathbf{CP}^n$ and $\mathbf{CP}^\infty$ as pointed C_2-spaces under the action of complex conjugation, with $\mathbf{CP}^0$ as the base point. The fixed point spaces are $\mathbf{RP}^n$ and $\mathbf{RP}^\infty$. There are homeomorphisms

$$(4.3) \qquad\qquad \mathbf{CP}^n/\mathbf{CP}^{n-1} \equiv S^{n\rho_2},$$

and in particular an identification $\mathbf{CP}^1 \equiv S^{\rho_2}$.

Definition 4.4 (Araki [1]). Let E be a C_2-equivariant homotopy commutative ring spectrum. A *real orientation* of E is a class $\bar{x} \in \tilde{E}_{C_2}^{\rho_2}(\mathbf{CP}^\infty)$ whose restriction to

$$\tilde{E}_{C_2}^{\rho_2}(\mathbf{CP}^1) = \tilde{E}_{C_2}^{\rho_2}(S^{\rho_2}) \approx E_{C_2}^0(\mathrm{pt})$$

is a unit. A *real oriented spectrum* is a C_2-equivariant ring spectrum E equipped with a real orientation.

If $(E, \bar{x})$ is a real oriented spectrum and $f : E \to E'$ is an equivariant multiplicative map, then the composite

$$f_*(\bar{x}) \in (E')^{\rho_2}_{C_2}(\mathbf{CP}^\infty)$$

is a real orientation of E'. We will often not distinguish in notation between $\bar{x}$ and $f_*\bar{x}$.

Example 4.5. The zero section $\mathbf{CP}^\infty \to MU(1)$ is an equivariant equivalence, and defines a real orientation

$$\bar{x} \in MU_{\mathbb{R}}^{\rho_2}(\mathbf{CP}^\infty),$$

making $MU_{\mathbb{R}}$ into a real oriented spectrum.

Example 4.6. From the map

$$MU_{\mathbb{R}} \to i_1^* MU^{((G))}$$

provided by (4.1), the spectrum $MU^{((G))}$ gets a real orientation which we'll also denote

$$\bar{x} \in MU^{((G))\rho_2}(\mathbf{CP}^\infty).$$

Example 4.7. If $(H, \bar{x}_H)$ and $(E, \bar{x}_E)$ are two real oriented spectra then $H \wedge E$ has two real orientations given by

$$\bar{x}_H = \bar{x}_H \otimes 1 \text{ and } \bar{x}_E = 1 \otimes \bar{x}_E.$$

The following result of Araki follows easily from the homeomorphisms (4.3).

Theorem 4.8 (Araki [1]). *Let E be a real oriented cohomology theory. There are isomorphisms*

$$E^\star(\mathbf{CP}^\infty) \approx E^\star[\![\bar{x}]\!]$$
$$E^\star(\mathbf{CP}^\infty \times \mathbf{CP}^\infty) \approx E^\star[\![\bar{x} \otimes 1, 1 \otimes \bar{x}]\!]$$

Because of Theorem 4.8, the map $\mathbf{CP}^\infty \times \mathbf{CP}^\infty \to \mathbf{CP}^\infty$ classifying the tensor product of the two tautological line bundles defines a formal group law over $\pi_\star^G E$. Using this, much of the theory relating formal groups, complex cobordism, and complex oriented cohomology theories works for C_2-equivariant spectra, with $MU_{\mathbb{R}}$ playing the role of MU. For information beyond the discussion below, see [1, 16].

The standard formulae for formal groups give elements in the $RO(C_2)$-graded homotopy groups $\pi_\star^{C_2} E$ of real oriented E. For example, there is a map from the Lazard ring to $\pi_\star^{C_2} E$ classifying the formal group law. Using Quillen's theorem to identify the Lazard ring with the complex cobordism ring this map can be written as

$$MU_* \to \pi_\star^{C_2} E.$$

It sends MU_{2n} to $\pi_{n\rho_2}^{C_2} E$. When $E = MU_{\mathbb{R}}$ this splits the forgetful map

$$(4.9) \qquad \pi_{n\rho_2}^{C_2} MU_{\mathbb{R}} \to \pi_{2n}^u MU_{\mathbb{R}} = \pi_{2n} MU,$$

which is therefore surjective. A similar discussion applies to iterated smash products of $MU_{\mathbb{R}}$ giving

Proposition 4.10. *For every $m > 0$, the theory of formal groups and complex cobordism gives a ring homomorphism*

$$(4.11) \qquad \pi_*^u \bigwedge^m MU_{\mathbb{R}} \to \bigoplus_j \pi_{j\rho_2} \bigwedge^m MU_{\mathbb{R}}$$

splitting the forgetful map

$$(4.12) \qquad \bigoplus_j \pi_{j\rho_2} \bigwedge^m MU_{\mathbb{R}} \to \pi_*^u \bigwedge^m MU_{\mathbb{R}}.$$

In particular, (4.12) is a split surjection. □

It is a result of Hu-Kriz[16] that (4.12) is in fact an isomorphism. This result, and a generalization to $MU^{((G))}$ can be recovered from the slice spectral sequence.

The class

$$\bar{x}_H \in H_{C_2}^{\rho_2}(\mathbf{CP}^{\infty}; \underline{\mathbb{Z}}_{(2)})$$

corresponding to $1 \in H_{C_2}^0(\mathrm{pt}, \underline{\mathbb{Z}}_{(2)})$ under the isomorphism

$$H_{C_2}^{\rho_2}(\mathbf{CP}^{\infty}; \underline{\mathbb{Z}}_{(2)}) \approx H_{C_2}^{\rho_2}(\mathbf{CP}^2; \underline{\mathbb{Z}}_{(2)}) \approx H_{C_2}^0(\mathrm{pt}, \underline{\mathbb{Z}}_{(2)})$$

defines a real orientation of $H\underline{\mathbb{Z}}_{(2)}$. As in Example 4.7, the classes $\bar{x}$ and $\bar{x}_H$ give two orientations of $E = H\underline{\mathbb{Z}}_{(2)} \wedge MU_{\mathbb{R}}$. By Theorem 4.8 these are related by a power series

$$\bar{x}_H = \log_F(\bar{x})$$
$$= \bar{x} + \sum_{i>0} \bar{m}_i \bar{x}^{i+1},$$

with

$$\bar{m}_i \in \pi_{i\rho_2}^{C_2} H\underline{\mathbb{Z}}_{(2)} \wedge MU_{\mathbb{R}}.$$

This power series is the *logarithm* of F. Similarly, the invariant differential on F gives classes $(n+1)\bar{m}_n \in \pi_{n\rho_2}^{C_2} MU_{\mathbb{R}}$. The coefficients of the formal sum give

$$\bar{a}_{ij} \in \pi_{(i+j-1)\rho_2}^{C_2} MU_{\mathbb{R}}.$$

If $(E, \bar{x}_E)$ is a real oriented spectrum then $E \wedge MU_{\mathbb{R}}$ has two orientations

$$\bar{x}_E = \bar{x}_E \otimes 1$$
$$\bar{x}_R = 1 \otimes \bar{x}.$$

These two orientations are related by a power series

$$(4.13) \qquad \bar{x}_R = \sum \bar{b}_i x_E^{i+1}$$

defining classes

$$\bar{b}_i = \bar{b}_i^E \in \pi_{i\rho_2}^{C_2} E \wedge MU_{\mathbb{R}}.$$

The power series (4.13) is an isomorphism over $\pi_\star^{C_2} E \wedge MU_{\mathbb{R}}$

$$F_E \to F_R$$

of the formal group law for $(E, \bar{x}_E)$ with the formal group law for $(MU_{\mathbb{R}}, \bar{x})$.

Theorem 4.14 (Araki [1]). *The map*

$$E_\star[\bar{b}_1, \bar{b}_2, \dots] \to \pi_\star^{C_2} E \wedge MU_{\mathbb{R}}$$

is an isomorphism. □

Araki's theorem has an evident geometric counterpart. For each j choose a map

$$S^{j\rho_2} \to E \wedge MU_{\mathbb{R}}$$

representing $\bar{b}_j$. Let

$$S[\bar{b}_j] = \bigvee_{k \geq 0} S^{k \cdot j\rho_2}$$

be the free associative algebra on $S^{j\rho_2}$ and

$$S[\bar{b}_j] \to E \wedge MU_{\mathbb{R}}$$

the homotopy associative algebra map extending (4.35). Using the multiplication map, smash these together to form a map of spectra

(4.15) $$E[\bar{b}_1, \bar{b}_2 \ldots] \to E \wedge MU^{((G))},$$

where

$$E[\bar{b}_1, \bar{b}_2 \ldots] = E \wedge \varinjlim_{k} S[\bar{b}_1] \wedge S\bar{b}_2] \wedge \cdots \wedge S[\bar{b}_k].$$

The map on $RO(C_2)$-graded homotopy groups induced by (4.15) is the isomorphism of Araki's theorem. This proves

Corollary 4.16. *If E is a real oriented spectrum then there is a weak equivalence*

$$E \wedge MU_{\mathbb{R}} \approx E[\bar{b}_1, \bar{b}_2, \ldots].$$

$\square$

Remark 4.17. If E is a (homotopy) commutative algebra then (4.15) is a map of (homotopy) associative algebras.

For each j write

$$S^0[G \cdot \bar{b}_j] = N_{C_2}^G S^0[S^{j\rho_2}].$$

The spectrum $S^0[G \cdot \bar{b}_j]$ is a G-equivariant associative algebra. Since

$$i_1^* S^0[H \cdot \bar{b}_j] = \bigwedge_{k=0}^{g/2-1} S^0[\gamma^k \bar{b}_j],$$

it can be thought of as an equivariant polynomial algebra on the variables $\gamma^k \bar{b}_j$, with $k = 0, \ldots, g/2 - 1$. Finally, let

$$MU^{((G))}[G \cdot \bar{b}_1, G \cdot \bar{b}_2, \ldots] = \varinjlim_{m} MU^{((G))} \wedge S^0[G \cdot \bar{b}_1] \wedge \cdots \wedge S^0[G \cdot \bar{b}_m].$$

Corollary 4.18. *For $H \subset G$ of index 2, there is an equivalence of associative algebras*

$$i_H^* MU^{((G))} \approx MU^{((H))}[H \cdot \bar{b}_1, H \cdot \bar{b}_2, \ldots].$$

Proof: Apply $N_{C_2}^H$ to the decomposition of Corollary 4.16 with $E = MU_{\mathbb{R}}$. $\square$

Corollary 4.19. *For any $H \subset G$ there is a decomposition*

$$i_H^* MU^{((G))} \approx MU^{((H))} \wedge (S^0 \vee W)$$

where W is a wedge of isotropic, even dimensional slices cells of positive dimension.

$\square$

If E is a real oriented spectrum with formal group law F^E, then over $\pi_\star^{C_2} E \wedge MU^{((G))}$ there is the formal group law F^E of E, the formal group law F with coordinate $\bar{x}$ coming from

$$MU_\mathbb{R} \to i_1^* MU^{((G))} \approx \bigwedge_{j=0}^{g/2-1} \gamma^j MU_\mathbb{R},$$

and the formal group laws F^{γ^j} with coordinate $\gamma^j \bar{x}$ associated to the other inclusions $MU_\mathbb{R} \to i_1^* MU^{((G))}$. There is also a diagram of isomorphisms

The isomorphism $\bar{b}$ is given by

$$\bar{x} = \bar{b}(\bar{x}_E) = \bar{x}_E + \sum \bar{b}_j \bar{x}_E^{j+1},$$

and the others are gotten by applying γ^j

$$\gamma^j \bar{x} = \gamma^j \bar{b}(\bar{x}_E) = \bar{x}_E + \sum \gamma^j \bar{b}_j \bar{x}_E^{j+1}.$$

Write

$$G \cdot \bar{b}_i$$

for the sequence

$$\bar{b}_i, \gamma \bar{b}_i, \dots, \gamma^{g/2-1} \bar{b}_i.$$

Using the decomposition (4.2) and iterating Araki's theorem gives

Proposition 4.20. *There is an isomorphism*

$$\pi_\star^{C_2} E \wedge i_1^* MU^{((G))} \approx E_\star[G \cdot \bar{b}_1, G \cdot \bar{b}_2, \dots].$$

$\square$

4.1.2. *The unoriented cobordism ring.* Passing to geometric fixed points from

$$\bar{x} : \mathbf{CP}^\infty \to \Sigma^{\rho_2} MU_\mathbb{R}$$

gives the canonical inclusion

$$a : \mathbf{RP}^\infty = MO(1) \to \Sigma MO,$$

defining the MO Euler-class of the tautological line bundle. There are isomorphisms

$$MO^*(\mathbf{RP}^\infty) \approx MO^*[\![a]\!]$$
$$MO^*(\mathbf{RP}^\infty \times \mathbf{RP}^\infty) \approx MO^*[\![a \otimes 1, 1 \otimes a]\!]$$

and the multiplication map $\mathbf{RP}^\infty \times \mathbf{RP}^\infty \to \mathbf{RP}^\infty$ gives a formal group law over MO_*. By Quillen [24], it is the universal formal group law F over a ring of characteristic 2 for which $F(a, a) = 0$.

The formal group can be used to give convenient generators for the unoriented cobordism ring. Over $\pi_* H\mathbb{Z}/2 \wedge MO$ there is a power series relating e and the image of the class a

$$e = \ell(a) = a + \sum \alpha_n a^{n+1}.$$

Lemma 4.21. *The composite series*

$$(4.22) \qquad (a + \sum \alpha_{2^j - 1} a^{2^j})^{-1} \circ \ell(a) = a + \sum_{j > 0} h_j a^j$$

has coefficients in $\pi_* MO$. The classes h_j with $j + 1 = 2^k$ are zero. The remaining h_j are polynomial generators for the unoriented cobordism ring

$$(4.23) \qquad \pi_* MO = \mathbb{Z}/2[h_j, j \neq 2^k - 1].$$

Proof: The assertion that $h_j = 0$ for $j + 1 = 2^k$ is straightforward. Since the sequence

$$(4.24) \qquad \pi_* MO \to \pi_* H\mathbb{Z}/2 \wedge MU \rightrightarrows \pi_* H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \wedge MO$$

is a split equalizer, to show that the remaining h_j are in $\pi_* MJ$ it suffices to show that they are equalized by the parallel maps in (4.24). This works out to showing that the series (4.22) is invariant under substitutions of the form

$$(4.25) \qquad e \mapsto e + \sum \zeta_m e^{2^m},$$

The series (4.22) is characterized as the unique isomorphism of the formal group law for unoriented cobordism with the additive group, having the additional property that the coefficients of a^{2^k} are zero. This condition is stable under the substitutions (4.25). The last assertion follows from Quillen's characterization of $\pi_* MO$. $\qquad \square$

Remark 4.26. Recall the real orientation $\bar{x}$ of $i_1^* MU^{((G))}$ of Example 4.6. Applying the $RO(G)$-graded cohomology norm (§2.3.4) to $\bar{x}$, and then passing to geometric fixed points, gives a class

$$\Phi^G N(\bar{x}) \in MO^1(\mathbf{RP}^\infty).$$

One can easily check that $\Phi^G N(\bar{x})$ coincides with the MO Euler class a defined at the beginning of this section. Similarly one has

$$\Phi^G N(x_H) = e.$$

Applying $\Phi^G N$ to $\log_{\bar{F}}$ and using the fact that it is a ring homomorphism (Proposition 2.35) gives

$$e = a + \sum \Phi^G N(\bar{m}_k) a^{k+1}.$$

It follows that

$$\Phi^G N(\bar{m}_k) = \alpha_k.$$

4.2. **Refinement of homotopy groups.** We begin by focusing on a simple consequence of Proposition 4.10.

Proposition 4.27. *For every $m > 1$, every element of*

$$\pi_{2k} \left(\bigwedge^m MU \right)$$

can be refined to an equivariant map

$$S^{k\rho_2} \to \bigwedge^m MU_{\mathbb{R}}.$$

$\qquad \square$

This result expresses an important property of the C_2-spectra given by iterated smash products of $MU_{\mathbb{R}}$. Our goal in this section is to formulate a generalization to the case $G = C_{2^n}$.

Definition 4.28. Suppose X is a G-spectrum with the property that $\pi_k^u X$ is a free abelian group. A *refinement of* $\pi_k^u X$ is a map

$$c : \widehat{W} \to X$$

in which $\widehat{W}$ is a wedge of slice cells of dimension k, with the property that the summands in $i_0^* \widehat{W}$ represent a basis of $\pi_k^u X$. A *refinement of the homotopy groups of X* (or a *refinement of homotopy of X*) is a map

$$\widehat{W} = \bigvee \widehat{W}_k \to X$$

whose restriction to each $\widehat{W}_k$ is a refinement of π_k^u.

The splitting (4.11) used to prove Proposition 4.27 is multiplicative. This too has an important analogue.

Definition 4.29. Suppose that R is an equivariant associative algebra. A *multiplicative refinement of homotopy* is an associative algebra map $\widehat{W} \to R$ which, when regarded as a map of G-spectra is a refinement of homotopy.

Proposition 4.30. *For every $m \geq 1$ there exists a multiplicative refinement of homotopy*

$$\bigwedge^m MU^{((G))}.$$

Two ingredients form the proof of Proposition 4.30. The first, Lemma 4.31 below, is a description of $\pi_*^u MU^{((G))}$ as a G-module. The computation is of interest in its own right, and is used elsewhere in this paper. It is proved in §4.3. The second is the classical description of $\pi_*^u(\bigwedge^m MU^{((G))})$, $m > 1$, as a $\pi_*^u MU^{((G))}$-module.

Lemma 4.31. *There is a sequence of elements $r_i \in \pi_{2i}^u MU^{((G))}$ with the property that*

$$\pi_*^u MU^{((G))} = \mathbb{Z}_2[r_1, \gamma r_1, \ldots, r_2, \gamma r_2 \ldots],$$

in which $(r_i, \gamma r_i, \ldots)$ stands for the sequence

$$(r_i, \ldots \gamma^{\frac{g}{2}-1} r_i)$$

of length $g/2$.

For example, Lemma 4.31 asserts

$$\pi_* MU = \mathbb{Z}_{(2)}[r_1, r_2, \ldots]$$

$$\pi_*^u MU^{((1))} = \pi_* MU \wedge MU = \mathbb{Z}_{(2)}[r_1, \gamma r_1, r_2, \gamma r_2, \ldots]$$

$$\pi_*^u MU^{((2))} = \pi_* \bigwedge^4 MU = \mathbb{Z}_{(2)}[r_1, \ldots, \gamma^3 r_1, r_2 \ldots]$$

Over $\pi_*^u MU^{((G))} \wedge MU^{((G))}$, there are two formal group laws, F_L and F_R coming from the canonical orientations of the left and right factors. There is also a canonical isomorphism between them, which can be written as

$$x_R = \sum b_j x_L^{j+1}.$$

As in the statement of Proposition 4.20 write

$$G \cdot b_i$$

for the sequence

$$b_i, \gamma b_i, \ldots, \gamma^{g/2-1} b_i.$$

The following result is a standard computation in complex cobordism. See for example [27].

Lemma 4.32. *The ring* $\pi_*^u MU^{((G))} \wedge MU^{((G))}$ *is given by*

$$\pi_*^u MU^{((G))} \wedge MU^{((G))} = \pi_*^u MU^{((G))}[G \cdot b_1, G \cdot b_2, \ldots].$$

For $m > 1$,

$$\pi_*^u \bigwedge^m MU^{((G))} = \pi_*^u MU^{((G))} \wedge \bigwedge^{m-1} MU^{((G))}$$

is the polynomial ring

$$\pi_*^u MU^{((G))}[G \cdot b_i^{(j)}],$$

with

$$i = 1, 2, \ldots$$
$$j = 1, \ldots, m - 1.$$

The element $b_i^{(j)}$ *is the element* b_i *corresponding to the* j^{th} *factor of* $MU^{((G))}$ *in* $\bigwedge^{m-1} MU^{((G))}$. $\qquad \square$

We also need

Lemma 4.33. *If* $\widehat{W}$ *is a wedge of even dimensional isotropic* C_2-*slices cells, then* $N_{C_2}^G \widehat{W}$ *is a wedge of even dimensional isotropic* G-*slice cells.*

Proof: This follows from the decomposition given by Proposition 2.11. Resorting to the notation used there, suppose

$$\widehat{W} = \bigvee_{j \in J} X_j$$

and that each X_j is an even dimensional C_2 slice cell of the form

$$X_j = S^{m_j \rho_2}.$$

Then for $\phi \in K = \mathrm{hom}(G/C_2, J)$ the spectrum

$$X_{\bar{\phi}} = \bigwedge_{t \in G/H_\phi} X_{\bar{\phi}(t)}$$

is of the form $S^{m \rho_2}$, hence an even dimensional isotropic C_2 slice cells. The spectrum $X_\phi = N_{C_2}^{H_\phi}$ is then of the form

$$S^{m \rho_{h_\phi}}, \quad h_\phi = |H_\phi|,$$

with H_ϕ non-trivial since it contains C_2. It then follows that

$$G_+ \underset{H_\phi}{\wedge} X_\phi$$

is an even dimensional, isotropic slice cell. $\qquad \square$

Remark 4.34. Using a generalization of Proposition 2.11 one can show that for any $H \subset G$, the norm of a wedge of even dimensional (isotropic) slice cells is a wedge of (isotropic) slice cells.

Proof of Proposition 4.30, assuming Lemma 4.31:. To keep the notation simple we begin with the case $m = 1$. Choose a sequence $r_i \in \pi^u_{2i} MU^{((G))}$ with the property described in Lemma 4.31. Let

(4.35) $$\bar{r}_i : S^{i\rho_2} \to i_1^* MU^{((G))},$$

be a representative of the image of r_i under the splitting (4.11). Let

$$S[\bar{r}_i] = \bigvee_{k \geq 0} S^{k \cdot i\rho_2}$$

be the free associative algebra on $S^{i\rho_2}$ and

$$S[\bar{r}_i] \to i_1^* MU^{((G))}$$

the associative algebra map extending (4.35). Since $i_1^* MU^{((G))}$ is a commutative algebra these maps can be smashed together to give an associative algebra map

$$S[\bar{r}_1, \bar{r}_2 \ldots] \to i_1^* MU^{((G))},$$

where

$$S[\bar{r}_1, \bar{r}_2 \ldots] = \varinjlim_k S[\bar{r}_1] \wedge S[\bar{r}_2] \wedge \cdots \wedge S[\bar{r}_k].$$

Finally, applying the norm and the co-unit of the restriction-norm adjunction gives an associative algebra map

(4.36) $$S[G \cdot \bar{r}_1, G \cdot \bar{r}_2 \ldots] \to N i_1^* MU^{((G))} \to MU^{((G))},$$

where

$$S[G \cdot \bar{r}_1, G \cdot \bar{r}_2 \ldots] = S[\bar{r}_1, \gamma\bar{r}_1, \ldots, \gamma^{g/2-1}\bar{r}_1, \bar{r}_2, \gamma\bar{r}_2, \ldots]$$
$$= N^G_{C_2} S[\bar{r}_1, \bar{r}_2 \ldots].$$

By Lemma 4.33, the spectrum $S[G \cdot \bar{r}_1, G \cdot \bar{r}_2 \ldots]$ is a wedge of even isotropic G-slice cells. Using Lemma 4.31 one then easily checks that (4.36) is multiplicative refinement of homotopy. The case $m \geq 1$ is similar, using in addition Lemma 4.32 and the collection $\{r_i, b_i(j)\}$. □

Remark 4.37. The structure of $\pi^u_* MU^{((G))}$ and the fact that it admits a refinement of its homotopy groups actually determines the slice cells involved in the refinement. To describe them explicitly, notice that the monomials in the $\{\gamma^i r_j\} \subset \pi^u_* MU^{((G))}$ are permuted, up to sign, by the action of G. For instance, the G-orbit through r_1 consists of the elements

$$G \cdot r_1 = \{\pm r_1, \ldots, \pm \gamma^{\frac{g}{2}-1} r_1\},$$

while the orbit through r_i^2 is half as large

$$G \cdot r_1^2 = \{r_1^2, \ldots, \gamma^{\frac{g}{2}-1} r_1^2\}.$$

We'll call the list of monomials which occur up to sign in a G-orbit a *mod 2 orbit of monomials* and indicate it with absolute value signs. For instance the mod 2 orbit of monomials containing b_1 is the set

(4.38) $$|G \cdot r_1| = \{r_1, \ldots, \gamma^{\frac{g}{2}-1} r_1\}.$$

The slice cells refining the homotopy groups of $MU^{((G))}$ correspond to the mod 2-orbits in the monomials in $\{\gamma^i r_j\}$. For example the mod 2 orbit (4.38) is refined by a map from the slice cell

$$G_+ \underset{C_2}{\wedge} S^{\rho_2}.$$

The monomial $r_1 \ldots \gamma^{\frac{g}{2}-1} r_1$ is itself a mod 2-orbit and corresponds to the slice cell

$$S^{\rho_G}.$$

A similar remark applies to the refinement of

$$\pi_*^u \bigwedge^m MU^{((G))}.$$

One important consequence of this is

Proposition 4.39. *The slices cells occurring in the refinement of $\pi_* \bigwedge^m MU^{((G))}$ are isotropic.* $\qquad\square$

The series of examples below explicitly describe the refinement of $\pi_*^u MU^{((G))}$ for $G = C_8$, and $* \le 4$.

Example 4.40. In π_0^u we have the monomial 1, corresponding to an equivariant map $S^0 \to MU^{((G))}$. So $\widehat{W}_0 = S^0$.

Example 4.41. The group π_2^u consists of the single mod 2 orbit

$$\{x_1, \gamma x_1, \gamma^2 x_1, \gamma^3 x_1\}$$

and so the homotopy in this dimension refines with

$$\widehat{W}_2 = G_+ \underset{Z/2}{\wedge} S^{\rho_2} \leftrightarrow \{x_1, \gamma x_1, \gamma^2 x_1, \gamma^3 x_1\}.$$

Example 4.42. The group $\pi_4^u MU^{((G))}$ has rank 14, and decomposes into the mod 2 orbits

$$\{x_1, \gamma x_1, \gamma^2 x_1, \gamma^3 x_1\}, \quad \{x_1 \gamma(x_1), \gamma(x_1)\gamma^2(x_1), \gamma^2(x_1)\gamma^3(x_1), \gamma^3(x_1)x_1\},$$
$$\{x_2, \gamma x_2, \gamma^2 x_2, \gamma^3 x_2\}, \quad \{x_1 \, \gamma^2 x_1, \gamma x_1 \, \gamma^3 x_1\}.$$

The homotopy in this dimension refines to an equivariant map $\widehat{W}_4 \to MU^{((G))}$ with $\widehat{W}_4$ the wedge of

$$G_+ \underset{C_2}{\wedge} S^{2\rho_2} \leftrightarrow \{x_1, \gamma x_1, \gamma^2 x_1, \gamma^3 x_1\}$$
$$G_+ \underset{C_2}{\wedge} S^{2\rho_2} \leftrightarrow \{x_1 \gamma(x_1), \gamma(x_1)\gamma^2(x_1), \gamma^2(x_1)\gamma^3(x_1), \gamma^3(x_1)x_1\}$$
$$G_+ \underset{C_2}{\wedge} S^{2\rho_2} \leftrightarrow \{x_2, \gamma x_2, \gamma^2 x_2, \gamma^3 x_2\}$$
$$G_+ \underset{C_4}{\wedge} S^{\rho_4} \leftrightarrow \{x_1 \, \gamma^2(x_1), \gamma(x_1) \, \gamma^3(x_1)\}.$$

4.3. **Algebra generators for $\pi_*^u MU^{((G))}$.** In this section we will describe convenient algebra generators for $\pi_*^u MU^{((G))}$. Our main results are Proposition 4.47 (giving a criterion for a sequence of elements r_i to "generate" $\pi_*^u MU^{((G))}$ as a G-algebra, as in Lemma 4.31) and Corollary 4.50 (specifying a particular sequence of r_i). Proposition 4.47 directly gives Lemma 4.31.

We remind the reader that the notation $H_*^u X$ refers to the homology groups $H_*(i_0^* X)$ of the non-equivariant spectrum underlying X.

4.3.1. *A criterion for a generating set.* Let $m_i \in H_{2i}MU$ be the coefficient of the universal logarithm. As in Proposition 4.20, using the identification (4.2)

$$i_1^* MU^{((G))} = \bigwedge_{j=0}^{g/2-1} \gamma^j MU_{\mathbb{R}}$$

one has

$$H_*^u MU^{((G))} = \mathbb{Z}_{(2)}[\gamma^j m_k],$$

where

$$k = 1, 2, \ldots,$$
$$j = 0, \ldots, g/2 - 1.$$

By definition, the action of G on $H_*^u MU^{((G))}$ is given by

$$(4.43) \qquad \gamma \cdot \gamma^j m_k = \begin{cases} \gamma^{j+1} m_k & j < g/2 - 1 \\ (-1)^k m_k & j = g/2 - 1. \end{cases}$$

Let

$$I = \ker \pi_*^u MU^{((G))} \to \mathbb{Z}_{(2)}$$
$$I_H = \ker H_*^u MU^{((G))} \to \mathbb{Z}_{(2)}$$

denote the augmentation ideals, and

$$Q_* = I/I^2$$
$$QH_* = I_H/I_H^2$$

the modules of indecomposable, with Q_{2m} and QH_{2m} indicating the homogeneous parts of degree $2m$ (the odd degree parts are zero). The module QH_* is the free abelian group with basis $\{\gamma^j m_k\}$, and from Milnor [23], one knows that the Hurewicz homomorphism gives an isomorphism

$$Q_{2k} \to QH_{2k}$$

if $2k$ is not of the form $2(2^\ell - 1)$, and an exact sequence

$$(4.44) \qquad Q_{2(2^\ell-1)} \rightarrowtail QH_{2(2^\ell-1)} \twoheadrightarrow \mathbb{Z}/2$$

in which the rightmost map is the one sending each $\gamma^j m_k$ to 1.

Formula (4.43) implies that the G-module QH_{2k} is the module induced from the sign representation of C_2 if k is odd and from the trivial representation if k is even.

Lemma 4.45. *Let $r = \sum a_j \gamma^j m_k \in QH_{2k}$. The unique G-module map*

$$\mathbb{Z}_{(2)}[G] \to QH_{2k}$$
$$1 \mapsto r$$

factors through a map

$$\mathbb{Z}_{(2)}[G]/(\gamma^{g/2} - (-1)^k) \to QH_{2k}$$

which is an isomorphism if and only if $\sum a_j \equiv 1 \mod 2$.

Proof: The factorization is clear, since $\gamma^{g/2}$ acts with eigenvalue $(-1)^k$ on QH_{2k}. Use the unique map $\mathbb{Z}_{(2)}[G] \to QH_{2k}$ sending 1 to m_k to identify QH_{2k} with $A = \mathbb{Z}_{(2)}[G]/(\gamma^{g/2} - (-1)^k)$. The main assertion is then that an element $r = \sum a_j \gamma^j \in A$ is a unit if and only if $\sum a_j \equiv 1 \mod 2$. Since A is a finitely generated free module over the Noetherian local ring $\mathbb{Z}_{(2)}$, Nakayama's lemma implies that the map $A \to A$ given by multiplication by r is an isomorphism if and only if it is after reduction modulo 2. So r is a unit if and only if it is after reduction modulo 2. But $A/(2) = \mathbb{Z}/2[\gamma]/(\gamma^{g/2} - 1)$ is a local ring with nilpotent maximal ideal $(\gamma - 1)$. The residue map

$$A/(2) \to A/(2, \gamma - 1) = \mathbb{Z}/2$$

sends $\sum a_j \gamma^j m_k$ to $\sum a_j$. The result follows. $\qquad\square$

Lemma 4.46. *The G-module $Q_{2(2^\ell - 1)}$ is isomorphic to the module induced from the sign representation of C_2. The unique G-map*

$$\mathbb{Z}_{(2)}[G] \to QH_{2(2^\ell - 1)}$$
$$1 \mapsto y$$

factors through a map

$$A = \mathbb{Z}_{(2)}[G]/(\gamma^{g/2} + 1) \to Q_{2(2^\ell - 1)}$$

which is an isomorphism if and only if $y = (1 - \gamma)r$ where $r \in QH_{2(2^\ell - 1)}$ satisfies the condition $\sum a_j = 1 \mod 2$ of Lemma 4.45.

Proof: Identify $QH_{2(2^\ell - 1)}$ with A by the map sending 1 to $m_{2^\ell - 1}$. In this case A is isomorphic to $\mathbb{Z}_{(2)}[\zeta]$, with ζ a primitive $(g/2)^{\text{th}}$ root of unity, and in particular is an integral domain. Under this identification, the rightmost map in (4.44) is the quotient of A by the principal ideal $(\zeta - 1)$. Since A is an integral domain, this ideal is a rank 1 free module generated by any element of the form $(1 - \gamma)r$ with $r \in A$ a unit. The result follows. $\qquad\square$

This discussion proves

Proposition 4.47. *Let*

$$\{r_1, r_2, \dots\} \subset \pi_*^u MU^{((G))}$$

be any sequence of elements whose image

$$s_k \in QH_{2k}$$

has the property that for $j \neq 2^\ell - 1$, $s_j = \sum a_j \gamma^j m_k$ with

$$\sum a_j \equiv 1 \mod 2,$$

and $s_{2^\ell - 1} = (1 - \gamma)\left(\sum a_j \gamma^j m_k\right)$, with

$$\sum a_j \equiv 1 \mod 2.$$

Then the sequence

$$\{r_1, \dots \gamma^{\frac{g}{2} - 1} r_1, r_2, \dots, \gamma^{\frac{g}{2} - 1} r_2, \dots\}$$

generates the ideal I, and so

$$\mathbb{Z}_{(2)}[r_1, \dots \gamma^{\frac{g}{2} - 1} r_1, r_2, \dots, \gamma^{\frac{g}{2} - 1} r_2, \dots] \to \pi_*^u MU^{((G))}$$

is an isomorphism. $\qquad\square$

4.3.2. *Specific generators.* We now use the action of G on $i_1^* MU^{((G))}$ to define specific elements $\bar{r}_i \in \pi_{i\rho_2}^G MU^{((G))}$ refining a sequence satisfying the condition of Proposition 4.47.

Write

$$\bar{F}(\bar{x}, \bar{y})$$

for the real formal group law over $\pi_*^{C_2} MU^{((G))}$, and

$$\log \bar{F}(\bar{x}) = \bar{x} + \sum_{i>0} \bar{m}_k \bar{x}^{k+1}$$

for its logarithm. This defines elements

$$\bar{m}_k \in \pi_{k\rho_2}^{C_2} H\underline{\mathbb{Z}}_{(2)} \wedge MU^{((G))}.$$

We define the elements

$$\bar{r}_k \in \pi_{k\rho_2}^{C_2} MU^{((G))}$$

to be the coefficients of the universal natural isomorphism of $\bar{F}$ with its 2-typification. The Hurewicz images

$$\bar{r}_k \in \pi_{k\rho_2}^{C_2} H\underline{\mathbb{Z}}_{(2)} \wedge MU^{((G))}$$

are given by the power series identity

$$(4.48) \qquad \sum \bar{r}_k \bar{x}^{k+1} = \left(\bar{x} + \sum \gamma(\bar{m}_{2^\ell-1}) \bar{x}^{2^\ell} \right)^{-1} \circ \log_{\bar{F}}(\bar{x}).$$

Modulo decomposables this becomes

$$(4.49) \qquad \bar{r}_k \equiv \begin{cases} \bar{m}_k - \gamma\bar{m}_k & k = 2^\ell - 1 \\ \bar{m}_k & \text{otherwise.} \end{cases}$$

This proves

Corollary 4.50. *The classes* $r_k = i_0^* \bar{r}_k$ *defined above satisfy the condition of Proposition 4.47.* □

These are the specific generators with which we shall work. Though it does not appear in the notation, the classes $\bar{r}_i$ depend on the group G. In §6 and §10 we will need to consider the classes $\bar{r}_i$ for a group G and for a subgroup $H \subset G$. We will then use the notation

$$\bar{r}_i^H \text{ and } \bar{r}_i^G$$

to distinguish them.

The following result establishes an important property of these specific $\bar{r}_k$.

Proposition 4.51. *For all* k

$$\Phi^G N(\bar{r}_k) = h_k \in \pi_k MO,$$

where the h_k *are the classes defined in §4.1.2. In particular, the set*

$$\{\Phi^G N(\bar{r}_k) \mid k \neq 2^\ell - 1\}$$

is a set of polynomial algebra generators of $\pi_* MO$, *and for all* ℓ

$$\Phi^G N(\bar{r}_{2^\ell-1}) = h_{2^\ell-1} = 0.$$

Proof: From Remark 4.26 we know that

$$\Phi^G N \bar{x} = a$$
$$\Phi^G N \bar{x}_H = e$$
$$\Phi^G N \bar{m}_n = \alpha_n.$$

Corollary 2.14 implies that

$$\Phi^G N \gamma \bar{m}_n = \Phi^G N \bar{m}_n,$$

so we also know that

$$\Phi^G N \gamma \bar{m}_n = \alpha_n.$$

Since the Hurewicz homomorphism

$$
\begin{array}{ccc}
\pi_* \Phi^G MU^{((G))} & \longrightarrow & \pi_* \Phi^G (H\underline{\mathbb{Z}}_{(2)} \wedge MU^{((G))}) \\
\approx \downarrow & & \downarrow \approx \\
\pi_* MO & \longrightarrow & \pi_* H\mathbb{Z}/2[b] \wedge MO
\end{array}
$$

is a monomorphism, we can calculate $\Phi^G N \bar{r}_k$ using (4.48). Applying $\Phi^G N$ to (4.48), and using the fact that it is a ring homomorphism gives

$$a + \sum (\Phi^G N \bar{r}_k) a^{k+1}$$
$$= \left(a + \sum (\Phi^G N \gamma \bar{m}_{2^\ell - 1}) a^{2^\ell} \right)^{-1} \circ \left(a + \sum (\Phi^G N \bar{m}_k) a^{k+1} \right)$$
$$= \left(a + \sum \alpha_{2^\ell - 1} a^{2^\ell} \right)^{-1} \circ \left(a + \sum \alpha_k a^{k+1} \right).$$

But this is the identity defining the classes h_k. $\qquad\qquad\square$

In addition to

$$h_k = \Phi^G N(\bar{r}_k) \in \pi_k \Phi^G MU^{((G))} = \pi_k MO$$

there are some important classes f_k attached to these specific $\bar{r}_k$.

Definition 4.52. Set

$$f_k = a_{\bar{\rho}_G}^k N \bar{r}_k \in \pi_k^G MU^{((G))},$$

where $\bar{\rho}_G = \rho_G - 1$ is the reduced regular representation.

The relationship between these classes is displayed by the following commutative diagram.

$$
\begin{array}{ccc}
 & S^k & \\
a_{\bar{\rho}_G}^k \swarrow & \downarrow f_k & \searrow h_k \\
S^{k\rho_G} \xrightarrow{N\bar{r}_k} & MU^{((G))} \longrightarrow & \tilde{E}C_2 \wedge MU^{((G))}
\end{array}
$$

4.4. The spectra $R(m)$, K_m, and K'_m. Our proof of the Slice Theorem makes use of some auxiliary spectra which we describe in this section.

4.4.1. *Definition of the spectra and the Reduction Theorem.* Define equivariant $MU^{((G))}$-modules $R(m) = R^G(m)$, $m = 0, 1, \ldots$ by

$$R(0) = MU^{((G))}$$

$$R(m) = MU^{((G))}/(G \cdot \bar{r}_1, \ldots, G \cdot \bar{r}_m).$$

The homotopy groups $\pi_*^u R(m)$ form a ring, naturally identified with the quotient of $\pi_*^u MU^{((G))}$ by the ideal generated by the elements of the mod 2 G-orbits through $\{r_1, \ldots, r_m\}$. On the other hand, the spectrum $R(m)$ is itself not a ring spectrum (see Remark 7.4). The $R(m)$ fit into a sequence

$$\cdots \to R(m) \to R(m+1) \to \cdots,$$

whose homotopy colimit we denote

$$R(\infty) = \underrightarrow{\mathrm{holim}}\, R(m).$$

We'll also need the $MU^{((G))}$-module spectra $K_m = K_m^G$ defined by the cofibration sequence

$$K_m \to R(m-1) \to R(m),$$

and the $MU^{((G))}$-module spectra $K'_m = K'_{m,G}$ defined by the cofibration sequence

$$\Sigma^{m\rho_G} R(m-1) \to K_m \to K'_m,$$

in which the first map is given by multiplication by $N\bar{r}_m$.

Lemma 4.53. *The geometric fixed point spectrum*

$$\Phi^G K'_m$$

is contractible.

Proof: By Proposition 2.34 the sequence

$$\Sigma^{m\rho_G} R(m-1) \xrightarrow{N\bar{r}_m} R(m-1) \to R(m)$$

becomes a cofibration sequence after applying geometric fixed points. This implies that

$$\Phi^G \Sigma^{m\rho_G} R(m-1) \to \Phi^G K_m$$

is a weak equivalence and hence that its cofiber $\Phi^G K'_m$ is contractible. $\square$

Lemma 4.54. *For all m*

$$R(m-1) \geq 0 \quad and \quad K_m \geq 2m.$$

The map

$$P^n R(m-1) \to P^n R(m)$$

is therefore an equivalence for $n < 2m$.

Proof: This is most easily verified using the formula

$$R(m) = R(m-1) \underset{S^0[G\cdot\bar{r}_m]}{\wedge} S^0.$$

The bar construction implies that $R(m)$ is contained in the smallest subcategory of $\mathcal{S}^G$ which is closed under homotopy colimits and which contains all spectra of the form

$$R(m-1) \wedge \bigwedge^k S^0[G \cdot \bar{r}_m], \quad k \geq 0.$$

All of these spectra are ≥ 0 hence so is $R(m)$. Let M be the $S^0[G \cdot \bar{r}_m]$-module fiber of

$$S^0[G \cdot \bar{r}_m] \to S^0.$$

The G-spectrum underlying M has the homotopy type of a wedge of slice cells of dimensions greater than or equal to $2m$. The formula

$$K_m \approx R(m-1) \underset{S^0[G \cdot \bar{r}_m]}{\wedge} M$$

implies that K_m is contained in the smallest subcategory of $\mathcal{S}^G$ which is closed under homotopy colimits and which contains all spectra of the form

$$R(m-1) \wedge \bigwedge^k S^0[G \cdot \bar{r}_m] \wedge M$$

with $k \geq 0$. All of these spectra are $\geq 2m$, and hence so is K_m. $\qquad\square$

The Thom map

$$MU^{((G))} \to H\underline{\mathbb{Z}}_{(2)}$$

factors uniquely through an $MU^{((G))}$-module map

$$R(\infty) \to H\underline{\mathbb{Z}}_{(2)}.$$

The following important result will be proved in §7.

Theorem 4.55 (The Reduction Theorem)**.** *The map*

$$R(\infty) \to H\underline{\mathbb{Z}}_{(2)}$$

is a weak equivalence.

The case $G = C_2$ of the Reduction Theorem is Proposition 4.9 of Hu-Kriz[16]. Its analogue in motivic homotopy theory appears in unpublished work of the second author and Morel.

4.4.2. *Homotopy groups of $R(m)$, K_m and K'_m.*

Proposition 4.56. *The spectra $R(m)$, K_m and K'_m admit refinements of homotopy groups*

$$\widehat{W}(R(m)) = \bigvee \widehat{W}_{2k}(R(m)) \to R(m)$$
$$\widehat{W}(K_m) = \bigvee \widehat{W}_{2k}(K_m) \to K_m$$
$$\widehat{W}(K'_m) = \bigvee \widehat{W}_{2k}(K'_m) \to K'_m.$$

These refinements are compatible in the sense that the diagrams

(4.57)
$$
\begin{array}{ccccc}
\widehat{W}(K_m) & \longrightarrow & \widehat{W}(R(m-1)) & \longrightarrow & \widehat{W}(R(m)) \\
\downarrow & & \downarrow & & \downarrow \\
K_m & \longrightarrow & R(m-1) & \longrightarrow & R(m)
\end{array}
$$

and

$$(4.58) \qquad \begin{array}{ccccc} \Sigma^{m\rho_G}\widehat{W}(R(m-1)) & \longrightarrow & \widehat{W}(K_m) & \longrightarrow & \widehat{W}(K'_m) \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^{m\rho_G}R(m-1) & \longrightarrow & K_m & \longrightarrow & K'_m. \end{array}$$

commute, the rows are cofibrations, and the top rows are split cofibrations.

Proof: Let

$$(4.59) \qquad \widehat{W} = S[G \cdot \bar{r}_1, G \cdot \bar{r}_2, \ldots] \to MU^{((G))}$$

be the multiplicative refinement of homotopy groups given by Proposition 4.30, using the generators $\bar{r}_i$. By Lemma 2.20, the $MU^{((G))}$-module $R(m-1)$ can be constructed as

$$MU^{((G))} \underset{S^0[G \cdot \bar{r}_1, \ldots, G \cdot \bar{r}_{m-1}]}{\wedge} S^0.$$

Smashing (4.59) over $S^0[G \cdot \bar{r}_1, \ldots, G \cdot \bar{r}_{m-1}]$ with S^0 then gives a map

$$(4.60) \qquad S[G \cdot \bar{r}_m, \ldots] = \widehat{W} \underset{S[G \cdot \bar{r}_1, \ldots G \cdot \bar{r}_{m-1}]}{\wedge} S^0 \to R(m-1)$$

which one easily checks to be a refinement of homotopy. We define $\widehat{W}(R(m-1)) \to R(m-1)$ to be the map (4.60). These refinements are compatible as m varies in the sense that the following diagram commutes

$$(4.61) \qquad \begin{array}{ccc} \widehat{W}(R(m-1)) & \longrightarrow & \widehat{W}(R(m)) \\ \downarrow & & \downarrow \\ R(m-1) & \longrightarrow & R(m), \end{array}$$

in which the top map is the one sending $G \cdot \bar{r}_m$ to $*$. The map

$$\widehat{W}(R(m-1)) \to \widehat{W}(R(m))$$

is a map of associative algebras, split by the evident algebra map

$$S[G \cdot \bar{r}_m, \ldots] \to S[G \cdot \bar{r}_{m-1}, G \cdot \bar{r}_m, \ldots].$$

We define

$$\widehat{W}(K_m) \to K_m$$

to be the induced map between the homotopy fibers of the rows of (4.61). Multiplication by the slice cell in $\widehat{W}(R(m-1))$ corresponding to $N\bar{r}_m$ gives a diagram

$$\begin{array}{ccc} \Sigma^{m\rho_G}\widehat{W}(R(m-1)) & \longrightarrow & \widehat{W}(K_m) \\ \downarrow & & \downarrow \\ \Sigma^{m\rho_G}R(m-1) & \longrightarrow & K_m. \end{array}$$

We define

$$\widehat{W}(K'_m) \to K'_m$$

to be the induced map of cofibers of the two rows. The claims of the proposition are now easily checked. $\qquad\square$

Remark 4.62. The slice cells occurring in the refinement of $R(m-1)$ correspond to the mod 2 orbits through the monomials in the variables $\gamma^j \bar{r}_k$ with $k \geq m$. From this it follows that none of these slice cells is free. The same is therefore true of the slice cells in $\widehat{W}(K_m)$ and $\widehat{W}(K'_m)$ since they form subsets of those in $\widehat{W}(R(m-1))$. The slice cells in $\widehat{W}(K'_m)$ correspond to the mod 2 orbits through the monomials in the variables $\{\gamma^j \bar{r}_k \mid k \geq m\}$ which are in the ideal $(\gamma^j \bar{r}_m)$ and which are not divisible by $N\bar{r}_m$. This implies that each of these slice cells is induced from a non-trivial proper subgroup of G, and hence that the spectrum

$$\widehat{W}(K'_m)$$

is itself induced from a proper subgroup of G.

5. More on the slice filtration

Using the spectra $MU^{((G))}$ we now turn to a deeper study of the slice filtration and of slices in their own right. In this section we give criteria which enable one to determine the slices of equivariant spectra under favorable circumstances.

We remind the reader that everything is now localized at 2, and that we have restricted to the case $G = C_{2^n}$.

5.1. **The d-slice of a d-cell.** Our first order of business is to show that the d-slice of a d-dimensional slice cell $\widehat{S}$ is $H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S}$ (Corollary 5.3 below). Everything comes down to the special case $\widehat{S} = S^0$.

Proposition 5.1. *The zero slice of S^0 is $H\underline{\mathbb{Z}}_{(2)}$.*

Before starting the proof, let's verify that $H\underline{\mathbb{Z}}_{(2)}$ is, in fact a zero slice. We prove something a little more general.

Lemma 5.2. *If $\widehat{S}$ and $\widehat{T}$ are cells, with $\dim \widehat{S} > \dim \widehat{T}$, then*

$$\mathcal{S}^G(\widehat{S}, H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{T})$$

is contractible.

Proof: By induction on $|G|$ we may assume that $\widehat{S}$ and $\widehat{T}$ are not induced cells. The table below lists the possibilities:

$\widehat{S}$	$\widehat{T}$	
$\widehat{S}(p\rho_G)$	$\widehat{S}(q\rho_G)$	$p|G| > q|G|$
$\Sigma^{-1}\widehat{S}(p\rho_G)$	$\Sigma^{-1}\widehat{S}(q\rho_G)$	$p|G| - 1 > q|G| - 1$
$\Sigma^{-1}\widehat{S}(p\rho_G)$	$\widehat{S}(q\rho_G)$	$p|G| - 1 > q|G|$
$\widehat{S}(p\rho_G)$	$\Sigma^{-1}\widehat{S}(q\rho_G)$	$p|G| > q|G| - 1.$

In the first two cases

$$\mathcal{S}^G(\widehat{S}(p\rho_G), H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S}(q\rho_G)) \approx \mathcal{S}^G(\widehat{S}((p-q)\rho_G), H\underline{\mathbb{Z}}_{(2)}).$$

is contractible, since for $k > 0$, the zero-skeleton of $\widehat{S}(k\rho_G)$ consists only of the basepoint. In the third case, we have

$$\mathcal{S}^G(\Sigma^{-1}\widehat{S}(p\rho_G), H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S}(q\rho_G)) \approx \mathcal{S}^G(\Sigma^{-1}\widehat{S}((p-q)\rho_G), H\underline{\mathbb{Z}}_{(2)}).$$

If G is the trivial group, then $(p - q) > 1$ and the space is contractible. If G is non-trivial then $(p - q) \geq 1$, and $\Sigma^{-1}\widehat{S}(p\rho_G)$ is the suspension spectrum of a G-space whose zero-skeleton consists only of the base point. The fourth case is the loop space of the first two cases. $\square$

Corollary 5.3. *If $\widehat{S}$ is a cell of dimension d then $H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S}$ is a d-slice. Thus the d-slice of $\widehat{S}$ is $H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S}$.*

Proof: By Proposition 3.13 we know that $H\underline{\mathbb{Z}}_{(2)} \geq 0$. It then follows from Lemma 3.28 that $H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S} \geq d$. So it suffices to show that $H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S} \approx P^d(H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S})$. But that is the content of Lemma 5.2. $\square$

We'll also need one more computational fact.

Lemma 5.4. *The Thom class*

$$(5.5) \qquad\qquad MU^{((G))} \to H\underline{\mathbb{Z}}_{(2)}$$

induces an isomorphism of Mackey functors

$$\underline{\pi}_0 MU^{((G))} \xrightarrow{\approx} \underline{\pi}_0 H\underline{\mathbb{Z}}_{(2)}.$$

Proof: We'll prove this by induction on $g = |G|$, the case of the trivial group being trivial. From Corollary 4.19, for $H \subset G$ there is a decomposition

$$i_H^* MU^{((G))} \approx MU^{((H))} \wedge (S^0 \vee W)$$

with W a wedge of slice cells of strictly positive, even dimension. Since $MU^{((H))} \geq 0$ it follows that $\underline{\pi}_0 MU^{((H))} \to \underline{\pi}_0 i_H^* MU^{((G))}$ is an isomorphism. We may therefore assume by induction that for H a proper subgroup of G, the map

$$(5.6) \qquad\qquad \pi_0^H(MU^{((G))}) \xrightarrow{\approx} \pi_0^H(H\underline{\mathbb{Z}}_{(2)})$$

is an isomorphism. We need to show that

$$(5.7) \qquad\qquad \pi_0^G(MU^{((G))}) \to \pi_0^G(H\underline{\mathbb{Z}}_{(2)})$$

is.

The isomorphism (5.6) implies that (5.5) induces an isomorphism on π_0 after smashing with G/H_+. Since $MU^{((G))}$ and $H\underline{\mathbb{Z}}_{(2)}$ are (-1)-connected this implies that (5.5) induces an isomorphism on π_0 after smashing with any X built from cells of the form $G/H_+ \wedge S^m$ with $m \geq 0$ and H a proper subgroup of G. This applies in particular to $X = EC_{2+}$.

Consider the map of long exact homotopy sequences gotten by smashing $MU^{((G))}$ and $H\underline{\mathbb{Z}}_{(2)}$ with the isotropy separation sequence

$$EC_{2+} \to S^0 \to \tilde{E}C_2.$$

We know that $EC_{2+} \wedge MU^{((G))}$ and $EC_{2+} \wedge H\underline{\mathbb{Z}}_{(2)}$ are both (-1)-connected and so have no negative homotopy groups. We also know by the induction hypothesis, as described above, that (5.5) induces an isomorphism on π_0 after smashing with EC_{2+}. Since $\Phi^G MU^{((G))} = MO$,

$$\pi_1^G \tilde{E}C_2 \wedge MU^{((G))} = \pi_1^G \tilde{E}C_2 \wedge H\underline{\mathbb{Z}}_{(2)} = 0,$$

and
$$\pi_0^G \tilde{E}C_2 \wedge MU^{((G))} = \pi_0^G \tilde{E}C_2 \wedge MU^{((G))} = \mathbb{Z}/2.$$
The map above is an isomorphism since it is a ring homomorphism. The interesting portion of the long exact sequences is therefore a map of short exact sequences

$$
\begin{array}{ccccc}
\pi_0^G EC_{2+} \wedge MU^{((G))} & \longrightarrow & \pi_0^G MU^{((G))} & \longrightarrow & \pi_0^G \tilde{E}C_2 \wedge MU^{((G))} \\
\approx \downarrow & & \downarrow & & \approx \downarrow \\
\pi_0^G EC_{2+} \wedge H\underline{\mathbb{Z}}_{(2)} & \longrightarrow & \pi_0^G H\underline{\mathbb{Z}}_{(2)} & \longrightarrow & \pi_0^G \tilde{E}C_2 \wedge H\underline{\mathbb{Z}}_{(2)}.
\end{array}
$$

We've shown that the left and right vertical maps are isomorphisms so the claim follows. $\qquad\square$

Proof of Proposition 5.1: Since $S^0 \geq 0$ and $H\underline{\mathbb{Z}}_{(2)} \geq 0$ it suffices to show that the map $S^0 \to H\underline{\mathbb{Z}}_{(2)}$ is an equivalence after applying P^0. We'll do this in two steps. First note that since $MU^{((G))}$ is built from S^0 by attaching slice cells of dim > 0, the inclusion of the unit $S^0 \to MU^{((G))}$ induces an equivalence

$$P^0 S^0 \to P^0 MU^{((G))}.$$

It therefore suffice to show that the Thom map $MU^{((G))} \to H\underline{\mathbb{Z}}_{(2)}$ induces an equivalence after applying P^0. But that is a consequence of Lemma 5.4 and Corollary 3.43. $\qquad\square$

5.2. **More general slices.** We now turn to criteria for determining the slices of other spectra.

5.2.1. *Cellular slices, isotropic and perfect spectra.*

Definition 5.8. A d-slice M is *cellular* if $M \approx H\mathbb{Z}_{(2)} \wedge \widehat{W}$, where $\widehat{W}$ is a wedge of slice cells of dimension d. A cellular slice is *isotropic* if $\widehat{W}$ can be written as a wedge of slice cells, none of which is free (i.e., of the form $G_+ \wedge S^n$).

Definition 5.9. A G-spectrum X *has cellular slices* if $P_n^n X$ is cellular for all n, and is *isotropic* if its slices are isotropic. A G-spectrum X is *perfect* if it has cellular slices and has the property that $P_n^n X$ is contractible for n odd. Finally, we'll say X has cellular slices (is perfect, isotropic) below dimension k if $P^k X$ is has cellular slices (is perfect, isotropic).

The problem of determining the slices $P_k^k X$ of a G-spectrum is handled in two different ways, depending on the parity of k.

5.2.2. *Even slices.* There is a miraculous condition that often makes quick work out of the problem of determining the even slices of a G-spectrum. We call it the *strange condition*, for lack of a better name. The condition applies to any homotopy group $\pi_{2k}^u X$ admitting an equivariant refinement in the sense of Definition 4.28 and leads to the following result.

Theorem 5.10. *Suppose that X is a G-spectrum and that $\widehat{W} \to X$ is a refinement of $\pi_{2k}^u X$. Then the canonical map*

$$H\mathbb{Z}_{(2)} \wedge \widehat{W} \to P_{2k}^{2k} X$$

is an equivalence.

The proof of Theorem 5.10 is given below, as well as an explication of the "canonical map" in its statement. The theorem asserts that if a spectrum admits a refinement of its homotopy groups, then the even slices are the "obvious" ones, given by the refinement.

Corollary 5.11. *Suppose that $f : X \to Y$ is a map of perfect spectra, and that X admits a refinement of its homotopy groups. If $\pi_*^u f$ is an isomorphism then f is a weak equivalence.*

Proof: It suffices to show that for each n, that $P_n^n X \to P_n^n Y$ is a weak equivalence. If n is odd, both sides are contractible by assumption. The case when n is even follows from Theorem 5.10. $\square$

Remark 5.12. Using the slice spectral sequence one can easily show that a perfect spectrum always admits a refinement of homotopy groups. So that condition on X in Corollary 5.11 is actually superfluous.

We now turn to some material needed for the proof of Theorem 5.10. In addition to the regular representation ρ_G of G, we will need the *sign representation* $\sigma = \sigma_G$ associated to the unique non-trivial homomorphism $G \to \mathbb{Z}/2 = O(1)$.

Definition 5.13. Let d be an *even* integer. A G-spectrum M which is a d-slice satisfies the *strange condition* (for G) if d is not divisible by g, or if $d = kg$ and

$$\pi_{k\rho_G + 2\sigma - 1}^G M = 0.$$

We'll say that a G-spectrum M satisfies the *strange condition for $H \subset G$* if $i_H^* M$ satisfies the strange condition for H.

Example 5.14. If $\widehat{S}$ is a slice cell, then $H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S}$ satisfies the strange condition. To see this, first use the isomorphism

$$\pi_{k\rho_G + 2\sigma - 1}^G H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S} = \pi_{2\sigma - 1}^G H\underline{\mathbb{Z}}_{(2)} \wedge S^{-k\rho_G} \wedge \widehat{S}$$

and the fact that $S^{-k\rho_G} \wedge \widehat{S}$ is a slice cell (Lemma 3.25) to reduce to the case $\dim \widehat{S} = 0$. If $\widehat{S} = G/H_+$ and $H \neq G$, then

$$\pi_{2\sigma - 1}^G H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S} = \pi_1^H H\underline{\mathbb{Z}}_{(2)} = 0.$$

It remains to check that

$$\pi_{2\sigma - 1}^G H\underline{\mathbb{Z}}_{(2)} = H_G^1(S^{2\sigma}; \mathbb{Z}_{(2)}) = 0$$

which is easily done, using the methods described in §2.6.

Example 5.15. One can check by explicit computation that the spectrum $M = H\underline{\mathbb{Z}}_{(2)} \wedge S^{2-2\sigma}$ is a 0-slice which does not satisfy the strange condition.

The real work behind Theorem 5.10 is

Proposition 5.16. *Let $H \subset G$ be a subgroup. Suppose that M and M' are two k-slices and $f : M \to M'$ is a G-equivariant map with the property that $i_H^* f$ is an H-equivalence. If for all $H \subset H' \subseteq G$ the spectrum M satisfies the strange condition for H' then f is an equivalence.*

In Proposition 5.16 think of M as a guess for what the even slice of a spectrum X might be, and of M' as the actual slice. The force of the proposition is that it enables one to determine an even slice, under favorable conditions, entirely in terms of a guess for it. Before turning to the proof, we state an important consequence

Proof of Theorem 5.10, assuming Prop. 5.16: Since $\widehat{W} \geq 2k$, applying the functor P_{2k} to $\widehat{W} \to X$ gives a factorization

$$\widehat{W} \to P_{2k}X \to X.$$

Applying P^{2k} and using Corollary 5.3 then gives

(5.17) $$H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{W} \to P_{2k}^{2k}X.$$

This is the "canonical map" in the statement of the theorem. By Example 5.14 the spectrum $H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{W}$ satisfies the strange condition for all $H' \subseteq G$. Since $\widehat{W} \to X$ is a refinement of π_{2k}^u, and the non-equivariant tower underlying the slice tower is the Postnikov tower (Proposition 3.23), the map (5.17) is a equivalence of underlying spectra. Applying Proposition 5.16 with $H = \{0\}$ then gives that (5.17) is an equivalence. $\qquad\square$

Proof of Proposition 5.16: We reduce immediately to the case in which $H \subset G$ has index 2, and M satisfies the strange condition for G. By Proposition 3.16 it suffices to show that

(5.18) $$[\widehat{S}, M]^G \to [\widehat{S}, M']^G$$

is an isomorphism whenever $\widehat{S}$ is a slice cell of dimension $\geq k$. If $\dim \widehat{S} > k$ then both sides are zero since M and M' are k-slices. If $k \not\equiv 0 \mod g$ then $\widehat{S}$ is an induced cell, and the map is a bijection by assumption. If k is divisible by g we may smash with $S^{-(k/g)\rho_G}$ and assume $k = 0$. Again, if $\widehat{S}$ is induced, the map (5.18) is an isomorphism by assumption. So we're down to showing that that

$$\pi_0^G M \to \pi_0^G M'$$

is an isomorphism. For this, map the cofiber sequence

$$S(2\sigma)_+ \to S^0 \to S^{2\sigma}$$

into M and M', resulting in a diagram

$$
\begin{array}{ccccccc}
\pi_{2\sigma}^G M & \longrightarrow & \pi_0^G M & \longrightarrow & [S(2\sigma)_+, M]^G & \longrightarrow & \pi_{2\sigma-1}^G M \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \approx} & & \downarrow \\
\pi_{2\sigma}^G M' & \longrightarrow & \pi_0^G M' & \longrightarrow & [S(2\sigma)_+, M']^G & \longrightarrow & \pi_{2\sigma-1}^G M'.
\end{array}
$$

in which the rows are exact. The column labeled an isomorphism is so because $S(2\sigma)_+$ is built entirely from induced G-cells. By Lemma 5.19 below, the leftmost terms are zero. By the *strange* assumption, the upper rightmost term is zero. This

implies that the middle arrow in the top row is an isomorphism, and diagram is in fact as shown below:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_0 M & \overset{\approx}{\longrightarrow} & [S(2\sigma)_+, M] & \longrightarrow & 0 \\
& & \downarrow & & \downarrow {\scriptstyle \approx} & & \downarrow \\
0 & \longrightarrow & \pi_0 M' & \longrightarrow & [S(2\sigma)_+, M'] & \longrightarrow & \pi_{2\sigma-1} M'.
\end{array}
$$

The result follows.   □

We have used

Lemma 5.19. *If M is a 0-slice, then $\pi_{2\sigma}^G M = 0$.*

Proof: First note that the restriction map

$$[S^{2(\rho_G-1)}, M]^G \to [S^{2\sigma}, M]^G$$

is an isomorphism. Indeed, if $G = C_2$, then $S^{2\sigma} = S^{2(\rho_G-1)}$. Otherwise, the spectrum $S^{2(\rho_G-1)}$ is built from $S^{2\sigma}$ by attaching G-cells of the form $G/K_+ \wedge S^m$ with $m \geq 3$, and the claim follows from Corollary 3.42. But

$$S^{2(\rho_G-1)} = \Sigma^{-1} S^{\rho_G} \wedge S^{\rho_G-1} \geq g - 1 > 0,$$

and so

$$[S^{2(\rho_G-1)}, M]^G = 0.$$

□

5.2.3. *Odd slices.* The above results place the even slices of an equivariant spectrum X under control. The odd slices are something of a different story, and getting at them requires some knowledge of the equivariant homotopy theory of X. Note that by Corollary 3.44 any Mackey functor can occur in an odd slice. On the other hand, only special ones can occur in even slices.

The situation most of interest to us in this paper is when the odd slices are contractible. Proposition 5.20 below gives useful criterion.

Proposition 5.20. *For a G spectrum X and an odd integer d, the following are equivalent*

 i) *The d-slice of X is contractible;*

 ii) *For every slice cell $\widehat{S}$ of dimension d, $[\widehat{S}, X]^G = 0$.*

Proof: By Corollary 3.45, there is an isomorphism

$$[\widehat{S}, X]^G = [\widehat{S}, P_d^d X].$$

The result then follows from Proposition 3.16.   □

Corollary 5.21. *If $X \to Y \to Z$ is a cofibration sequence and the odd slices of X and Z are contractible below dimension d, then the odd slices of Y are contractible below dimension d.*

Proof: This is immediate from Proposition 5.20 and the long exact sequence of homotopy classes of maps. □

The next result has a very technical statement. It is more or less tailor made to apply to the spectra K'_m.

Proposition 5.22. *Suppose that K is a G-spectrum with the following properties*

i) *For every proper $H \subset G$, the odd slices of $i^*_H K$ are contractible.*

ii) *The even slices of K are induced from a proper subgroup: $P^{2k}_{2k} K = G_+ \wedge_H M$.*

iii) *The geometric fixed point spectrum $\Phi^G K$ is contractible.*

Then the odd slices of K are contractible. If the even slices of K are cellular, then K is perfect.

Proof: The condition on geometric fixed points is equivalent to the requirement that $\tilde{E}C_2 \wedge K$ be contractible. Now consider the spectrum $\tilde{E}C_2 \wedge P^d_d K$. If d is even this is contractible, by assumption ii). If d is odd, then $EC_{2+} \wedge P^d_d K$ is contractible, by assumption i) and so $P^d_d K \to \tilde{E}C_2 \wedge P^d_d K$ is an equivalence. This gives a decreasing filtration of $\tilde{E}C_2 \wedge K$ whose associated graded is a wedge of $P^d_d K$ with d odd. Proposition 3.24 then implies that this is the slice filtration of $\tilde{E}C_2 \wedge K$. But $\tilde{E}C_2 \wedge K$ is contractible, and hence so are its slices. This implies that $P^d_d K$ is contractible for d odd. □

5.3. Further multiplicative properties of the slice filtration. In this section we show that the filtration has the expected multiplicative properties for perfect spectra. Our main result is Proposition 5.23 below. It has the consequence that if X and Y are perfect spectra, and $E^{s,t}_r(-)$ is the slice spectral sequence, then there is a map of spectral sequences

$$E^{s,t}_r(X) \otimes E^{s',t'}_r(Y) \to E^{s+s',t+t'}_r(X \wedge Y)$$

representing the pairing $\underline{\pi}_* X \wedge \underline{\pi}_* Y \to \underline{\pi}_*(X \wedge Y)$. In other words, multiplication in the slice spectral sequence of perfect spectra behaves in the expected manner. We leave the deduction of this property from Proposition 5.23 to the reader.

Proposition 5.23. *If $X \geq n$ is perfect and $Y \geq m$ has cellular slices, then $X \wedge Y \geq n + m$.*

Proof: We need to show that $P^{n+m-1}(X \wedge Y)$ is contractible. By Lemma 3.33 the map

$$X \wedge Y \to P^{n+m-1} X \wedge P^{n+m-1} Y$$

is a P^{n+m-1}-equivalence, so we may reduce to the case in which the slice filtrations of X and Y are finite. That case in turn reduces to the situation in which

$$X = H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S}$$
$$Y = H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S}'$$

where $\widehat{S}$ is an even dimensional slice cell and $\widehat{S}'$ is any slice cell. By induction on g the assertion further reduces to the case in which neither $\widehat{S}$ nor $\widehat{S}'$ is induced. Thus we may assume

$$X = H\underline{\mathbb{Z}}_{(2)} \wedge S^{k\rho_G}$$

$$Y = H\underline{\mathbb{Z}}_{(2)} \wedge S^{\ell\rho_G} \quad \text{or} \quad H\underline{\mathbb{Z}}_{(2)} \wedge \Sigma^{-1} S^{\ell\rho_G}$$

in which case the result follows from Corollary 3.29. $\square$

6. Slice Theorem I: the induction

In the language of §5.2.1 the Slice Theorem asserts that $MU^{((G))}$ is isotropic and perfect. That, and a little more, is the content of

Theorem 6.1 (Slice Theorem). *The spectra $MU^{((G))}$, $R^G(m)$, K_m^G, and $K_{m,G}'$ are isotropic perfect spectra.*

We remind the reader that the spectra $R^G(m)$ are defined by

$$R^G(0) = MU^{((G))}$$

$$R^G(m) = MU^{((G))}/(G \cdot \bar{r}_1, \ldots, G \cdot \bar{r}_m),$$

and that K_m^G and $K_{m,G}'$ are defined by the cofibration sequences

$$K_m^G \to R^G(m-1) \to R^G(m)$$

$$\Sigma^{\rho_G} R^G(m-1) \xrightarrow{N\bar{r}_m} K_m^G \to K_{m,G}'.$$

Here is the idea of the proof of the Slice Theorem. Imagine one could construct a filtration of $MU^{((G))}$ by powers of the ideal $I = (G \cdot \bar{r}_1, \ldots)$. Then the associated graded spectrum of $MU^{((G))}$ by powers of this ideal would be

$$R^G(\infty)[G \cdot \bar{r}_1, \ldots].$$

By the Reduction Theorem, $R^G(\infty) = H\underline{\mathbb{Z}}_{(2)}$, so the above can be rewritten

$$H\underline{\mathbb{Z}}_{(2)}[G \cdot \bar{r}_1, \ldots].$$

But this expression is exactly what is claimed to the wedge of the slices of $MU^{((G))}$. It is not difficult to go from the decomposition just described to the Slice Theorem. Among other things, this makes clear the important role played by the Reduction Theorem.

The proof of Theorem 6.1 is by induction on $g = |G|$ and will be completed in §8. When G is trivial the result is classical, so the induction starts. The purpose of this section is to derive several consequences of the induction hypothesis. So until the end of §8 we will assume that G is a non-trivial group, and that Theorem 6.1 has been proved for all proper $H \subset G$.

The following consequences of this hypothesis will be established in this section.

Proposition 6.2. *Suppose $H \subset G$ has index 2. If Theorem 6.1 holds for H then the spectra $i_H^* MU^{((G))}$ and $i_H^* R^G(m)$ are isotropic perfect spectra.*

Proposition 6.3. *Under the hypotheses of Proposition 6.2, $i_H^* K_m^G$ and $i_H^* K_{m,G}'$ are isotropic perfect spectra.*

Proposition 6.4. *Under hypotheses of Proposition 6.2, there is an equivalence $i_H^* R^G(\infty) \approx R^H(\infty)$.*

In §7 the Reduction Theorem for G will be proved assuming the induction hypothesis. Using this, the induction step is completed in §8 and Theorem 6.1 is proved.

Before turning to the proof of the four propositions above we need to establish that certain polynomial extensions of $MU^{((G))}$ are weakly commutative (§2.3.7).

6.1. **Weak commutativity of special polynomial extensions.**

Lemma 6.5. *For $i > 0$, the spectrum*

$$MU_{\mathbb{R}}[S^{i\rho_2}]$$

is weakly commutative.

Proof: Write

$$\pi_\star MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} = MU_{\mathbb{R}\star}[\bar{b}_1, \dots]$$

and for each $j > 0$ choose a map

$$(6.6) \qquad\qquad S^{j\rho_2} \to MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}$$

representing $\bar{b}_j$. As described above, this gives an algebra map

$$MU_{\mathbb{R}}[\bar{b}_1, \bar{b}_2, \dots] \to MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}$$

which is a weak equivalence since, by construction it induces an isomorphism on $\pi_\star$. Using the maps $S^{j\rho_2} \to *$ for $j \neq i$ defines a map

$$MU_{\mathbb{R}}[\bar{b}_1, \bar{b}_2, \dots] \to MU_{\mathbb{R}}[\bar{b}_i],$$

splitting the evident inclusion. This exhibits

$$MU_{\mathbb{R}}[S^{i\rho_2}] = MU_{\mathbb{R}}[\bar{r}_i]$$

as an algebra retract of $MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}$. $\qquad\square$

Corollary 6.7. *If R is a commutative $MU_{\mathbb{R}}$-algebra then every polynomial extension of the form*

$$R[\bar{x}_1, \dots], \quad \bar{x}_i \leftrightarrow S^{k_i\rho_2}, \quad k_i > 0$$

is weakly commutative. $\qquad\square$

By taking norms one gets

Corollary 6.8. *If R is a commutative $MU^{((G))}$-algebra then every polynomial extension of the form*

$$R[G \cdot \bar{x}_1, \dots], \quad \bar{x}_i \leftrightarrow S^{k_i\rho_2}, \quad k_i > 0$$

is weakly commutative. $\qquad\square$

6.2. **Proofs of Propositions 6.2 and 6.4.** We begin with Proposition 6.2. For convenience, recall the statement.

Proposition. *Suppose $H \subset G$ has index 2. If Theorem 6.1 holds for H then the spectra $i_H^* MU^{((G))}$ and $i_H^* R^G(m)$ are isotropic perfect spectra.*

The case of $MU^{((G))}$ is an easy consequence of Corollary 4.18. As a reminder, one starts with the C_2-equivariant equivalence

$$MU_{\mathbb{R}}[\bar{b}_1, \bar{b}_2, \dots] \to MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}$$

and applies the norm up to H to get an equivalence

$$i_H^* MU^{((G))} \approx MU^{((H))}[H \cdot \bar{b}_1, H \cdot \bar{b}_2, \dots].$$

This shows that $i_H^* MU^{((G))}$ is a wedge of smash products of even dimensional isotropic slice cells with $MU^{((H))}$, and hence an isotropic perfect spectrum since $MU^{((H))}$ is.

For the remaining assertion we need to compare the spectra $i_H^* R^G(m)$ and $R^H(m)$, and for that we need to relate the classes r_i^H and the r_i^G. The map (4.1)

$$MU^{((H))} \to i_H^* MU^{((G))}$$

induces an embedding

$$\pi_*^u MU^{((H))} \to \pi_*^u MU^{((G))}$$

and so there is a formula expressing r_i^H in terms of elements in $\pi_*^u MU^{((G))}$. We need to know a little about it. Let $r_{\leq m}^H$ and $r_{\leq m}^G$ stand for the sequences

$$r_1^H, \dots, r_m^H \quad \text{and} \quad r_1^G, \dots, r_m^G$$

respectively.

Lemma 6.9. *In*

$$\mathbb{Z}_{(2)}[H \cdot r_{\leq m}^G, H \cdot \gamma r_{\leq m}^G] = \pi_*^u MU^{((G))}$$

there is an identity

$$r_i^H = \begin{cases} r_i^G + \gamma r_i^G \mod I^2 & i = 2^k - 1 \\ r_i^G \mod I^2 & otherwise \end{cases}$$

where I is the ideal generated by the r_i and γr_i.

Proof: The formula is in pure algebra and since the ring in question is torsion free we can invert 2 and make use of the log. The result is then a simple consequence of (4.49). $\square$

Corollary 6.10. *For every m, the map*

$$\mathbb{Z}_{(2)}[H \cdot r_{\leq m}^H, H \cdot \gamma r_{\leq m}^G] \to \mathbb{Z}_{(2)}[H \cdot r_{\leq m}^G, H \cdot \gamma r_{\leq m}^G]$$

is an isomorphism. $\square$

We now realize the above by a map of H-equivariant associative algebras. Let $R = i_H^* MU^{((G))}$. For each i define a C_2-equivariant associative algebra map

(6.11) $$S^0[\bar{r}_i^H] \to i_1^* R[H \cdot \bar{r}_{\leq m}^G, H \cdot \gamma \bar{r}_{\leq m}^G]$$

using the map

$$S^{i\rho_2} \to i_1^* R[H \cdot \bar{r}_{\leq m}^G, H \cdot \gamma \bar{r}_{\leq m}^G]$$

representing the element

$$\bar{r}_i^H \in \mathbb{Z}_{(2)}[H \cdot \bar{r}_{\leq m}^G, H \cdot \gamma \bar{r}_{\leq m}^G] \subset \pi_\star^{C_2} R[H \cdot \bar{r}_{\leq m}^G, H \cdot \gamma \bar{r}_{\leq m}^G]$$
$$= R_\star^{C_2}[\bar{r}_{\leq m}^G, \gamma \bar{r}_{\leq m}^G].$$

From Corollary 6.8, $R[H \cdot \bar{r}_{\leq m}^G, H \cdot \gamma \bar{r}_{\leq m}^G]$ is weakly commutative. We can therefore smash these maps (6.11) together for $i = 1, \ldots, m$ and apply the norm to form an H-equivariant algebra map

$$R[H \cdot \bar{r}_{\leq m}^H, H \cdot \gamma \bar{r}_{\leq m}^G] \to R[H \cdot \bar{r}_{\leq m}^G, H \cdot \gamma \bar{r}_{\leq m}^G].$$

Proposition 6.12. *For each m the map*

$$R[H \cdot \bar{r}_{\leq m}^H, H \cdot \gamma \bar{r}_{\leq m}^G] \to R[H \cdot \bar{r}_{\leq m}^G, H \cdot \gamma \bar{r}_{\leq m}^G].$$

is a weak equivalence.

Proof: The Slice Theorem for H shows that both spectra are perfect. Both spectra admit refinements of homotopy (Proposition 4.30), and the map is an equivalence of underlying non-equivariant spectra by Corollary 6.10. The map is therefore an equivalence of equivariant spectra by Corollary 5.11. $\qquad\square$

The assertion about $i_H^* R^G(m)$ in Proposition 6.2 is an immediate consequence of the next result.

Proposition 6.13. *Suppose $H \subset G$ has index 2. There is an equivalence of $MU^{((H))}$-modules*

$$i_H^* R^G(m) \approx R^H(m) \wedge S[H \cdot \gamma \bar{r}_{m+1}^G, H \cdot \gamma \bar{r}_{m+2}^G, \ldots].$$

Under this equivalence, the map

$$i_H^* R^G(m) \to i_H^* R^G(m+1)$$

corresponds to the smash product of the map $R^H(m) \to R^H(m+1)$ with the map

$$S[H \cdot \gamma \bar{r}_{m+1}^G, H \cdot \gamma \bar{r}_{m+2}^G, \ldots] \to S[H \cdot \gamma \bar{r}_{m+2}^G, \ldots]$$

sending $H \cdot \gamma \bar{r}_{m+1}^G$ to $*$.

Proof: To simplify the notation write

$$A = i_H^* MU^{((G))}$$

The spectrum $i_H^* R^G(m)$ can be identified with

$$i_H^* R^G(m) = A \underset{A[H \cdot \bar{r}, H \cdot \gamma \bar{r}]}{\wedge} A,$$

where $\bar{r}$ and $\gamma \bar{r}$ stand for the sequences

$$\bar{r}_1^G, \ldots, \bar{r}_m^G \quad \text{and} \quad \gamma \bar{r}_1^G, \ldots, \gamma \bar{r}_m^G$$

respectively. The spectrum

$$R^H(m) \wedge S[H \cdot \gamma \bar{r}_{m+1}^G, H \cdot \gamma \bar{r}_{m+2}^G, \ldots]$$

is given by

$$A \underset{A[\bar{r}', \gamma \bar{r}]}{\wedge} A$$

where in which $\bar{r}'$ stands for the sequence

$$\bar{r}_1^H, \ldots \bar{r}_m^H.$$

The result is proved in this case if we can find an H-equivariant associative algebra equivalence

$$A[H \cdot \bar{r}', H \cdot \gamma \bar{r}] \to A[H \cdot \bar{r}, H \cdot \gamma \bar{r}]$$

fitting into a diagram of equivariant associative algebras

$$
\begin{array}{ccc}
A & \longleftarrow A[H \cdot \bar{r}', H \cdot \gamma \bar{r}] \longrightarrow & A \\
\| & \downarrow & \| \\
A & \longleftarrow A[H \cdot \bar{r}, H \cdot \gamma \bar{r}] \longrightarrow & A
\end{array}
$$

in which the left arrows are maps sending the variables $\bar{r}_i^H$, $\bar{r}_i^G$ and $\gamma \bar{r}_i^G$ to the classes in $\pi_\star A$ corresponding to their names, and the right maps are the ones sending them to 0. But that is exactly the content of Corollary 6.12. $\square$

Proposition 6.4 is an immediate consequence of Proposition 6.13. By way of reminder, here is the statement.

Proposition. *For $H \subset G$, there is an equivalence*

$$\varinjlim_m i_H^* R^G(m) = \varinjlim_m R^H(m)$$

$\square$

6.3. **Proof of Proposition 6.3.** For convenience, we recall the statement.

Proposition. *Under the hypotheses of Proposition 6.2, $i_H^* K_m^G$ and $i_H^* K'_{m,G}$ are isotropic perfect spectra.*

By Proposition 4.56 and Remark 4.62, the spectra K_m^G and $K'_{m,G}$ admit isotropic refinements of homotopy. The even slices are therefore cellular and isotropic by Theorem 5.10. What is to check then is that the induction hypothesis implies that the odd slices of $i_H^* K_m^G$ and $i_H^* K'_{m,G}$ are contractible.

We start with $i_H^* K_m^G$. From Proposition 6.13 the map

$$i_H^* R^G(m-1) \to i_H^* R^G(m)$$

can be factored as

$$R^H(m-1)[H \cdot \gamma r_m^G, \dots] \to R^H(m-1)[H \cdot \gamma r_{m+1}^G, \dots]$$
$$\to R^H(m)[H \cdot \gamma r_{m+1}^G, \dots].$$

This places $i_H^* K_m^G$ in a cofibration

$$Y \to i_H^* K_m^G \to K_m^H[H \cdot \gamma r_{m+1}^G, \dots],$$

with Y the fiber of

$$(6.14) \qquad R^H(m-1)[H \cdot \gamma r_m^G, \dots] \to R^H(m-1)[H \cdot \gamma r_{m+1}^G, \dots].$$

The spectrum

$$K_m^H[H \cdot \gamma r_{m+1}^G, \dots]$$

has no odd slices by the induction hypothesis. Since (6.14) is the projection to a wedge summand, the spectrum Y is a retract of

$$R^H(m-1)[H \cdot \gamma r_m^G, \dots],$$

which also has no odd slices by the induction hypothesis. It follows that $i_H^* K_m^G$ has no odd slices (Proposition 5.21).

For the spectrum $K'_{m,G}$, note that the map

$$i_H^* \Sigma^{\rho_G} R^G(m-1) \to K_m^G$$

factors through a map

(6.15) $$i_H^* \Sigma^{\rho_G} R^G(m-1) \to Y$$

which is the inclusion of a wedge summand. It follows that there is a cofibration sequence

$$Y' \to i_H^* K_{G,m} \to K_m^H[H \cdot \gamma r_{m+1}^G, \dots],$$

where Y' is the cofiber of (6.15). Since Y' is a retract of Y it has no odd slices. As above, it then follows from Proposition 5.21 that $i_H^* K'_{G,m}$ has no odd slices.

7. THE REDUCTION THEOREM

We now turn to the proof of the main technical result of this paper. For convenience we repeat the statement.

Theorem (The Reduction Theorem). *The map*

$$R(\infty) \to H\underline{\mathbb{Z}}_{(2)}$$

is a weak equivalence.

The proof proceeds by induction on $g = |G|$, the case in which G is the trivial group being classical. Because of the equivalence

$$\varinjlim_m i_H^* R^G(m) = \varinjlim_m R^H(m),$$

given by Proposition 6.4, and the obvious fact that $i_H^* H\underline{\mathbb{Z}}_{(2)} = H\underline{\mathbb{Z}}_{(2)}$, we may assume that for a proper subgroup $H \subset G$ the map

$$i_H^* R^G(\infty) \to H\underline{\mathbb{Z}}_{(2)}$$

is a weak equivalence.

For the induction step we smash the map in question with the isotropy separation sequence (2.27)

$$
\begin{array}{ccc}
EC_{2+} \wedge R(\infty) \twoheadrightarrow R(\infty) \twoheadrightarrow \tilde{E}C_2 \wedge R(\infty) \\
\Big\downarrow{\scriptstyle f} \qquad\qquad \Big\downarrow{\scriptstyle g} \qquad\qquad \Big\downarrow{\scriptstyle h} \\
EC_{2+} \wedge H\underline{\mathbb{Z}}_{(2)} \twoheadrightarrow H\underline{\mathbb{Z}}_{(2)} \twoheadrightarrow \tilde{E}C_2 \wedge H\underline{\mathbb{Z}}_{(2)}.
\end{array}
$$

By the induction hypothesis, the map f is an equivalence. It therefore suffices to show that the map h is, and that, as discussed in Remark 2.28, is equivalent to showing that

(7.1) $$\pi_*^G h : \pi_* \Phi^G R(\infty) \to \pi_* \Phi^G H\underline{\mathbb{Z}}_{(2)}$$

is an isomorphism.

We first show that the two groups are abstractly isomorphic.

Proposition 7.2. *The ring* $\pi_* \Phi^G H\underline{\mathbb{Z}}_{(2)}$ *is given by*

$$\pi_* \Phi^G H\underline{\mathbb{Z}}_{(2)} = \mathbb{Z}/2[b],$$

with

$$b = u_{2\sigma} a_\sigma^{-2} \in \pi_2 \Phi^G H\underline{\mathbb{Z}}_{(2)} \subset a_\sigma^{-1} \pi_\star^G H\underline{\mathbb{Z}}_{(2)}.$$

The groups $\pi_* \Phi^G R(\infty)$ are given by

$$\pi_n \Phi^G R(\infty) = \begin{cases} \mathbb{Z}/2 & n \geq 0 \ even \\ 0 & otherwise. \end{cases}$$

Proof: The first assertion is a restatement of Proposition 2.42. For the second assertion, start with the the equivalence (given by Lemma 2.22)

$$R(m) = R(m-1) \underset{MU^{((G))}}{\wedge} MU^{((G))}/(G \cdot \bar{r}_m)$$

and apply geometric fixed points to get

$$\Phi^G R(m) = \Phi^G R(m-1) \underset{MO}{\wedge} \Phi^G MU^{((G))}/(G \cdot \bar{r}_m).$$

By Proposition 2.34 there is a cofibration sequence

$$\Sigma^m MO \xrightarrow{h_m} MO \to \Phi^G MU^{((G))}/(G \cdot \bar{r}_m),$$

and therefore a cofibration sequence of MO-modules

$$\Sigma^m \Phi^G R(m-1) \xrightarrow{h_m} \Phi^G R(m-1) \to \Phi^G R(m).$$

By Proposition 4.51 have

$$\Phi^G \bar{r}_i = \begin{cases} h_i & i \neq 2^k - 1 \\ 0 & i = 2^k - 1. \end{cases}$$

From this it is an easy matter to compute the groups $\pi_* \Phi^G R(m)$ inductively, starting with

$$\pi_* \Phi^G R(0) = \pi_* MO = \mathbb{Z}/2[h_i, i \neq 2^k - 1].$$

The outcome is as asserted. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Before going further we record a simple consequence of the above discussion which will be used in §10.1.

Proposition 7.3. *The map*

$$\pi_* \Phi^G MU^{((G))} = \pi_* MO \to \pi_* \Phi^G H\underline{\mathbb{Z}}_{(2)}$$

is zero for $* > 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 7.4. Once we know that the map (7.1) is an isomorphism it will follow that the spectra $R(m)$ cannot be ring spectra. Indeed in any ring structure the Thom map $R(m) \to H\underline{\mathbb{Z}}_{(2)}$ would be multiplicative, and hence

$$\pi_* \Phi^G R(m) \to \pi_* \Phi^G H\underline{\mathbb{Z}}_{(2)}$$

would be a ring homomorphism. But the image contains the element b and only finitely many other powers, so it is not a subring.

A simple multiplicative property reduces the problem of showing that that (7.1) is an isomorphism to showing that it is surjective in dimensions which are a power of 2.

Lemma 7.5. *If for every $k \geq 1$, the class $b^{2^{k-1}}$ is in the image of*

$$(7.6) \qquad\qquad \pi_{2^k} \Phi^G MU^{((G))}/(G \cdot \bar{r}_{2^k - 1}) \to \pi_{2^k} \Phi^G H\underline{\mathbb{Z}}_{(2)},$$

then (7.1) is surjective, hence an isomorphism.

Proof: By writing

$$R(\infty) = \bigwedge_i^{MU^{((G))}} MU^{((G))}/(G \cdot \bar{r}_i)$$

in which the smash product is taken in the category of left $MU^{((G))}$-modules, we see that if for every $k \geq 1$, $b^{2^{k-1}}$ is in the image of (7.6), then all products of the $b^{2^{k-1}}$ are in the image of

(7.7) $$\pi_* \Phi^G R(\infty) \to \pi_* \Phi^G H\mathbb{Z}_{(2)}.$$

Hence every power of b is in the image of (7.7). $\qquad\square$

In view of Lemma 7.5, the Reduction Theorem follows from

Proposition 7.8. *For every $k \geq 1$, the class $b^{2^{k-1}}$ is in the image of*

$$\pi_{2^k} \Phi^G MU^{((G))}/(G \cdot \bar{r}_{2^k-1}) \to \pi_{2^k} \Phi^G H\mathbb{Z}_{(2)}.$$

To simplify some of the notation, write

$$c_k = 2^k - 1.$$

We start with the class

$$N\bar{r}_{c_k} \in \pi_{c_k \rho_G}^G MU^{((G))}.$$

By Proposition 4.51 its image in $\pi_{c_k} MO$ is zero. Its image in

$$\pi_{c_k \rho_G}^G \tilde{E}C_2 \wedge MU^{((G))}$$

is then also zero since by Remark 2.29, restriction along the fixed point inclusion

$$S^{c_k} \subset S^{c_k \rho_G}$$

is an isomorphism

$$\pi_{c_k \rho_G}^G \tilde{E}C_2 \wedge MU^{((G))} \approx \pi_{c_k}^G \tilde{E}C_2 \wedge MU^{((G))},$$

and

$$\pi_{c_k}^G \tilde{E}C_2 \wedge MU^{((G))} \approx \pi_{c_k} \Phi^G MU^{((G))} \approx \pi_{c_k} MO.$$

There is therefore a class

$$y_k \in \pi_{c_k \rho_G}^G EC_{2+} \wedge MU^{((G))}.$$

lifting $N\bar{r}_{c_k}$. The key computation, from which everything follows is

Proposition 7.9. *The image under*

$$\pi_{c_k \rho_G}^G EC_{2+} \wedge MU^{((G))} \to \pi_{c_k \rho_G}^G EC_{2+} \wedge H\mathbb{Z}_{(2)},$$

of any choice of y_k above, is non-zero.

Proof of Proposition 7.8 assuming Proposition 7.9: Write

$$M_k = MU^{((G))}/(G \cdot \bar{r}_{c_k})$$

and consider the following diagram

$$
\begin{array}{ccccc}
EC_{2+} \wedge MU^{((G))} & \twoheadrightarrow & MU^{((G))} & \twoheadrightarrow & \tilde{E}C_2 \wedge MU^{((G))} \\
\downarrow & & \downarrow & & \downarrow \\
EC_{2+} \wedge M_k & \longrightarrow & M_k & \longrightarrow & \tilde{E}C_2 \wedge M_k \\
\downarrow & & \downarrow & & \downarrow \\
EC_{2+} \wedge H\underline{\mathbb{Z}}_{(2)} & \longrightarrow & H\underline{\mathbb{Z}}_{(2)} & \longrightarrow & \tilde{E}C_2 \wedge H\underline{\mathbb{Z}}_{(2)}
\end{array}
$$

By construction the image of y_k in $\pi^G_{c_k\rho_G} EC_{2+} \wedge M_k$ maps to zero in $\pi^G_{c_k\rho_G} M_k$. It therefore comes from a class

$$
\tilde{y}_k \in \pi^G_{c_k\rho_G+1} \tilde{E}C_2 \wedge M_k.
$$

The image of $\tilde{y}_k$ in $\pi^G_{c_k\rho_G+1} \tilde{E}C_2 \wedge H\underline{\mathbb{Z}}_{(2)}$ is non-zero since it has a non-zero image in

$$
\pi^G_{c_k\rho_G} EC_{2+} \wedge H\underline{\mathbb{Z}}_{(2)}
$$

by Proposition 7.9. Now consider the commutative square below, in which the horizontal maps are the isomorphisms (Remark 2.29) given by restriction along the fixed point inclusion $S^{2^k} \subset S^{c_k\rho_G+1}$

$$
\begin{array}{ccc}
\pi^G_{c_k\rho_G+1} \tilde{E}C_2 \wedge M_k & \xrightarrow{\approx} & \pi^G_{2^k} \tilde{E}C_2 \wedge M_k \\
\downarrow & & \downarrow \\
\pi^G_{c_k\rho_G+1} \tilde{E}C_2 \wedge H\underline{\mathbb{Z}}_{(2)} & \xrightarrow{\approx} & \pi^G_{2^k} \tilde{E}C_2 \wedge H\underline{\mathbb{Z}}_{(2)}.
\end{array}
$$

The group on the bottom right is cyclic of order 2, generated by $b^{2^{k-1}}$. We've just shown that the image of $\tilde{y}_k$ under the left vertical map is non-zero. It follows that the right vertical map is non-zero and hence that $b^{2^{k-1}}$ is in its image. $\square$

The remainder of this section is devoted to the proof of Proposition 7.9.

The advantage of Proposition 7.9 is that it entirely involves G-spectra which have been smashed with EC_{2+}, and which therefore fall under the jurisdiction of the inductive hypothesis. In particular, the map

(7.10) $EC_{2+} \wedge MU^{((G))} \to EC_{2+} \wedge H\underline{\mathbb{Z}}_{(2)}$

can be studied by smashing the slice tower of $MU^{((G))}$ with EC_{2+}. For reasons that will become clear later it will be more convenient to factor (7.10) as

$$
EC_{2+} \wedge MU^{((G))} \to EC_{2+} \wedge R(c_k-1) \to EC_{2+} \wedge H\underline{\mathbb{Z}}_{(2)}
$$

and track what happens to y_k under

$$
\pi^G_{c_k\rho_G} EC_{2+} \wedge R(c_k-1) \to \pi^G_{c_k\rho_G} EC_{2+} \wedge H\underline{\mathbb{Z}}_{(2)}
$$

by studying the smash product of the slice tower for $R(c_k-1)$ with EC_{2+}. Because of the induction hypothesis, we know that the odd terms

$$
EC_{2+} \wedge P^{2n+1}_{2n+1} R(c_k-1)
$$

are contractible and that the even terms are given by

$$
EC_{2+} \wedge P^{2n}_{2n} R(c_k-1) \approx EC_{2+} \wedge H\mathbb{Z} \wedge \widehat{W},
$$

where $\widehat{W} \to R(c_k - 1)$ is a refinement of $\pi_{2n}^u R(c_k - 1)$. (We know this for the even terms before smashing with EC_{2+} by Proposition 5.10.)

The spectral sequence takes the form

$$E_2^{s,t} = \pi_{c_k \rho_G + t - s}^G EC_{2+} \wedge P_{c_k g + t}^{c_k g + t} R(c_k - 1) \implies \pi_{c_k \rho_G + t - s}^G EC_{2+} \wedge R(c_k - 1).$$

Note that it converges strongly. The fact that smashing commutes with homotopy colimits implies

$$\varinjlim_{m \to -\infty} \pi_\star^G EC_{2+} \wedge P^m R(c_k - 1) = 0.$$

Proposition 3.36 and the fact that $EC_{2+} \wedge P_m R(c_k - 1) \geq m$ imply that for a fixed index $\star$,

$$\pi_\star^G EC_{2+} \wedge P_m R(c_k - 1) = 0, \text{ for } m \gg 0.$$

The "edge homomorphism" (projection to $t = -c_k g$) represents the the map we wish to determine

$$\pi_{c_k \rho_G + *}^G EC_{2+} \wedge R(c_k - 1) \to \pi_{c_k \rho_G + *}^G EC_{2+} \wedge H\underline{\mathbb{Z}}_{(2)}.$$

There isn't much that can contribute to

$$\pi_{c_k \rho_G}^G EC_{2+} \wedge R(c_k - 1).$$

Since H acts trivially on EC_{2+}, for an induced slice cell $\widehat{S} = G \wedge_H \widehat{S}'$ one has

$$\pi_{c_k \rho_G}^G EC_{2+} \wedge H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S} = \pi_{c_k \rho_G}^H H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S}'$$

and the summands of the slices of this form contribute only to the restricted range of dimensions described in Theorem 3.38. In particular, the only ones contributing to $\pi_{c_k \rho_G}^G R(c_k - 1)$ are the ones occurring in $P_{c_k g}^{c_k g} R(c_k - 1)$. A simple inspection of

$$\pi_{c_k \rho_G + *}^G EC_{2+} \wedge H\underline{\mathbb{Z}}_{(2)} \wedge S^{m c_k \rho_G} \qquad m \geq 0$$

shows that the only non-induced terms which contribute to $\pi_{c_k \rho_G}^G$ are

$$\pi_{c_k \rho_G}^G EC_{2+} \wedge H\underline{\mathbb{Z}}_{(2)},$$

coming from $P_0^0 R(c_k - 1)$, and

$$\pi_{c_k \rho_G}^G EC_{2+} \wedge H\underline{\mathbb{Z}}_{(2)} \wedge S^{c_k \rho_G},$$

coming from the summand of $P_{c_k g}^{c_k g} R(c_k - 1)$ representing $N\bar{r}_{c_k}$.

The spectral sequence is depicted in Figure 2. The edge homomorphism is the projection map to the row of dots lying along the line of slope -1, and meeting the $(t - s)$-axis at $(t - s) = -c_k g$. Each dot represents a group $Z/2$, and the boxes represents copies of the group $\mathbb{Z}_{(2)}$. The refinement of $\pi_{c_k g}^u R(c_k - 1)$ has the form

$$S^{c_k \rho_G} \vee W \to R(c_k - 1)$$

in which W is a wedge of induced slice cells of dimension $c_k g$, and the map

$$S^{c_k \rho_G} \to R(c_k - 1)$$

is $N\bar{r}_{c_k}$. The chart in position $(0,0)$ has one box generated by $N\bar{r}_{c_k}$ and one box for each of the slice cells constituting W, altogether representing the group

$$\pi_{c_k \rho_G}^G EC_{2+} \wedge H\underline{\mathbb{Z}}_{(2)} \left(S^{c_k \rho_G} \vee W \right).$$

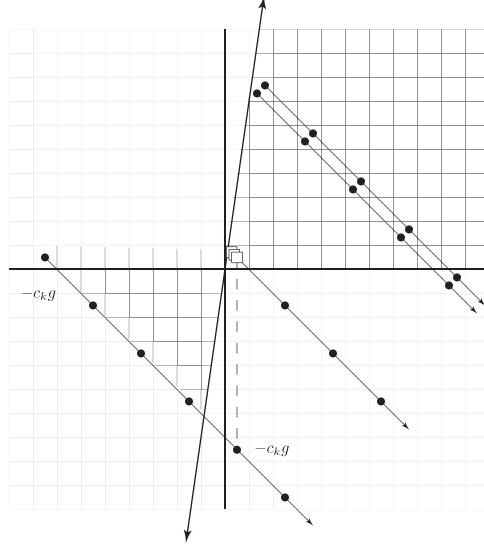From the E_2-term of the spectral sequence, as displayed in Figure 2 one reads off

FIGURE 2. The spectral sequence for

$$\pi^G_{c_k \rho_G + t - s} EC_{2+} \wedge R(c_k - 1)$$

Corollary 7.11. *The map*

$$\pi^G_{c_k \rho_G} EC_{2+} \wedge P_{c_k g} R(c_k - 1) \to \pi^G_{c_k \rho_G} EC_{2+} \wedge P^{c_k g}_{c_k g} R(c_k - 1)$$

is an isomorphism. There is a short exact sequence

$$\pi^G_{c_k \rho_G} EC_{2+} \wedge P_{c_k g} R(c_k - 1) \longrightarrow \pi^G_{c_k \rho_G} EC_{2+} \wedge R(c_k - 1)$$

$$\downarrow$$

$$\pi^G_{c_k \rho_G} EC_{2+} \wedge H\underline{\mathbb{Z}}_{(2)} = \mathbb{Z}/2.$$

$\square$

This analysis shows that the map of Proposition 7.9 is surjective, and it gives a description of the kernel. Corollary 7.11 converts the problem of showing that y_k has non-zero image in $\pi^G_{c_k \rho_G} EC_{2+} \wedge H\underline{\mathbb{Z}}_{(2)}$ to showing that it is not in the image of

$$\pi^G_{c_k \rho_G} P^{c_k g}_{c_k g} R(c_k - 1).$$

To get control over this we need to find a property of the image of $\pi^G_{c_k \rho_G} EC_{2+} \wedge P_{c_k g} R(c_k - 1)$ that is not shared by y_k.

Proposition 7.12. *The image of*

$$\pi^G_{c_k \rho_G} EC_{2+} \wedge P_{c_k g} R(c_k - 1) \to \pi^G_{c_k \rho_G} R(c_k - 1) \xrightarrow{i^*_0} \pi^u_{c_k g} R(c_k - 1)$$

is contained in the image of $(1 - \gamma)$.

The class y_k does not have the property described in Proposition 7.12. Its image in $\pi^u_{c_k g} R(c_k - 1)$ is $i^*_0 N \bar{r}_{c_k}$ which generates a sign representation of G occurring as

a summand of $\pi^u_{c_k g} R(c_k - 1)$. Thus once Proposition 7.12 is proved the proof of the Reduction Theorem is complete.

The proof of Proposition 7.12 is in two steps. First it is shown (Corollary 7.15) that the image of

$$\pi^G_{c_k \rho_G} EC_{2+} \wedge P_{c_k g} R(c_k - 1) \to \pi^G_{c_k \rho_G} R(c_k - 1)$$

is contained in the image of the transfer map

$$\pi^H_{c_k \rho_G} R(c_k - 1) \to \pi^G_{c_k \rho_G} R(c_k - 1)$$

from the subgroup $H \subset G$ of index 2. We then show (Lemma 7.16) that the image of the transfer map in $\pi^u_{c_k g} R(c_k - 1)$ is in the image of $(1 - \gamma)$.

We now turn to these steps.

Lemma 7.13. *Let $M \geq 0$ be a G-spectrum. The image of*

$$\pi^G_0 EC_{2+} \wedge M \to \pi^G_0 M$$

is the image of the transfer map

$$\pi^H_0 M \to \pi^G_0 M$$

where $H \subset G$ is the subgroup of index 2.

Proof: Since M is (-1)-connected (Proposition 3.13) the cell decomposition of EC_{2+} implies that $\pi^G_0 C_{2+} \wedge M \to \pi^G_0 EC_{2+} \wedge M$ is surjective. The composite

$$\pi^G_0 C_{2+} \wedge M \to \pi^G_0 EC_{2+} \wedge M \to \pi^G_0 M$$

is the transfer. $\square$

Corollary 7.14. *The image of*

$$\pi^G_{c_k \rho_G} EC_{2+} \wedge P_{c_k g} R(c_k - 1) \to \pi^G_{c_k \rho_G} P_{c_k g} R(c_k - 1)$$

is contained in the image of the transfer map.

Proof: This follows from Lemma 7.13 above, after the identification

$$\pi^G_{c_k \rho_G} P_{c_k g} R(c_k - 1) \approx \pi^G_0 S^{-c_k \rho_G} \wedge P_{c_k g} R(c_k - 1)$$

and the observation that

$$S^{-c_k \rho_G} \wedge P_{c_k g} R(c_k - 1) \approx P_0(S^{-c_k \rho_G} \wedge R(c_k - 1))$$

is ≥ 0. $\square$

Corollary 7.15. *The image of*

$$\pi^G_{c_k \rho_G} EC_{2+} \wedge P_{c_k g} R(c_k - 1) \to \pi^G_{c_k \rho_G} R(c_k - 1)$$

is contained in the image of the transfer map.

Proof: Immediate from Corollary 7.14 and the naturality of the transfer. $\square$

The next result is quite general. We are interested in the case $X = P_{c_k g} R(c_k - 1)$, $V = c_k \rho_G$.

Lemma 7.16. *Let X be a G-spectrum, V a virtual representation of G of virtual dimension d, and $H \subset G$ the subgroup of index 2. Write $\epsilon \in \{\pm 1\}$ for the degree of*

$$\gamma : i_0^* S^V \to i_0^* S^V.$$

The image of

$$\pi_V^H X \xrightarrow{\mathrm{Tr}} \pi_V^G X \to \pi_d^u X$$

is contained in the image of

$$(1 + \epsilon\gamma) : \pi_d^u X \to \pi_d^u X.$$

Proof: Consider the diagram

$$
\begin{array}{ccc}
\pi_V^G(C_{2+} \wedge X) & \longrightarrow & \pi_V^G X \\
\downarrow & & \downarrow \\
\pi_d^u(C_{2+} \wedge X) & \longrightarrow & \pi_d^u X,
\end{array}
$$

in which the map of the top row is induced by the projection $C_{2+} \to S^0$. By the Wirthmüller isomorphism, the term in the upper left is isomorphic to $\pi_V^H X$ and the map of the top row can be identified with the transfer map. The non-equivariant identification

$$C_{2+} \approx S^0 \vee S^0$$

gives an isomorphism of groups non-equivariant stable maps

$$[C_{2+} \wedge S^V, X] \approx [S^V, X] \oplus [S^V, X],$$

and so an isomorphism of the group in the lower left hand corner with

$$\pi_d^u X \oplus \pi_d^u X$$

under which the generator $\gamma \in G$ acts as

$$(a, b) \mapsto (\epsilon\gamma b, \epsilon\gamma a).$$

The map along the bottom is $(a, b) \mapsto a + b$. Now the image of the left vertical map is contained in the set of elements invariant under γ which, in turn, is contained in the set of elements of the form

$$(a, \epsilon\gamma a).$$

$\square$

Proof of Proposition 7.12: As described after its statement, Proposition 7.12 is a consequence of Corollary 7.15 and Lemma 7.16. $\square$

8. Slice Theorem II: the proof

In the language of §5.2.1 the Slice Theorem asserts that $MU^{((G))}$ is an isotropic perfect spectrum. That, and a little more is the content of

Theorem 8.1. *The spectra $MU^{((G))}$, $R(m)$, K_m^G are isotropic, perfect spectra.*

We begin with the even slices.

Proposition 8.2. *The even slices of $MU^{((G))}$, $R(m)$ and K_m and K_m' are isotropic cellular slices.*

Proof: It was shown in §4.3 and §4.4 that the spectra involved admit refinements of π_{2k} for every k, and that none of the cells constituting the refinement is G-free. The result then follows from Theorem 5.10. □

Proposition 8.3. *For all $m \geq 1$, the spectrum K_m' is perfect and isotropic.*

Proof: We may assume by induction on g that $K_{m,H}'$ is perfect for any proper subgroup $H \subset G$. By Proposition 6.3 $i_H^* K_G'$ is perfect and isotropic. It's now a simple matter to check that K' satisfies the conditions of Proposition 5.22. The fact that the even slices are induced follows from Remark 4.62, and the contractibility of the geometric fixed point spectrum is Lemma 4.53. □

Proposition 8.4. *If the odd slices of $R(m-1)$ are contractible in dimensions less than d, then the odd slices of K_m are contractible in dimensions less than $d + gm$ (hence also in dimensions less than $d + 2m$, since $g \geq 2$).*

Proof: Consider the cofibration sequence

$$\Sigma^{m\rho_G} R(m-1) \xrightarrow{Nr_m} K_m \to K_m'.$$

We know that K_m' is perfect by Proposition 8.3. Since for all N,

$$P^N(\Sigma^{m\rho_G} R(m-1)) \approx \Sigma^{m\rho_G} P^{N-mg}(R(m-1)),$$

our assumption implies that the odd slices of $\Sigma^{m\rho_G} R(m-1)$ are contractible in dimensions less than $d + mg$. It then follows from Corollary 5.21 that the odd slices of K_m are contractible in dimensions less than $d + mg$. □

Proof of the Theorem 8.1: We've already shown that the even slices are cellular and isotropic, so what remains to prove is that the odd slices are contractible. Let A_k stand for the assertion: "for all $m \geq 1$, the odd slices of $R(m-1)$ and K_m are contractible in dimensions less than $2(m + k)$." The proof will be complete if we can establish A_k for all k. The maps

$$P^d R(m-1) \to P^d R(m) \to \cdots \to P^d R(\infty)$$

are equivalences for $d < 2m$ by Lemma 4.54. The Reduction Theorem therefore gives the assertion A_0. Assuming A_k, Proposition 8.4 above gives that the odd slices of K_m are contractible below dimension $2(m + k) + 2m \geq 2(m + k + 1)$. Applying Corollary 5.21 to the cofibration

$$K_m \to R(m-1) \to R(m)$$

then gives that the odd slices of $R(m-1)$ are contractible in dimensions less than $2(m+k+1)$. Thus

$$A_k \implies A_{k+1}.$$

So we know A_k for all k. □

9. The Gap Theorem

The proof of the Gap Theorem was sketched in the introduction, and various supporting details were scattered throughout the paper. We collect the story here for convenient reference.

Given the Slice Theorem, the Gap Theorem is a consequence of the following simple computation.

Proposition 9.1. *Suppose that $G = C_{2^n}$ is a non-trivial group, and $m \geq 0$. Then*

$$H_G^i(S^{m\rho_G}; \underline{\mathbb{Z}}_{(2)}) = 0 \qquad \text{for } 0 < i < 4.$$

Proof: First suppose that $G = C_2$ and $m = 1$. As described in §2.6, the complex for computing $H_G^*(S^{\rho_2}; \underline{\mathbb{Z}}_{(2)})$ is

$$\mathbb{Z}_{(2)} \xrightarrow{1} \mathbb{Z}_{(2)}$$

and so in fact $H_G^*(S^{\rho_2}; \underline{\mathbb{Z}}_{(2)}) = 0$. For $G = C_{2^n}$ with $n > 1$ the complex starts out

$$\mathbb{Z}_{(2)} \xrightarrow{1} \mathbb{Z}_{(2)} \xrightarrow{0} \mathbb{Z}_{(2)} \xrightarrow{2} \mathbb{Z}_{(2)} \longrightarrow \cdots$$

$$1 \qquad\quad 2 \qquad\quad 3 \qquad\quad 4 \qquad \cdots$$

and again the claim holds. For $m = 2$ and any G, the complex is

$$\mathbb{Z}_{(2)} \xrightarrow{1} \mathbb{Z}_{(2)} \longrightarrow \cdots$$

$$2 \qquad\quad 3 \qquad \cdots$$

while for $m = 3$ and any G it is

$$\mathbb{Z}_{(2)} \xrightarrow{1} \mathbb{Z}_{(2)} \longrightarrow \cdots$$

$$3 \qquad\quad 4 \qquad \cdots$$

The claim follows. □

Lemma 9.2 (The Cell Lemma). *Let $G = C_{2^n}$ for some $n > 0$. If $\widehat{S}$ is an isotropic slice cell of even dimension, then the groups $\pi_k^G H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S}$ are zero for $-4 < k < 0$.*

Proof: Suppose that

$$\widehat{S} = G_+ \wedge_H S^{m\rho_H}$$

with $H \subset G$ non-trivial. By the Wirthmüller isomorphism

$$\pi_k^G H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S} \approx \pi_k^H H\underline{\mathbb{Z}}_{(2)} \wedge S^{m\rho_H},$$

so the assertion is reduced to the case $\widehat{S} = S^{m'\rho_G}$ with G non-trivial. If $m' \geq 0$ then $\pi_k^G H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S} = 0$ for $k < 0$. For the case $m' < 0$ write $i = -k$, $m = -m' > 0$, and

$$\pi_k^G H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S} = H_G^i(S^{m\rho_G}; \underline{\mathbb{Z}}_{(2)}).$$

The result then follows from Proposition 9.1. $\square$

Theorem 9.3. *If X is perfect and isotropic, then*

$$\pi_i^G X = 0 \qquad -4 < i < 0.$$

Proof: Immediate from the slice spectral sequence for X and the Cell Lemma. $\square$

Corollary 9.4. *If Y can be written as a directed homotopy colimit of isotropic perfect spectra, then*

$$\pi_i^G X = 0 \qquad -4 < i < 0.$$

$\square$

Theorem 9.5 (The Gap Theorem)**.** *Let $G = C_{2^n}$ with $n > 0$ and $D \in \pi_{\ell\rho_G} MU^{((G))}$ be any class. Then for $-4 < i < 0$*

$$\pi_i^G D^{-1} MU^{((G))} = 0.$$

Proof: The spectrum $D^{-1} MU^{((G))}$ is the homotopy colimit

$$\varinjlim_j \Sigma^{-j\,\ell\rho_G} MU^{((G))}.$$

By the Slice Theorem, $MU^{((G))}$ is perfect and isotropic. But then the spectrum

$$\Sigma^{-j\,\ell\rho_G} MU^{((G))}$$

is also perfect and isotropic, since for any X

$$P_m^m \Sigma^{\rho_G} X \approx \Sigma^{\rho_G} P_{m-g}^{m-g} X$$

by Corollary 3.27. The result then follows from Corollary 9.4. $\square$

10. The Periodicity Theorem

In this section we will describe a general method for producing periodicity results for spectra obtained from $MU^{((G))}$ by inverting suitable elements of $\pi_\star^G MU^{((G))}$. The Periodicity Theorem (Theorem 10.15) used in the proof of Theorem 1.1 is a special case. The proof relies on a small amount of computation of $\pi_\star^G MU^{((G))}$.
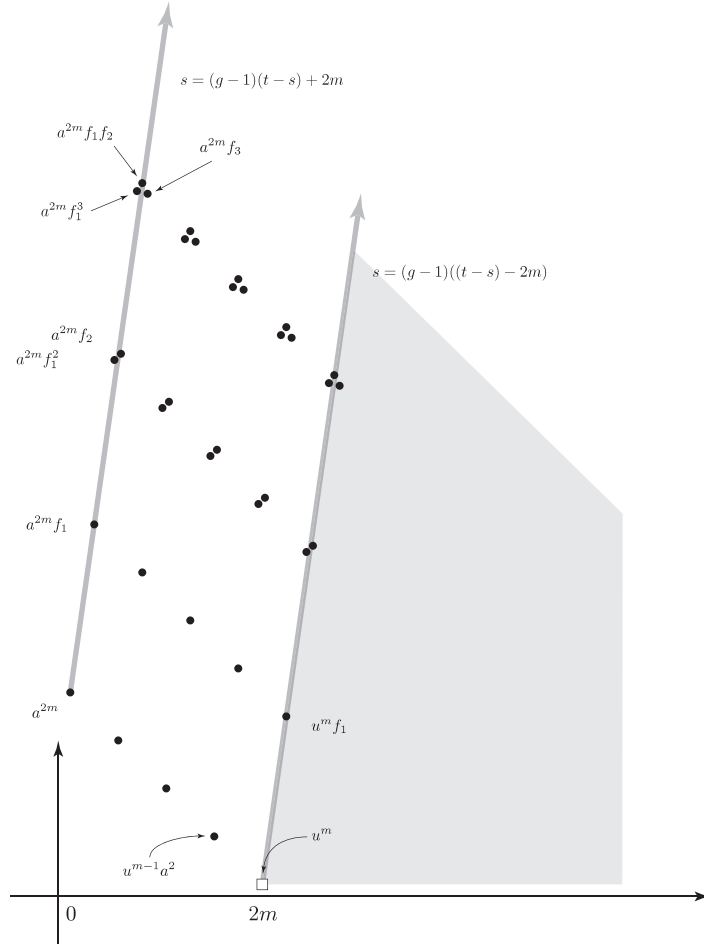
FIGURE 3. The slice spectral sequence for $\Sigma^{2m\sigma} MU^{((G))}$

10.1. **The $RO(G)$-graded slice spectral sequence for $MU^{((G))}$.** Let $\sigma = \sigma_G$ be the real sign representation of G. The key issues surrounding the periodicity results reduce to questions about the the $RO(G)$-graded homotopy groups

$$\pi^G_{p+q\sigma} MU^{((G))} \quad p, q \in \mathbb{Z}.$$

We study these groups using the $RO(G)$-graded slice spectral sequence, with the conventions described in §3.5.

Certain elements play an important role. The classes

$$f_i \in \pi^G_i MU^{((G))}$$

described in Definition 4.52 are represented at the E_2-term of the slice spectral sequence by elements we will also call

$$f_i \in E_2^{i(g-1),ig} = \pi^G_i P^{ig}_{ig} MU^{((G))}$$
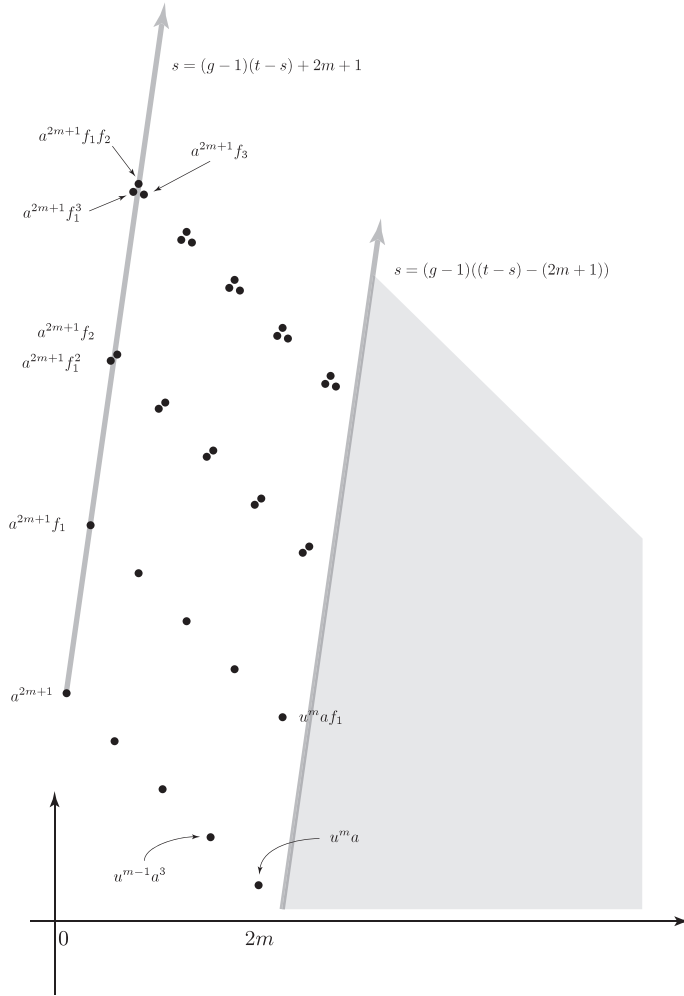
FIGURE 4. The slice spectral sequence for $\Sigma^{(2m+1)\sigma} MU^{((G))}$

given by

$$S^i \xrightarrow{a_\rho^i} S^{i\rho_G} \xrightarrow{N\bar{r}_i} P_{ig}^{ig} MU^{((G))}.$$

By construction, the classes f_i are permanent cycles.

We also need the the Hurewicz image of the classes a_σ defined by (2.37)

(10.1) $$S^0 \xrightarrow{a_\sigma} S^\sigma \to S^\sigma \wedge MU^{((G))}.$$

We won't distinguish in notation between the classes a_σ and the composite (10.1). The class a_σ is represented in the E_2-term of the slice spectral sequence by

$$S^0 \to S^\sigma \wedge P_0^0 MU^{((G))} = S^\sigma \wedge H\underline{\mathbb{Z}}_{(2)}.$$

We denote this representing element

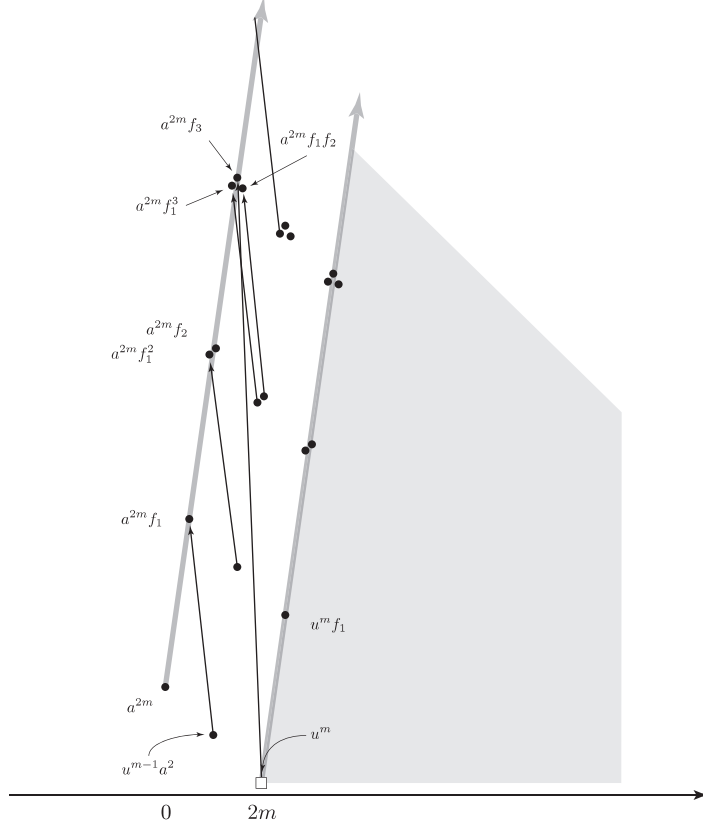$$a \in E_2^{1,1-\sigma} = \pi_{-\sigma}^G P_0^0 MU^{((G))}.$$

FIGURE 5. Differentials on u^m

With our conventions it is displayed on the $(t-s, s)$-plane in position $s = 1$, $t-s = 0$ (see Figure 3).

Finally there is the class

$$u = u_{2\sigma} \in E_2^{0,2\sigma-2} = \pi_{2-2\sigma}^G H\underline{\mathbb{Z}}_{(2)} = \pi_{2-2\sigma}^G P_0^0 MU^{((G))},$$

defined by (2.38).

Taking products defines a map

$$(10.2) \qquad\qquad \mathbb{Z}_{(2)}[a, f_i, u]/(2a, 2f_i) \to \bigoplus_{s,t,k \geq 0} E_2^{s,t-k\sigma}.$$

Proposition 10.3. *The map* (10.2) *is an isomorphism for*

$$s \geq (g-1)((t-s) - k).$$

Proof: The proof makes use of Lemmas 10.4, 10.5, 10.6, and 10.7 below. By Lemma 10.4 the only contributions to the E_2-term in the region $s \geq (g-1)(t-k-s)$ come from the summands $H\underline{\mathbb{Z}}_{(2)} \wedge S^{m\rho_G}$ occurring in the even slices of $MU^{((G))}$. These are the summands indexed by the monomials in the r_i invariant up to sign

under the action of G, or in other words, the ones which refine to monomials in the elements $N\bar{r}_i$. The result is then straightforward application of Lemmas 10.5, 10.6 and 10.7. $\qquad\square$

We have used

Lemma 10.4. *Suppose that $\widehat{S}$ is an even dimensional slice cell of dimension $d \geq 0$. Then for $k \geq 0$,*

$$\pi_t^G S^{k\sigma} \wedge \underline{\mathbb{Z}}_{(2)} \wedge \widehat{S} = 0 \text{ for } \begin{cases} t < \frac{d}{g} & \text{always} \\ t < \frac{2d}{g} + k & \text{if } \widehat{S} \text{ is induced.} \end{cases}$$

Proof: The first assertion follows from part iii) of Proposition 3.36. For the second, suppose that $\widehat{S} = G_+ \underset{H}{\wedge} \widehat{S}'$, with $\widehat{S}'$ an even dimensional slice cell of dimension d for a proper subgroup $H \subset G$. Since H is proper, the restriction of σ to H is trivial, and so

$$\pi_t^G S^{k\sigma} \wedge \underline{\mathbb{Z}}_{(2)} \wedge \widehat{S} \approx \pi_t^H S^k \wedge \underline{\mathbb{Z}}_{(2)} \wedge \widehat{S}' \approx \pi_{t-k}^H \underline{\mathbb{Z}}_{(2)} \wedge \widehat{S}'.$$

Using this, the second assertion then follows from the first, applied to the group H. $\qquad\square$

Lemma 10.5. *The map "multiplication by $a_{\bar{\rho}_G}^k$" ($\bar{\rho}_G = \rho_G - 1$) is an isomorphism*

$$\pi_t^G \underline{\mathbb{Z}}_{(2)} \wedge S^{m\sigma+k} \to \pi_t^G \underline{\mathbb{Z}}_{(2)} \wedge S^{m\sigma+k\rho_G}$$

for $t < m + k$ and an epimorphism for $t = m + k$.

Proof: This is immediate from the fact that $S^{m\sigma+k\rho_G}$ is constructed from $S^{m\sigma+k}$ by attaching equivariant cells $G_+ \underset{H}{\wedge} D^p$ with $p > m + k$ $\qquad\square$

The next two results are straightforward computations, using the standard equivariant cell decomposition of $S^{d\sigma}$, and the technique described in §2.6.

Lemma 10.6. *Suppose that $\ell \geq 0$. Then*

$$\pi_t^G(H\underline{\mathbb{Z}}_{(2)} \wedge S^{2\ell\sigma}) = 0$$

unless $t = 2j$, with $0 \leq j \leq \ell$. One has

$$\pi_{2\ell}^G(\underline{\mathbb{Z}}_{(2)} \wedge S^{2\ell\sigma}) = \mathbb{Z}_{(2)}$$

generated by $u_{2\ell\sigma} = u_{2\sigma}^\ell$, and for $0 \leq j < \ell$,

$$\pi_{2j}^G(H\underline{\mathbb{Z}}_{(2)} \wedge S^{2\ell\sigma}) = \mathbb{Z}/2$$

generated by $a_\sigma^{2\ell-2j} u_{2\sigma}^j$. $\qquad\square$

Lemma 10.7. *Suppose that $\ell \geq 0$. Then*

$$\pi_t^G(H\underline{\mathbb{Z}}_{(2)} \wedge S^{(2\ell+1)\sigma}) = 0$$

unless $t = 2j$ with $0 \leq j \leq \ell$. In that case

$$\pi_{2j}^G(H\underline{\mathbb{Z}}_{(2)} \wedge S^{2\ell\sigma}) = \mathbb{Z}/2$$

generated by $a_\sigma^{2\ell-2j+1} u_{2\sigma}^j$. $\qquad\square$

Remark 10.8. It follows from Proposition 10.3 that the groups $E_2^{s,t-k\sigma}$ are zero for $s > (g-1)(t-s)$, and that what lies on the "vanishing line"

$$s = (g-1)(t-s)$$

is the algebra

$$\mathbb{Z}_{(2)}[a, f_i]/(2a, 2f_i).$$

In Proposition 4.51 it was shown that the kernel of the map

$$\mathbb{Z}_{(2)}[a_\sigma, f_i]/(2a, 2f_i) \to \pi_\star^G MU^{((G))} \to \pi_\star^G \Phi^G MU^{((G))} = \pi_* MO[a_\sigma^{\pm 1}]$$

is the ideal $(2, f_1, f_3, f_7, \dots)$. The only possible non-trivial differentials into the vanishing line must therefore land in this ideal.

Having described the E_2-term in the range of interest, we now turn to some differentials. The case $G = C_2$ of the following result appears in unpublished work of Araki and in Hu-Kriz [16].

Theorem 10.9 (Slice Differentials Theorem). *In the slice spectral sequence for* $\pi_\star^G MU^{((G))}$ *the differentials* $d_i u^{2^{k-1}}$ *are zero for* $i < r = 1 + (2^k - 1)g$, *and*

$$d_r u^{2^{k-1}} = a^{2^k} f_{2^k - 1}.$$

Proof: The simplest way to check the index r is the correct one for the asserted differential is to recall that a has s-filtration 1, and that f_i has s-filtration $i(g-1)$. The differential goes from s-filtration 0 to s-filtration

$$2^k + (2^k - 1)(g - 1) = (2^k - 1)g + 1.$$

We'll establish the differential by induction on k, and refer the reader to Figure 3. Assume the result for $k' < k$. Then what's left in the range $s \geq (g-1)(t-s-k)$ after the differentials assumed by induction is the sum of two modules over $\mathbb{Z}_{(2)}[f_i]/(2f_i)$. One is generated by a^{2^k} and is free over the quotient ring

$$\mathbb{Z}/2[f_i]/(f_1, f_3, \dots, f_{2^{k-1}-1}).$$

The other is generated by $u^{2^{k-1}}$. Since the differential must take its value in the ideal $(2, a, f_1, f_3, \dots)$, the next (and only) possible differential on $u^{2^{k-1}}$ is the one asserted in the theorem. So all we need do is show that the classes $u^{2^{k-1}}$ do not survive the spectral sequence. For this it suffices to do so after inverting a. Consider the map

$$a_\sigma^{-1} \pi_\star^G MU^{((G))} \to a_\sigma^{-1} \pi_\star^G H\underline{\mathbb{Z}}_{(2)}.$$

We know the $\mathbb{Z}$-graded homotopy groups of both sides, since they can be identified with the homotopy groups of the geometric fixed point spectrum. If $u^{2^{k-1}}$ is a permanent cycle, then the class $a^{-2^k} u^{2^{k-1}}$ is as well, and represents a class with non-zero image in $\pi_*^G \Phi^G H\underline{\mathbb{Z}}_{(2)}$. This contradicts Proposition 7.3. □

Remark 10.10. After inverting a_σ, the differentials described in Theorem 10.9 describe completely the $RO(G)$-graded slice spectral sequence. The spectral sequence starts from

$$\mathbb{Z}/2[f_i, a_\sigma^{\pm 1}, u].$$

The class $u^{2^{k-1}}$ hits a unit multiple of $f_{2^k - 1}$, and so the E_∞-term is

$$\mathbb{Z}/2[f_i, i \neq 2^k - 1][a^{\pm 1}] = MO_*[a^{\pm 1}]$$

which we know to be the correct answer since $\Phi^G MU^{((G))} = MO$. This also shows that the class $u^{2^{k-1}}$ is a permanent cycle modulo $(\bar{r}_{2^k-1})$. This fact corresponds to the main computation in the proof of Theorem 4.55 (which, of course we used in the above proof). The logic can be reversed, and in [16] the results are established in the reverse order (for the group $G = C_2$).

Write
$$\bar{\mathfrak{d}}_k = N\bar{r}_{2^k-1} \in \pi^G_{(2^k-1)\rho_G} MU^{((G))},$$
and note that with this notation
$$f_{2^k-1} = a_{\bar{\rho}}^{2^k-1} \bar{\mathfrak{d}}_k.$$

Corollary 10.11. *In the $RO(G)$-graded slice spectral sequence for $\bar{\mathfrak{d}}_k^{-1} MU^{((G))}$, the class u^{2^k} is a permanent cycle.*

Proof: Because of the vanishing regions in the slice spectral sequence for
$$\pi^G_\star \bar{\mathfrak{d}}_k^{-1} MU^{((G))},$$
the differentials described in Theorem 10.9 are the last possible differentials on the u^{2^j}, even after inverting $\bar{\mathfrak{d}}_k$. The result thus follows from Theorem 10.9 if we can show that
$$a^{2^{k+1}} f_{2^{k+1}-1} = 0$$
before the $E_{r'}$-term of the slice spectral sequence, where $r' = 1 + (2^{k+1} - 1)g$. Unpacking the notation we find that
$$f_{2^{k+1}-1} \bar{\mathfrak{d}}_k = a_{\bar{\rho}}^{2^{k+1}-1} \bar{\mathfrak{d}}_{k+1} \bar{\mathfrak{d}}_k = f_{2^k-1} a_{\bar{\rho}}^{2^k} \bar{\mathfrak{d}}_{k+1}$$
and so
$$d_r u^{2^{k-1}} \cdot \epsilon = a^{2^{k+1}} f_{2^{k+1}-1},$$
where
$$r = 1 + (2^k - 1)g < r'$$
and
$$\epsilon = a^{2^k} a_{\bar{\rho}}^{2^k} \bar{\mathfrak{d}}_{k+1} \bar{\mathfrak{d}}_k^{-1}.$$
$\square$

10.2. **Periodicity theorems.** We now turn to our main periodicity theorem. As will be apparent to the reader, the technique can be used to get a much more general result. We have chosen to focus on a case which contains the result needed for the proof of Theorem 1.1, and yet can be stated for general $G = C_{2^n}$. In order to formulate it we need to consider all of the spectra $MU^{((H))}$ for $H \subseteq G$, and we'll need to distinguish some of the important elements of the homotopy groups we've specified. Let
$$\bar{r}_i^H \in \pi^{C_2}_{i\rho_2} MU^{((H))}$$
be the element defined in §4.3.2,
$$\bar{\mathfrak{d}}_k^H = N_{C_2}^H(\bar{r}_{2^k-1}^H) \in \pi^H_{(2^k-1)\rho_H} MU^{((H))},$$
and let Δ_k^H be the element of the E_2-term of the slice spectral sequence for $MU^{((H))}$ given by
$$\Delta_k^H = u_{2(2^k-1)\rho_H}(\bar{\mathfrak{d}}_k^H)^2.$$

We'll use (4.1) to map
$$\pi_\star^H MU^{((H))} \to \pi_\star^H MU^{((G))}.$$
Finally, in addition to $g = |G|$ we'll use $h = |H|$ for $H \subseteq G$.

Theorem 10.12. *Let $D \in \pi_{\ell\rho_G}^G MU^{((G))}$ be any class whose image in $\pi_\star^H MU^{((G))}$ is divisible by $\bar{\mathfrak{d}}_{g/h}^H$, for all $0 \neq H \subseteq G$. In the slice spectral sequence for $\pi_*^G D^{-1} MU^{((G))}$ the element $u_{2\rho_G}^{2^{g/2}}$ is a permanent cycle.*

Proof: For simplicity write
$$\sigma_h = \sigma_H$$
$$\rho_h = \rho_H$$
for the sign and regular representations of a subgroup H. By Corollary 10.11, for each nontrivial subgroup $H \subseteq G$, the class $u_{2\sigma_h}^{2^{g/h}}$ is a permanent cycle, and therefore so is the class $u_{2\sigma_h}^{2^{g/2}}$. Starting with
$$u_{2\sigma_2}^{2^{g/2}} = u_{2\rho_2}^{2^{g/2}},$$
norm up to C_4, multiply by $u_{4\sigma_4}^{2^{g/2}} = (u_{2\sigma_4}^{2^{g/2}})^2$ and use Lemma 2.39 to conclude that
$$u_{4\sigma_4}^{2^{g/2}} N u_{2\rho_2}^{2^{g/2}} = u_{2\rho_4}^{2^{g/2}}$$
is a permanent cycle. Norming and continuing we find that
$$u_{2\rho_G}^{2^{g/2}}$$
is a permanent cycle, as claimed. $\square$

Corollary 10.13. *In the situation of Theorem 10.12 the class*

(10.14)
$$(\Delta_1^G)^{2^{g/2}} = u_{2\rho_G}^{2^{g/2}} (\bar{\mathfrak{d}}_1^G)^{2 \cdot 2^{g/2}}$$

is a permanent cycle. Any class in $\pi_{2 \cdot g \cdot 2^{g/2}}^G D^{-1} MU^{((G))}$ represented by (10.14) restricts to a unit in $\pi_*^u D^{-1} MU^{((G))}$.

Proof: The fact that (10.14) is a permanent cycle is immediate from Theorem 10.12. Since the slice tower refines the Postnikov tower, the restriction of an element in the $RO(G)$-graded group $\pi_\star^G D^{-1} MU^{((G))}$ to $\pi_*^u D^{-1} MU^{((G))}$ is determined entirely by any representative at the E_2-term of the slice spectral sequence. Since $u_{2\rho_G}$ restricts to 1, the restriction of any representative of (10.14) is equal to the restriction of $(\bar{\mathfrak{d}}_1^G)^{2 \cdot 2^{g/2}}$ which is a unit since $\bar{\mathfrak{d}}_1^G$ divides D. $\square$

This gives

Theorem 10.15. *With the notation of Theorem 10.12, if M is any equivariant $D^{-1} MU^{((G))}$-module, then multiplication by $(\Delta_1^G)^{2^{g/2}}$ is a weak equivalence*
$$\Sigma^{2 \cdot g \cdot 2^{g/2}} i_0^* M \to i_0^* M$$
and hence an isomorphism
$$(\Delta_1^G)^{2^{g/2}} : \pi_* M^{hG} \to \pi_{*+2 \cdot g \cdot 2^{g/2}}^G M^{hG}.$$

$\square$

For example, in the case of $G = C_2$ the groups $\pi_*(MU^{((G))})^{hG}$ are periodic with period $2 * 2 * 2 = 8$ and for $G = C_4$ there is a periodicity of $2 * 4 * 2^2 = 32$. For $G = C_8$ we have a period of $2 * 8 * 2^4 = 256$.

Corollary 10.16 (The Periodicity Theorem). *Let $G = C_8$, and*

$$D = (N_{C_2}^{C_8} \bar{\mathfrak{d}}_4^{C_2}) \, (N_{C_4}^{C_8} \bar{\mathfrak{d}}_2^{C_4}) \, (\bar{\mathfrak{d}}_1^{C_8}) \in \pi_{19\rho_G}^G MU^{((G))}.$$

Then multiplication by $(\Delta_1^G)^{16}$ gives an isomorphism

$$\pi_*(D^{-1}MU^{((G))})^{hG} \to \pi_{*+256}(D^{-1}MU^{((G))})^{hG}.$$

$\square$

11. The Homotopy Fixed Point Theorem

We study a situation similar to the one in §10.2, and fix a class

$$D \in \pi_{\ell\rho_G}^G MU^{((G))}$$

with the property that for all $0 \neq H \subseteq G$ the restriction of D to $\pi_*^H MU^{((G))}$ is divisible by $\bar{\mathfrak{d}}_k^H$ for some k which may depend on H.

Definition 11.1. A G-spectrum X is *cofree* if the map

$$X \to F(EG_+, X)$$

is a weak equivalence.

Remark 11.2. If X is cofree then the map

$$\pi_*^G X \to \pi_*^G F(EG_+, X) = \pi_* X^{hG}$$

is an isomorphism.

Theorem 11.3 (Homotopy Fixed Point Theorem). *If M is a module over $D^{-1}MU^{((G))}$ then M is cofree, and so*

$$\pi_*^G M \to \pi_* M^{hG}$$

is an isomorphism.

Corollary 11.4. *In the situation of Corollary 10.16, the map "multiplication by Δ_1^G" gives an isomorphism*

$$\pi_*^G(D^{-1}MU^{((G))}) \to \pi_{*+256}^G(D^{-1}MU^{((G))}).$$

Proof: In the diagram

$$
\begin{array}{ccc}
\pi_*^G(D^{-1}MU^{((G))}) & \longrightarrow & \pi_{*+256}^G(D^{-1}MU^{((G))}) \\
\downarrow & & \downarrow \\
\pi_*(D^{-1}MU^{((G))})^{hG} & \longrightarrow & \pi_{*+256}^G(D^{-1}MU^{((G))})^{hG}
\end{array}
$$

the vertical maps are isomorphism by Theorem 11.3, and the bottom horizontal map is an isomorphism by Corollary 10.16. $\square$

The proof of Theorem 11.3 involves studying the geometric fixed point spectra by the subgroups of G. We remind the reader of the comments made in Remarks 2.30, that we use $\Phi^H X$ in favor of the more correct $\Phi^H i_H^* X$, and that for any group G, a map $X \to Y$ induces a weak equivalence $\tilde{E}C_2 \wedge X \to \tilde{E}C_2 \wedge Y$ if and only if it induces a weak equivalence $\Phi^G X \to \Phi^G Y$.

Lemma 11.5. *For every subgroup $H \subset G$, the geometric fixed point spectrum*

$$\Phi^H D^{-1} MU^{((G))}$$

is contractible.

Proof: The class

$$\Phi^H \bar{\mathfrak{d}}_k^H = \Phi^{C_2} \bar{r}_{2^k-1}^H$$

is zero by Proposition 4.51, and hence so is $\Phi^H D$ since D is divisible by $\bar{\mathfrak{d}}_k^H$. The claim follows. $\square$

Proposition 11.6. *For any module M over $D^{-1} MU^{((G))}$ the spectrum $\tilde{E}G \wedge M$ is contractible.*

Proof: Since $\tilde{E}G \wedge M$ is a module over $\tilde{E}G \wedge D^{-1} MU^{((G))}$ it suffices to show that $\tilde{E}G \wedge D^{-1} MU^{((G))}$ is contractible. We'll prove by induction on $H \subset G$ that $i_H^* \tilde{E}G \wedge D^{-1} MU^{((G))}$ is contractible. The statement is trivial when H is the trivial group, since $i_0^* \tilde{E}G$ is contractible. Smashing $i_H^* \tilde{E}G \wedge D^{-1} MU^{((G))}$ with the isotropy separation sequence for H reduces the induction step to the assertion that

$$\Phi^H \tilde{E}G \wedge D^{-1} MU^{((G))}$$

is contractible. Using the fact that $\Phi^H \tilde{E}G \approx S^0$ one finds

$$\Phi^H \tilde{E}G \wedge D^{-1} MU^{((G))} = \Phi^H \tilde{E}G \wedge \Phi^H D^{-1} MU^{((G))}$$
$$= \Phi^H D^{-1} MU^{((G))},$$

which is contractible by Lemma 11.5. $\square$

Proof of Theorem 11.3: Since EG is non-equivariantly contractible, the map $i_0^* M \to i_0^* F(EG_+, M)$ is a weak equivalence. It follows that $G_+ \wedge M \to G_+ \wedge F(EG_+, M)$ is a weak equivalence and hence so is $EG_+ \wedge M \to EG_+ \wedge F(EG_+, M)$, since EG is built from free G-cells. Smashing $M \to F(EG_+, M)$ with the cofibration sequence

$$EG \to S^0 \to \tilde{E}G$$

gives the diagram

$$
\begin{array}{ccccc}
EG_+ \wedge M & \longrightarrow & M & \longrightarrow & \tilde{E}G \wedge M \\
\downarrow & & \downarrow & & \downarrow \\
EG_+ \wedge F(EG_+, M) & \longrightarrow & F(EG_+, M) & \longrightarrow & \tilde{E}G \wedge F(EG_+, M).
\end{array}
$$

The left column is an equivalence as we've just noted. The spectra on the right are equivariantly contractible, by Proposition 11.6. It follows that the middle arrow is an equivalence. $\square$

12. The Detection Theorem

12.1. θ_j in the Adams-Novikov spectral sequence. Browder's theorem says that θ_j is detected in the classical Adams spectral sequence by

$$h_j^2 \in \mathrm{Ext}_A^{2,2^{j+1}}(\mathbb{Z}/2, \mathbb{Z}/2).$$

This element is known to be the only one in its bidegree.

It is more convenient for us to work with the Adams-Novikov spectral sequence, which maps to the Adams spectral sequence. It has a family of elements in filtration 2, namely

$$\beta_{i/j} \in \mathrm{Ext}_{MU_*(MU)}^{2,6i-2j}(MU_*, MU_*)$$

for certain values of of i and j. When $j = 1$, it is customary to omit it from the notation. The definition of these elements can be found in [27, Chapter 5].

Here are the first few of these in the relevant bidegrees.

$$
\begin{aligned}
\text{bidegree of } \theta_2 : & \quad \beta_{2/2} \\
\text{bidegree of } \theta_3 : & \quad \beta_{4/4} \text{ and } \beta_3 \\
\text{bidegree of } \theta_4 : & \quad \beta_{8/8} \text{ and } \beta_{6/2} \\
\text{bidegree of } \theta_5 : & \quad \beta_{16/16}, \beta_{12/4} \text{ and } \beta_{11}
\end{aligned}
$$

and so on. In the bidegree of θ_j, only $\beta_{2^{j-1}/2^{j-1}}$ has a nontrivial image (namely h_j^2) in the Adams spectral sequence. There is an additional element in this bidegree, namely $\alpha_1 \alpha_{2^j-1}$. According to [27, Corollary 5.4.5], [30], a basis for

$$\mathrm{Ext}_{MU_*MU}^{2,2^{j+2}}(MU_*, MU_*)$$

for $j > 0$ is given by

$$(12.1) \qquad \left\{ \alpha_1 \alpha_{2^j-1} \right\} \cup \left\{ \beta_{c(j,k)/2^{j-1-2k}} : 0 \le k < j/2 \right\},$$

where $c(j,k) = 2^{j-1-2k}(1+2^{2k+1})/3$. For $j = 1$, the second set is empty. For $j > 1$, $\alpha_1 \alpha_{2^j-1}$ supports a nontrivial d_3 for $j > 1$. None of these elements is divisible by 2. The element $\beta_{c(j,k)/2^{j-1-2k}}$ is represented by the chromatic fraction

$$\frac{v_2^{c(j,k)}}{2v_1^{2^{j-1-2k}}};$$

no correction terms are needed.

We need to show that any element mapping to h_j^2 in the classical Adams spectral sequence has nontrivial image the Adams-Novikov spectral sequence for $M = (D^{-1}MU^{((C_8))})^{hC_8}$, the spectrum in the Periodicity Theorem (Corollary 10.16).

Theorem 12.2 (The Detection Theorem). *Let*

$$x \in \mathrm{Ext}_{MU_*(MU)}^{2,2^{j+1}}(MU_*, MU_*)$$

be any element whose image in $\mathrm{Ext}_A^{2,2^{j+1}}(\mathbb{Z}/2, \mathbb{Z}/2)$ is h_j^2 with $j \ge 6$. (Here A denotes the mod 2 Steenrod algebra.) Then the image of x in $H^{2,2^{j+1}}(C_8; \pi_* D^{-1}MU^{((C_8))})$ is nonzero.

We will prove this by showing the same is true after we map the latter to a simpler object involving another algebraic tool, *the theory of formal A-modules*, where A is the ring of integers in a suitable field.

12.2. **Formal A-modules.** Recall the a formal group law over a ring R is a power series

$$F(x, y) = x + y + \sum_{i,j>0} a_{i,j} x^i y^j \in R[[x, y]]$$

with certain properties.

For positive integers m one has power series $[m](x) \in R[[x]]$ defined recursively by $[1](x) = x$ and

$$[m](x) = F(x, [m-1](x)).$$

These satisfy

$$[m + n](x) = F([m](x), [n](x)) \text{ and } [m]([n](x)) = [mn](x).$$

With these properties we can define $[m](x)$ uniquely for all integers m, and we get a homomorphism τ from $\mathbb{Z}$ to $End(F)$, the endomorphism ring of F.

If the ground ring R is an algebra over the p-local integers $\mathbb{Z}_{(p)}$ or the p-adic integers $\mathbb{Z}_p$, then we can make sense of $[m](x)$ for m in $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$.

Now suppose R is an algebra over a larger ring A, such as the ring of integers in a number field or a finite extension of the p-adic numbers. We say that the formal group law F is a *formal A-module* if the homomorphism τ extends to A in such a way that

$$[a](x) \equiv ax \bmod (x^2) \text{ for } a \in A.$$

The theory of formal A-modules is well developed. Lubin-Tate [20] used them to do local class field theory, and a good reference for the theory is Hazewinkel's book [12, Chapter 21].

The example of interest to us is $A = \mathbb{Z}_2[\zeta_8]$, where ζ_8 is a primitive 8th root of unity. The maximal ideal of A is generated by $\pi = \zeta_8 - 1$, and π^4 is a unit multiple of 2. There is a formal A-module F over $R_* = A[w^{\pm 1}]$ (with $|w| = 2$) satisfying

$$\log_F(F(x, y)) = \log_F(x) + \log_F(y)$$

where

(12.3)
$$\log_F(x) = x + \sum_{k>0} \frac{w^{2^k - 1} x^{2^k}}{\pi^k}.$$

12.3. $\pi_* MU^{(4)}$ **and** R_*. What does all this have to do with our spectrum $\widetilde{\Omega} = D^{-1} MU^{(4)}$? Recall that

$$\bar{\mathfrak{d}}_k^H \quad = \quad N_2^h \bar{r}_{2^k-1}^H \in \pi_{(2^k-1)\rho_H}^H MU^{((H))} \qquad \text{for } h = |H|$$

and

$$\begin{aligned}
D \quad &= \quad N_2^8 \left(\bar{\mathfrak{d}}_4^{C_{2n}} \right) N_4^8 \left(\bar{\mathfrak{d}}_2^{C_4} \right) \bar{\mathfrak{d}}_1^{C_8} \\
&= \quad N_2^8 \left(\bar{r}_{15}^{C_2} \right) N_4^8 \left(N_2^4 \bar{r}_3^{C_4} \right) N_2^8 \left(\bar{r}_1^{C_8} \right) \\
&= \quad N_2^8 \left(\bar{r}_{15}^{C_2} \bar{r}_3^{C_4} \bar{r}_1^{C_8} \right) \\
&\in \quad \pi_{19\rho_G}^G MU^{((C_8))}.
\end{aligned}$$

We saw earlier that inverting a product of this sort is needed to get the Periodicity Theorem, but we did not explain the choice of subscripts of $\bar{\mathfrak{d}}$. They are the smallest ones that satisfy the second part of the following.

Lemma 12.4. *The classifying homomorphism* $\lambda : \pi_* MU \to R_*$ *for* F *factors through* $\pi_* MU^{(4)}$ *in such a way that*

   (i) *the homomorphism* $\lambda^{(4)} : \pi_* MU^{(4)} \to R_*$ *is equivariant, where* C_8 *acts on* $\pi_* MU^{(4)}$ *as before, it acts trivially on* A *and* $\gamma w = \zeta_8 w$ *for a generator* γ *of* C_8.

   (ii) *The element* $i_0^* D \in \pi_* MU^{(4)}$ *that we invert to get* $i_0^* \widetilde{\Omega}$ *goes to a unit in* R_*.

We will prove this later.

12.4. **The proof of the Detection Theorem.** It follows from Lemma 12.4 that we have a map

$$H^*(C_8; \pi_*(i_0^* D)^{-1} MU^{(4)}) \to H^*(C_8; R_*).$$

The source here is the E_2-term of the homotopy fixed point spectral sequence for M, and the target is easy to calculate. We will use it to prove the Detection Theorem above by showing that the image of x in $H^{2,2^{j+1}}(C_8; R_*)$ is nonzero.

We will calculate with BP-theory. Recall that

$$BP_*(BP) = BP_*[t_1, t_2, \dots] \quad \text{where } |t_n| = 2(2^n - 1).$$

We will abbreviate $\text{Ext}^{s,t}_{BP_*)(BP)} (BP_*, BP_*)$ by $\text{Ext}^{s,t}$.

The Hopf algebroid associated with $H^*(C_8; R_*)$ has the form $(R_*, R_*(C_8))$, where $R_*(C_8)$ denotes the ring of R_*-valued functions on C_8. Its left unit sends R_* to the set of constant functions, and the right unit is determined by the group action on R_* via the formula

$$\eta_R(r)(\gamma) = \gamma(r) \qquad \text{for } r \in R_* \text{ and } \gamma \in C_8.$$

This map is A-linear and C_8 has a generator γ for which $\eta_R(w)(\gamma^k) = \zeta_8^k w$.

The coproduct

$$\Delta : R_*(C_8) \to R_*(C_8) \otimes_{R_*} R_*(C_8)$$

sends a function f to the function $\Delta(f)$ on two variables defined by

$$\Delta(f)(\gamma_1, \gamma_2) = f(\gamma_1 \gamma_2).$$

There is a map to this Hopf algebroid from $BP_*(BP)$ in which t_n maps to an R_*-valued function on C_8 (regarded as the group of 8th roots of unity) determined by

$$[\zeta](x) = \sum_{i \geq 0}^{F} t_i(\zeta)(\zeta x)^{2^i}$$

for each root ζ, where t_0 is the constant function with value 1. (For an analogous description of the t_i as functions on the full Morava stabilizer group, see the last two paragraphs of the proof of [27, Theorem 6.2.3].) Taking the log of each side we get

$$\zeta \sum_{n \geq 0} \frac{w^{2^n - 1} x^{2^n}}{\pi^n} = \sum_{i,j \geq 0} \frac{w^{2^i - 1}}{\pi^i} t_i(\zeta)^{2^i} (\zeta x)^{2^{i+j}}$$

$$\zeta \frac{w^{2^n - 1}}{\pi^n} = \zeta^{2^n} \sum_{0 \leq i \leq n} \frac{w^{2^i - 1}}{\pi^i} t_{n-i}(\zeta)^{2^i}$$

For $n = 0$ this gives $t_0(\zeta) = 1$ as expected. For $n = 1$ we get

$$\frac{\zeta w}{\pi} = \zeta^2 \left(t_1(\zeta) + \frac{w}{\pi} \right)$$

(12.5)
$$t_1(\zeta) = \frac{w(1 - \zeta)}{\pi \zeta} \in R_*$$

This a unit when ζ is a primitive eighth root of unity.

Lemma 12.6. *Let*

$$b_{1,j-1} = \frac{1}{2} \sum_{0 < i < 2^j} \binom{2^j}{i} \left[t_1^i | t_1^{2^j - i} \right] \in \mathrm{Ext}^{2, 2^{j+1}}$$

Its image in $H^{2, 2^{j+1}}(C_8; R_*)$ is nontrivial for $j \geq 2$.

This element is known to be cohomologous to $\beta_{2^{j-1}/2^{j-1}}$ and to have order 2; see [27, Theorem 5.4.6(a)].

Proof. Let $\gamma \in C_8$ be the generator with $\gamma(w) = \zeta_8 w$. Then $H^*(C_8; R_*)$ is the cohomology of the cochain complex of $R_*[C_8]$-modules

(12.7)
$$R_* \xrightarrow{\ \gamma - 1\ } R_* \xrightarrow{\ \mathrm{Trace}\ } R_* \xrightarrow{\ \gamma - 1\ } \cdots$$

where Trace is multiplication by $1 + \gamma + \cdots + \gamma^7$.

The cohomology groups $H^s(C_8; R_*)$ for $s > 0$ are periodic in s with period 2. We have

$$H^0(C_8; R_{2m}) = \ker(\zeta_8^m - 1)$$
$$= \begin{cases} A & \text{for } m \equiv 0 \bmod 8 \\ 0 & \text{otherwise} \end{cases}$$

$$H^1(C_8; R_{2m}) = \ker(1 + \zeta_8^m + \cdots + \zeta_8^{7m})/\mathrm{im}(\zeta_8^m - 1)$$
$$= \begin{cases} w^m A/(\pi) & \text{for } m \text{ odd} \\ w^m A/(\pi^2) & \text{for } m \equiv 2 \bmod 4 \\ w^m A/(2) & \text{for } m \equiv 4 \bmod 8 \\ 0 & \text{for } m \equiv 0 \bmod 8 \end{cases}$$

$$H^2(C_8; R_{2m}) = \ker(\zeta_8^m - 1)/\mathrm{im}(1 + \zeta_8^m + \cdots + \zeta_8^{7m})$$
$$= \begin{cases} w^m A/(8) & \text{for } m \equiv 0 \bmod 8 \\ 0 & \text{otherwise} \end{cases}$$

Note that the A-modules occurring above are

$$A/(\pi) \cong \mathbb{Z}/2$$
$$A/(\pi^2) \cong \mathbb{Z}/2[\pi]/(\pi^2)$$
$$A/(2) \cong \mathbb{Z}/2[\pi]/(\pi^4)$$
$$A/(8) \cong \mathbb{Z}/8[\pi]/(\pi^4 - 2)$$

We also have a map

$$\mathrm{Ext}^{*,*}_{MU_* MU}(MU_*, MU_*/2) \to H^{*,*}(C_8; R_*/(2))$$

Reducing the complex of (12.7) modulo (2) makes the trace map trivial, so for $t \geq 0$ we have

$$H^{2t}(C_8; R_{2m}/2) = \begin{cases} \pi^3 A/(2) & \text{for } m \text{ odd} \\ \pi^2 A/(2) & \text{for } m \equiv 2 \bmod 4 \\ A/(2) & \text{for } m \equiv 0 \bmod 4 \end{cases}$$

$$H^{2t+1}(C_8; R_{2m}/2) = \begin{cases} A/(\pi) & \text{for } m \text{ odd} \\ A/(\pi^2) & \text{for } m \equiv 2 \bmod 4 \\ A/(2) & \text{for } m \equiv 0 \bmod 4 \end{cases}$$

For $j \geq 0$, the image of the class $[t_1^{2^j}] \in \mathrm{Ext}^{1,2^{j+1}}_{MU_*MU}(MU_*, MU_*/2)$ in $H^1(C_8; R_{2^{j+1}}/(2))$ is a unit in $A/(2) = \mathbb{Z}/2[\pi]/(\pi^4)$ since the function t_1 is not divisible by π by (12.5). For $j \geq 2$, consider the following diagram in which we abbreviate $\mathrm{Ext}^{s,t}_{MU_*MU}(MU_*, M)$ by $H^{s,t}(M)$.

$$
\begin{array}{ccccc}
H^{1,2^{j+1}}(MU_*) & \longrightarrow & H^{1,2^{j+1}}(MU_*/(2)) & \overset{\delta}{\longrightarrow} & H^{2,2^{j+1}}(MU_*) \\
\downarrow{\lambda} & & \downarrow{\lambda} & & \downarrow{\lambda} \\
H^1(C_8; R_{2^j}) & \longrightarrow & H^1(C_8; R_{2^j}/(2)) & \overset{\delta'}{\longrightarrow} & H^2(C_8; R_{2^j}) \\
\| & & \| & & \| \\
0 & \longrightarrow & A/(2) & \longrightarrow & A/(8)
\end{array}
$$

Here δ and δ' are the evident connecting homomorphisms, $\lambda : MU_* \to R_*$ is the classifying map for our formal A-module and the rows are exact. $\delta([t_1^{2^j}]) = b_{1,j-1}$, and $\lambda([t_1^{2^j}])$ is a unit, so $\lambda(b_{1,j-1})$ has the desired property. $\qquad\square$

To finish the proof of the Detection Theorem we need to show that the other βs in the same bidegree map to zero. We will do this for $j \geq 6$. The appropriate Ext group was described in (12.1). Note that $\beta_{c(j,0)/2^{j-1}} = \beta_{2^{j-1}/2^{j-1}}$, so we need to show that the elements with $k > 0$ map to zero.

We will see in the proof of Lemma 12.4 below that v_1 and v_2 map to unit multiples (with the units in A) of $\pi^3 w$ and $\pi^2 w^3$ respectively. This means we can define a valuation $\| \cdot \|$ on BP_* compatible with the one on A in which $\|2\| = 1$, $\|\pi\| = 1/4$, $\|v_1\| = 3/4$ and $\|v_2\| = 1/2$. We extend the valuation on A to R_* by setting $\|w\| = 0$. We will show that the valuation of the relevant chromatic fractions is ≥ 3. This valuation is a lower bound on the one in $H^*(C_8, R_*)$, where every group has exponent at most 8. Hence a valuation ≥ 3 means the β-element has trivial image.

Hence for $k \geq 1$ and $j \geq 6$ we have

$$
\begin{aligned}
||\beta_{c(j,k)/2^{j-1-2k}}|| &\geq \left|\left|\frac{v_2^{c(j,k)}}{2v_1^{2^{j-1-2k}}}\right|\right| \\
&= \frac{c(j,k)}{2} - \frac{3 \cdot 2^{j-1-2k}}{4} - 1 \\
&= \frac{2^j + 2^{j-1-2k}}{6} - \frac{3 \cdot 2^{j-1-2k}}{4} - 1 \\
&= (2^{j-1} - 7 \cdot 2^{j-3-2k})/3 - 1 \\
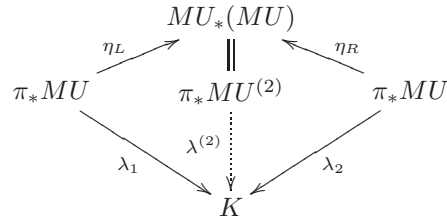&\geq 5.
\end{aligned}
$$

This means $\beta_{c(j,k)/2^{j-1-2k}}$ maps to an element that is divisible by 8 and therefore zero.

We have to make a similar computation with the element $\alpha_1\alpha_{2^j-1}$. We have

$$
\begin{aligned}
||\alpha_{2^j-1}|| &\geq \left|\left|\frac{v_1^{2^j-1}}{2}\right|\right| \\
&= \frac{3(2^j-1)}{4} - 1 \\
&\geq \frac{21}{4} - 1 > 4 \quad \text{for } j \geq 3.
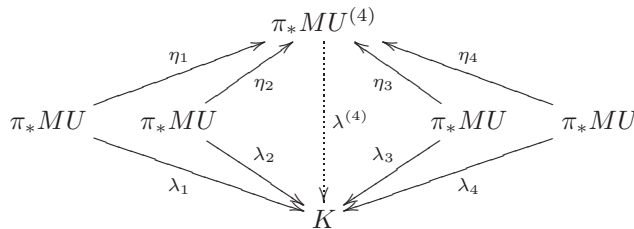\end{aligned}
$$

This completes the proof of the Detection Theorem modulo Lemma 12.4.

12.5. **The proof of Lemma 12.4.** To prove the first part, consider the following diagram for an arbitrary ring K.

The maps λ_1 and λ_2 classify two formal group laws F_1 and F_2 over K. The Hopf algebroid $MU_*(MU)$ represents strict isomorphisms between formal group laws. Hence the existence of $\lambda^{(2)}$ is equivalent to that of a strict isomorphism between F_1 and F_2 compatible with the maps λ_1 and λ_2.

Similarly consider the diagram

where the homomorphisms η_j are unit maps corresponding to the four smash product factors of $MU^{(4)}$. The existence of $\lambda^{(4)}$ is equivalent to that of compatible strict isomorphisms between the formal group laws F_j classified by the λ_j.

Now suppose that K has a C_8-action and that $\lambda^{(4)}$ is equivariant with respect to the previously defined C_8-action on $MU^{(4)}$. Then the isomorphism induced by the fourth power of a generator $\gamma \in C_8$ is the isomorphism sending x to its formal inverse on each of the F_j.

This means that the existence of an equivariant $\lambda^{(4)}$ is equivalent to that of a formal $\mathbb{Z}[\zeta_8]$-module structure on each of the F_j, and compatible strict isomorphisms between them. This proves the first part of the Lemma.

For the second part of Lemma 12.4, recall that

$$D = N_2^8(\overline{r}_{15}^{C_2}\overline{r}_3^{C_4}\overline{r}_1^{C_8}).$$

The norm sends products to products, and $N(x)$ is a product of conjugates of x under the action of C_8. Hence its image in R_* is a unit multiple of that of a power of x, so it suffices to show that each of the three elements $\overline{r}_{15}^{C_2}$, $\overline{r}_3^{C_4}$ and $\overline{r}_1^{C_8}$ maps to a unit in R_*.

The generators $\overline{r}_i^H$ are defined by (4.48), which we rewrite as

$$\bar{x} + \sum_{i>0} \bar{m}_i \bar{x}^{i+1} = \left(\bar{x} + \sum_{k>0} \gamma_H(\bar{m}_{2^k-1})\bar{x}^{2^k}\right) \circ \left(\bar{x} + \sum_{i>0} \bar{r}_i^H \bar{x}^{i+1}\right)$$

where $\gamma_H = \gamma^{8/h}$ denotes a generator of $H \subset G$ and γ is a generator of C_8. Note here that the $\bar{m}_i$ are independent of the choice of subgroup H. For our purposes we can replace this by the corresponding equation in underlying homotopy, namely

$$x + \sum_{i>0} m_i x^{i+1} = \left(x + \sum_{k>0} \gamma^{8/h}(m_{2^k-1})x^{2^k}\right) \circ \left(x + \sum_{i>0} r_i^H x^{i+1}\right)$$

Applying the homomorphism $\lambda^{(4)} : \pi_* MU^{(4)} \to R_*$, we get

$$x + \sum_{k>0} \frac{w^{2^k-1}}{\pi^k} x^{2^k} = \left(x + \sum_{j>0} \frac{\zeta^{8/h}w^{2^j-1}}{\pi^j} x^{2^j}\right) \circ \left(x + \sum_{i>0} \lambda^{(4)}(r_i^H)x^{i+1}\right).$$

Solving for $\lambda^{(4)}\left(r_{2^k-1}^H\right)$ for various H and k, we find that

$$\lambda^{(4)}\left(r_1^{C_2}\right) = \left(-\pi^3 - 4\pi^2 - 6\pi - 4\right)w = \pi^3 \cdot \text{unit} \cdot w$$

$$\lambda^{(4)}\left(r_3^{C_2}\right) = \left(-4\pi^3 - 5\pi^2 + 14\pi + 26\right)w^3 = \pi^2 \cdot \text{unit} \cdot w^3$$

$$\lambda^{(4)}\left(r_7^{C_2}\right) = \left(-6182\pi^3 - 21426\pi^2 - 22171\pi - 1052\right)w^7$$

$$= \pi \cdot \text{unit} \cdot w^7$$

$$\lambda^{(4)}\left(r_{15}^{C_2}\right) = \left(306347134\pi^3 - 3700320563\pi^2\right.$$

$$\left. -15158766469\pi - 16204677587\right)w^{15}$$

$$= \text{unit} \cdot w^{15}$$

$$\lambda^{(4)}\left(r_1^{C_4}\right) = \left(-\pi - 2\right)w = \pi \cdot \text{unit} \cdot w$$

$$\lambda^{(4)}\left(r_3^{C_4}\right) = \left(8\pi^3 + 26\pi^2 + 25\pi - 1\right)w^3 = \text{unit} \cdot w^3$$

$$\lambda^{(4)}\left(r_1^{C_8}\right) = -w,$$

where each unit is in A. Hence the images under $\lambda^{(4)}$ of $r_1^{C_2}$, $r_3^{C_2}$, $r_7^{C_2}$, and $r_1^{C_4}$ are not units. For this reason, smaller subscripts of $\bar{\mathfrak{d}}$ in the definition of D would not work. On the other hand, the images of $r_{15}^{C_2}$, $r_3^{C_4}$, and $r_1^{C_8}$ are units as required. Thus we have shown that each factor of i_0^*D and hence i_0^*D itself maps to a unit in R_*, thereby proving the lemma. $\qquad\square$

References

1. Shôrô Araki, *Orientations in τ-cohomology theories*, Japan. J. Math. (N.S.) **5** (1979), no. 2, 403–430. MR MR614829 (83a:55010)
2. M. G. Barratt, J. D. S. Jones, and M. E. Mahowald, *Relations amongst Toda brackets and the Kervaire invariant in dimension 62*, Journal of the London Mathematical Society **30** (1985), 533–550.
3. W. Browder, *The Kervaire invariant of framed manifolds and its generalization*, Annals of Mathematics **90** (1969), 157–186.
4. Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. I*, Ann. of Math. (2) **128** (1988), no. 2, 207–241. MR 89m:55009
5. Andreas W. M. Dress, *Contributions to the theory of induced representations*, Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 183–240. Lecture Notes in Math., Vol. 342. MR MR0384917 (52 #5787)
6. E. Dror Farjoun, *Cellular inequalities*, The Čech centennial (Boston, MA, 1993), Contemp. Math., vol. 181, Amer. Math. Soc., Providence, RI, 1995, pp. 159–181. MR MR1320991 (96g:55011)
7. Daniel Dugger, *An Atiyah-Hirzebruch spectral sequence for KR-theory*, K-Theory **35** (2005), no. 3-4, 213–256 (2006). MR MR2240234 (2007g:19004)
8. Leonard Evens, *A generalization of the transfer map in the cohomology of groups*, Trans. Amer. Math. Soc. **108** (1963), 54–65. MR MR0153725 (27 #3686)
9. Michikazu Fujii, *Cobordism theory with reality*, Math. J. Okayama Univ. **18** (1975/76), no. 2, 171–188. MR MR0420597 (54 #8611)
10. J. P. C. Greenlees and J. P. May, *Equivariant stable homotopy theory*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 277–323. MR MR1361893 (96j:55013)
11. _____, *Localization and completion theorems for MU-module spectra*, Ann. of Math. (2) **146** (1997), no. 3, 509–544. MR MR1491447 (99i:55012)
12. M. Hazewinkel, *Formal groups and applications*, Academic Press, New York, 1978.

13. Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. MR MR1944041 (2003j:18018)

14. Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) **148** (1998), no. 1, 1–49. MR 99h:55009

15. Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR MR1650134 (99h:55031)

16. Po Hu and Igor Kriz, *Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence*, Topology **40** (2001), no. 2, 317–399. MR MR1808224 (2002b:55032)

17. Michel A. Kervaire, *A manifold which does not admit any differentiable structure*, Comment. Math. Helv. **34** (1960), 257–270.

18. Michel A. Kervaire and John W. Milnor, *Groups of homotopy spheres. I*, Ann. of Math. (2) **77** (1963), 504–537.

19. Peter S. Landweber, *Conjugations on complex manifolds and equivariant homotopy of MU*, Bull. Amer. Math. Soc. **74** (1968), 271–274. MR MR0222890 (36 #5940)

20. J. Lubin and J. Tate, *Formal complex multiplication in local fields*, Annals of Mathematics **81** (1965), 380–387.

21. M. A. Mandell and J. P. May, *Equivariant orthogonal spectra and S-modules*, Mem. Amer. Math. Soc. **159** (2002), no. 755, x+108. MR MR1922205 (2003i:55012)

22. M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, *Model categories of diagram spectra*, Proc. London Math. Soc. (3) **82** (2001), no. 2, 441–512. MR MR1806878 (2001k:55025)

23. J. W. Milnor, *On the cobordism ring Ω^* and a complex analogue, Part I*, American Journal of Mathematics **82** (1960), 505–521.

24. D. G. Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bulletin of the American Mathematical Society **75** (1969), 1293–1298.

25. Daniel Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math. **7** (1971), 29–56 (1971). MR MR0290382 (44 #7566)

26. D. C. Ravenel, *The nonexistence of odd primary Arf invariant elements in stable homotopy theory*, Math. Proc. Cambridge Phil. Soc. **83** (1978), 429–443.

27. _____, *Complex cobordism and stable homotopy groups of spheres*, Academic Press, New York, 1986.

28. Charles Rezk, *Notes on the Hopkins-Miller theorem*, Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997), Contemp. Math., vol. 220, Amer. Math. Soc., Providence, RI, 1998, pp. 313–366. MR MR1642902 (2000i:55023)

29. Stefan Schwede and Brooke Shipley, *Equivalences of monoidal model categories*, Algebr. Geom. Topol. **3** (2003), 287–334 (electronic). MR MR1997322 (2004i:55026)

30. Katsumi Shimomura, *Novikov's* Ext^2 *at the prime* 2, Hiroshima Math. J. **11** (1981), no. 3, 499–513. MR MR635034 (83c:55027)

31. V. Voevodsky, *On the zero slice of the sphere spectrum*, Tr. Mat. Inst. Steklova **246** (2004), no. Algebr. Geom. Metody, Svyazi i Prilozh., 106–115. MR MR2101286 (2005k:14042)

32. Vladimir Voevodsky, *Open problems in the motivic stable homotopy theory. I*, Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998), Int. Press Lect. Ser., vol. 3, Int. Press, Somerville, MA, 2002, pp. 3–34. MR MR1977582 (2005e:14030)

33. _____, *A possible new approach to the motivic spectral sequence for algebraic K-theory*, Recent progress in homotopy theory (Baltimore, MD, 2000), Contemp. Math., vol. 293, Amer. Math. Soc., Providence, RI, 2002, pp. 371–379. MR MR1890744 (2003g:55011)

34. Klaus Wirthmüller, *Equivariant homology and duality*, Manuscripta Math. **11** (1974), 373–390. MR MR0343260 (49 #8004)

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