The Lost Telescope of Z

Electronic Computational Homotopy Theory Seminar

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This talk began in discussions last summer with

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Mark Behrens
Prasit Bhattacharya
Dominic Culver
Zhouli Xu

Introduction
The triple loop space approach
The construction of $y(n)$
The Adams-Novikov spectral sequence for $L_K(n)y(n)$
The Adams spectral sequences for $y(n)$ and $Y(n)$
Disproving the Telescope Conjecture for $n \geq 2$?
Going equivariant
What is $Z$ and what is its telescope?

Z is a finite CW spectrum constructed recently by Prasit Bhattacharya and Philip Egger. It has 32 cells in dimensions ranging from 0 to 16. Mahowald would say it is “half of $A(2)$.” It admits a self map $\Sigma^6Z \to Z$ realizing multiplication by $v_2$. Its homotopy of its $K(2)$-localization is very nice. It could be an interesting test case for the Telescope Conjecture, which says that its telescope and $K(2)$-localization are the same. Z might have a motivic analog. This could lead to additional structure in its Adams spectral sequence.
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What is the Telescope Conjecture?

I first made the Telescope Conjecture in the late '70s and published it in 1984. It has a version for each prime $p$ and each integer $n \geq 0$. The $n = 1$ case is due to Mahowald for $p = 2$ and to Miller for odd primes.
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Earthquake of October 17, 1989

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The triple loop space approach

Recall that the mod 2 dual Steenrod algebra is
\[ A^* = \mathbb{Z}/2[\xi_1, \xi_2, \ldots] \]
with \(|\xi_n| = 2^{n-1} - 1\).

Mahowald had a spectrum \(Y\) with
\[ H^* Y = \mathbb{Z}/2[\xi_1] / (\xi_4^2) \]
or "half" of \(A(1)^* = \mathbb{Z}/2[\xi_1, \xi_2] / (\xi_4, \xi_2^2)\).

It has a self map \(\Sigma^2 Y \rightarrow Y \rightarrow C_{v_1} = \text{cofiber}\) with
\[ H^* C_{v_1} = A(1)^* = \mathbb{Z}/2[\xi_1, \xi_2] / (\xi_4, \xi_2^2) \].

The Bhattacharya-Egger spectrum \(Z\) has
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Unlike $Y$ and $Z$, it is an associative ring spectrum.
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It has a self-map

$$\Sigma^{2(2^n - 1)} y(n) \xrightarrow{v_n} y(n)$$

inducing an isomorphism in $K(n)_*(-)$, the $n$th Morava K-theory.
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We have ways to study the homotopy groups of both of them.
The construction of $y(n)$

Consider the diagram

$$
\begin{array}{ccc}
S^1 & \xrightarrow{f} & BO \\
\downarrow i & & \downarrow g \\
\Omega^2 S^3 & \xrightarrow{f} & BO \\
\end{array}
$$

where

- $f$ represents the nontrivial element of $\pi_1 BO = \mathbb{Z}/2$,
- $i$ is the adjoint of the identity map on $\Sigma S^1 = S^3$ and
- $g$ is the extension of $f$ given by the infinite loop space structure on $BO$.
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We know that

\[ H_* \Omega^2 S^3 = \mathbb{Z}/2[u_1, u_2, \ldots] \quad \text{with} \quad |u_n| = 2^n - 1 = |\xi_n|. \]
The construction of $y(n)$ (continued)

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The construction of $y(n)$ (continued)

$$S^1 \xrightarrow{f} BO \xleftarrow{i} \Omega^2 S^3 \xrightarrow{g} \Omega^2 S^3$$

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Let $y(\infty)$ denote the Thom spectrum induced by $g$. 

Long ago Mahowald showed that it is the mod 2 Eilenberg-Mac Lane spectrum $H\mathbb{Z}/2$. 

We will construct subspaces $W_n$ of $\Omega^2 S^3$ with $H_* W_n = \mathbb{Z}/2[u_1, u_2, \ldots, u_n]$, and $y(n)$ will be the corresponding Thom spectrum.
The construction of $y(n)$ (continued)

\[ S^1 \xrightarrow{f} BO \]
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The construction of $y(n)$ (continued)

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In the early 50s Ioan James defined the reduced product $J_kX$ (for any space $X$) as a certain quotient of $X \times^k$ and showed that $J_\infty X$ is equivalent to $\Omega \Sigma X$.

He showed there is a 2-local fiber sequence

$$\Omega^2 S^{2n+1} + 1 \rightarrow J_{2n-1} S^2 \rightarrow \Omega S^3 \rightarrow \Omega S^{2n+1} + 1.$$
In the early 50s Ioan James defined the reduced product \( J_k X \) (for any space \( X \)) as a certain quotient of \( X \times^k \) and showed that \( J_{\infty} X \) is equivalent to \( \Omega \Sigma X \).

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$$\Omega^2 S^{2n+1+1} \to J_2 S^2 \to \Omega S^3 \to \Omega S^{2n+1+1}.$$

Note that $\Omega S^3$ is equivalent to a CW complex with a single cell in each even dimension.
The construction of \( y(n) \) (continued)

In the early 50s Ioan James defined the reduced product \( J_k X \) (for any space \( X \)) as a certain quotient of \( X \times k \) and showed that \( J_\infty X \) is equivalent to \( \Omega \Sigma X \).

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\Omega^2 S^{2n+1+1} \to J_{2n-1} S^2 \to \Omega S^3 \to \Omega S^{2n+1+1}.
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Note that \( \Omega S^3 \) is equivalent to a CW complex with a single cell in each even dimension. \( J_{2n-1} S^2 \) is its \((2^{n+1} - 1)\)-skeleton.
The construction of $y(n)$ (continued)

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Our space $W_n$ is $\Omega J_{2n-1} S^2$. 

1.12 The construction of $y(n)$
The construction of $y(n)$ (continued)

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$$\Omega^2 S^{2n+1} + 1 \to J_{2^n - 1} S^2 \to \Omega S^3 \to \Omega S^{2n+1} + 1.$$  

Note that $\Omega S^3$ is equivalent to a CW complex with a single cell in each even dimension. $J_{2^n - 1} S^2$ is its $(2^{n+1} - 1)$-skeleton.

Our space $W_n$ is $\Omega J_{2^n - 1} S^2$, so it maps to $\Omega^2 S^3$ as desired.
The construction of \( y(n) \) (continued)

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Note that \( \Omega S^3 \) is equivalent to a CW complex with a single cell in each even dimension. \( J_{2^n-1} S^2 \) is its \( (2^{n+1} - 1) \)-skeleton.

Our space \( W_n \) is \( \Omega J_{2^n-1} S^2 \), so it maps to \( \Omega^2 S^3 \) as desired. The MRS spectrum \( y(n) \) is the Thomification of

\[
\begin{array}{ccc}
\Omega J_{2^n-1} S^2 & \xrightarrow{g} & \Omega^2 S^3 & \xrightarrow{BO} \\
\end{array}
\]
The construction of $y(n)$ (continued)

The MRS spectrum $y(n)$ is the Thomification of

$$\Omega J_{2^n-1} S^2 \rightarrow \Omega^2 S^3 \xrightarrow{g} BO.$$
The construction of $y(n)$ (continued)

The MRS spectrum $y(n)$ is the Thomification of

$$
\Omega J_{2^n - 1} S^2 \rightarrow \Omega^2 S^3 \xrightarrow{g} BO.
$$

From James’ 2-local fiber sequence

$$
\Omega^3 S^{2n+1 + 1} \rightarrow \Omega J_{2^n - 1} S^2 \rightarrow \Omega^2 S^3
$$
The construction of $y(n)$ (continued)

The MRS spectrum $y(n)$ is the Thomification of

$$\Omega J_{2^n-1} S^2 \longrightarrow \Omega^2 S^3 \overset{g}{\longrightarrow} BO.$$ 

From James’ 2-local fiber sequence

$$\Omega^3 S^{2n+1+1} \longrightarrow \Omega J_{2^n-1} S^2 \longrightarrow \Omega^2 S^3$$

we get maps of spectra

$$\Sigma^\infty S^{|v_n|} \longrightarrow \Sigma^\infty \Omega^3 S^{2n+1+1} \longrightarrow y(n) \longrightarrow H\mathbb{Z}/2.$$
The construction of $y(n)$ (continued)

The MRS spectrum $y(n)$ is the Thomification of

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\Omega J_{2^n-1} S^2 \xrightarrow{} \Omega^2 S^3 \xrightarrow{g} BO.
$$

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\Sigma^\infty S^{|v_n|} \xrightarrow{} \Sigma^\infty \Omega^3 S^{2^{n+1}+1} \xrightarrow{} y(n) \xrightarrow{} \mathbb{H}\mathbb{Z}/2.
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where the map $S^{|v_n|} \xrightarrow{} \Omega^3 S^{2^{n+1}+1}$ is the inclusion of the bottom cell.
The construction of $y(n)$ (continued)

The MRS spectrum $y(n)$ is the Thomification of

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where the map $S|v_n| \rightarrow \Omega^3 S^{2^{n+1}+1}$ is the inclusion of the bottom cell. Since $y(n)$ is the Thom spectrum for a loop map, it is an associative ring spectrum.
The construction of $y(n)$ (continued)

The MRS spectrum $y(n)$ is the Thomification of

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where the map $S|v_n| \rightarrow \Omega^3 S^{2n+1+1}$ is the inclusion of the bottom cell. Since $y(n)$ is the Thom spectrum for a loop map, it is an associative ring spectrum. The composite map above leads to the desired $v_n$-self map of $y(n)$. 
The Adams-Novikov spectral sequence for $L_K(n)y(n)$

Let $Y(n)$ denote the telescope associated with $y(n)$. 
The Adams-Novikov spectral sequence for $L_K(n)y(n)$

Let $Y(n)$ denote the telescope associated with $y(n)$. Then we have

$$BP_* = \mathbb{Z}(2)[v_1, v_2, \ldots] \quad \text{where} \quad |v_i| = 2^{i+1} - 2$$

$$BP_*(BP) = BP_*[t_1, t_2, \ldots] \quad \text{where} \quad |t_i| = 2^{i+1} - 2$$

$$BP_*(y(n)) = (BP_* / I_n)[t_1, t_2, \ldots t_n]$$

$$BP_*(Y(n)) = BP_*(L_K(n)y(n)) = v_n^{-1}BP_*(y(n))$$
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where $I_n = (2, v_1, \ldots v_{n-1})$

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The Adams-Novikov $E_2$-term for $L_K(n)y(n)$ is

$$E_2 = \mathbb{Z}/2[v_n^{\pm 1}, v_{n+1}, \ldots v_{2n}] \otimes E(h_{n+i,j}: 1 \leq i \leq n, 0 \leq j < n)$$
The Adams-Novikov spectral sequence for $LK(n)y(n)$

Let $Y(n)$ denote the telescope associated with $y(n)$. Then we have

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where $h_{n+i,j} = [t_{n+i}^{2^j}]$. 
The Adams-Novikov spectral sequence for $L_K(n)y(n)$

Let $Y(n)$ denote the telescope associated with $y(n)$. Then we have

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The Adams-Novikov $E_2$-term for $L_K(n)y(n)$ is

\[ E_2 = \mathbb{Z}/2[v_n^{\pm 1}, v_{n+1}, \ldots v_{2n}] \otimes E(h_{n+i,j}: 1 \leq i \leq n, 0 \leq j < n) \]

where $h_{n+i,j} = [t^{2j}_{n+i}].$ The second factor is an exterior algebra on $n^2$ generators.
The Lost Telescope

The triple loop space approach

The construction of $y(n)$

The Adams-Novikov spectral sequence for $L_K(n)y(n)$

Let $Y(n)$ denote the telescope associated with $y(n)$. Then we have

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where $I_n = (2, v_1, \ldots v_{n-1})$

$$BP_*(Y(n)) = BP_*(L_K(n)y(n)) = v_n^{-1}BP_*(y(n))$$

The Adams-Novikov $E_2$-term for $L_K(n)y(n)$ is

$$E_2 = Z/2[v_n^{\pm 1}, v_{n+1}, \ldots v_{2n}] \otimes E(h_{n+i,j} : 1 \leq i \leq n, 0 \leq j < n)$$

where $h_{n+i,j} = [t_{n+i}^{2j}]$. The second factor is an exterior algebra on $n^2$ generators. This $E_2$-term is finitely generated as a module over the ring

$$R(n) = Z/2[v_n^{\pm 1}, v_{n+1}, \ldots v_{2n}]$$
The Adams spectral sequences for \( y(n) \) and \( Y(n) \)

Since

\[
H_* y(n) = \mathbb{Z}/2[\xi_1, \xi_2, \ldots, \xi_n],
\]
The Adams spectral sequences for $y(n)$ and $Y(n)$

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$$H_* y(n) = \mathbb{Z}/2[\xi_1, \xi_2, \ldots, \xi_n],$$

a standard change-or-rings argument shows that

$$\text{Ext}_{A_*} (\mathbb{Z}/2, H_* y(n)) \cong \text{Ext}_{A[n]_*} (\mathbb{Z}/2, \mathbb{Z}/2)$$
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This leads to an Adams $E_1$-term of the form

$$E_1 = P(v_n, v_{n+1}, \ldots) \otimes P(h_{n+i, j}: i > 0, j \geq 0)$$
The Adams spectral sequences for $y(n)$ and $Y(n)$

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This leads to an Adams $E_1$-term of the form
$$E_1 = P(v_n, v_{n+1}, \ldots) \otimes P(h_{n+i,j} : i > 0, j \geq 0)$$
where, for such $i$ and $j$,
$$v_{n+i-1} = [\xi_{n+i}] \in E_1^{1,2^{n+i}-1},$$
$$h_{n+i,j} = [\xi_{n+i}^{2^j+1}] \in E_1^{1,2^j(2^{n+i}-1)}$$
and
$$d(v_{2n+i}^j) = \sum_{0 \leq k < i} v_{n+k}^j h_{n+i+j-k, n+k} = v_n^j h_{n+i+j, n} + \ldots$$
Localizing the Adams spectral sequence for $y(n)$

The Adams spectral sequence for a spectrum $X$ is based on an Adams resolution,

$$X_0 \xleftarrow{\Sigma} X_1 \xleftarrow{\Sigma} X_2 \xleftarrow{\Sigma} \ldots$$

This leads to a localized Adams spectral sequence converging to the homotopy of $Y(n) = v^{-1} n y(n)$. 

The Adams spectral sequence for $y(n)$

The Adams-Novikov spectral sequence for $LK(n) y(n)$

The Adams spectral sequences for $y(n)$ and $Y(n)$

Disproving the Telescope Conjecture for $n \geq 2$?

Going equivariant
Localizing the Adams spectral sequence for $y(n)$

The Adams spectral sequence for a spectrum $X$ is based on an Adams resolution, which is a diagram of the form

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \ldots$$

with certain properties.
Localizing the Adams spectral sequence for $y(n)$

The Adams spectral sequence for a spectrum $X$ is based on an Adams resolution, which is a diagram of the form

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \ldots$$

with certain properties. When $X = y(2)$, the self map $\Sigma^6 X_i \to X_i$ lifts to $X_{i+1}$, and we get a diagram

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$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\Sigma^{-6} X_0 \leftarrow \Sigma^{-6} X_1 \leftarrow \Sigma^{-6} X_2 \leftarrow \Sigma^{-6} X_3 \leftarrow \Sigma^{-6} X_4 \leftarrow \ldots$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\Sigma^{-12} X_0 \leftarrow \Sigma^{-12} X_1 \leftarrow \Sigma^{-12} X_2 \leftarrow \Sigma^{-12} X_3 \leftarrow \Sigma^{-12} X_4 \leftarrow \Sigma^{-12} X_5 \leftarrow \ldots$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$
Localizing the Adams spectral sequence for $y(n)$

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$$
\begin{array}{ccccccc}
X_0 & \leftarrow & X_1 & \leftarrow & X_2 & \leftarrow & X_3 & \leftarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Sigma^{-6} X_0 & \leftarrow & \Sigma^{-6} X_1 & \leftarrow & \Sigma^{-6} X_2 & \leftarrow & \Sigma^{-6} X_3 & \leftarrow & \Sigma^{-6} X_4 & \leftarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Sigma^{-12} X_0 & \leftarrow & \Sigma^{-12} X_1 & \leftarrow & \Sigma^{-12} X_2 & \leftarrow & \Sigma^{-12} X_3 & \leftarrow & \Sigma^{-12} X_4 & \leftarrow & \Sigma^{-12} X_5 & \leftarrow & \ldots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\end{array}
$$

This leads to a localized Adams spectral sequence converging to the homotopy of

$$Y(n) = v_n^{-1} y(n).$$
Localizing the Adams spectral sequence for $y(n)$ (continued)

This localization converts

$$E_1 = P(v_n, v_{n+1}, \ldots) \otimes P(h_{n+i, j} : i > 0, j \geq 0)$$

converging to $\pi_* y(n)$
Localizing the Adams spectral sequence for $y(n)$ (continued)

This localization converts

$$E_1 = P(v_n, v_{n+1}, \ldots) \otimes P(h_{n+i,j} : i > 0, j \geq 0)$$

converging to $\pi_* y(n)$ to

$$E_2 = P(v_{n+1}^{\pm 1}, v_{n+1}, \ldots, v_{2n}) \otimes P(h_{n+i,j} : i > 0, 0 \leq j < n)$$

converging to $\pi_* Y(n)$. 
Localizing the Adams spectral sequence for $y(n)$ (continued)

This localization converts

$$E_1 = P(v_n, v_{n+1}, \ldots) \otimes P(h_{n+i, j} : i > 0, j \geq 0)$$

converging to $\pi_\ast y(n)$ to

$$E_2 = P(v_{n}^{\pm1}, v_{n+1}, \ldots, v_{2n}) \otimes P(h_{n+i, j} : i > 0, 0 \leq j < n)$$

converging to $\pi_\ast Y(n)$. For $n = 2$ this reads

$$E_2 = P(v_{2}^{\pm1}, v_{3}, v_{4}) \otimes P(h_{2+i,0}, h_{2+i,1} : i > 0).$$

It is likely that for $i > 0$ there are Adams differentials

$$d_2 h_{4+i,0} = v_2 h_{2+i,1}^2$$
$$d_4 h_{3+i,1} = v_2 h_{2+i,0}^4.$$
Localizing the Adams spectral sequence for $y(n)$ (continued)

In the localized Adams spectral sequence for $Y(2)$ we have

$$E_2 = P(v_2^{\pm 1}, v_3, v_4) \otimes P(h_{2+i}, 0, h_{2+i}, 1 : i > 0).$$

with likely differentials

$$d_2 h_{4+i,0} = v_2 h_{2+i,1}^2$$

and

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Localizing the Adams spectral sequence for $y(n)$ (continued)

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This would leave

$$E_5 = E_\infty = P(v_2^{\pm 1}, v_3, v_4) \otimes E(h_{3,0}, h_{3,1}, h_{4,0}) \otimes E(b_{3,0}, b_{4,0}, b_{5,0}, \ldots)$$

where $b_{i,0} = h_{i,0}^2$. This is infinitely generated over the ring

$$R(2) = P(v_2^{\pm 1}, v_3, v_4)$$
Localizing the Adams spectral sequence for $y(n)$ (continued)

In the localized Adams spectral sequence for $Y(2)$ we have

$$E_2 = P(v_2^{\pm 1}, v_3, v_4) \otimes P(h_{2+i,0}, h_{2+i,1} : i > 0).$$

with likely differentials

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where $b_{i,0} = h_{i,0}^2$. This is infinitely generated over the ring

$$R(2) = P(v_2^{\pm 1}, v_3, v_4)$$

while $\pi_* L_K(n) y(n)$ is finitely generated over it.
Disproving the Telescope Conjecture for $n \geq 2$?

We have just seen that, if all goes according to plan, the Adams-Novikov spectral sequence shows that

$$\pi_* L_K(2) y(2) = P(v_2^\pm 1, v_3, v_4) \otimes E(h_3,0, h_3,1, h_4,0, h_4,1)$$

while the localized Adams spectral sequence shows that

$$\pi_* Y(2) = P(v_2^\pm 1, v_3, v_4) \otimes E(h_3,0, b_3,0, b_4,0, b_5,0, \ldots).$$

There is a similar story for $n > 2$ and for odd primes.

The Telescope Conjecture says these two graded groups are the same, so this appears to disprove it. What could go wrong? We do not have complete control over differentials in the localized Adams spectral sequence. The ones we “know” about could be preempted by others that we don’t know about. Mahowald, Shick and I were unable to rule out this possibility.
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There is a similar story for $n > 2$ and for odd primes.
Disproving the Telescope Conjecture for $n \geq 2$?

We have just seen that, if all goes according to plan, the Adams-Novikov spectral sequence shows that

$$
\pi_* LK(2)y(2) = P(v_2^{\pm 1}, v_3, v_4) \otimes E(h_3,0, h_3,1, h_4,0, h_4,1)
$$

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$$
\pi_* Y(2) = P(v_2^{\pm 1}, v_3, v_4) \otimes E(h_3,0, h_3,1, h_4,0) \otimes E(b_3,0, b_4,0, b_5,0, \ldots).
$$

There is a similar story for $n > 2$ and for odd primes. The Telescope Conjecture says these two graded groups are the same, so this appears to disprove it.
Disproving the Telescope Conjecture for \( n \geq 2 \)?

We have just seen that, if all goes according to plan, the Adams-Novikov spectral sequence shows that

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What could go wrong? We do not have complete control over differentials in the localized Adams spectral sequence. The ones we “know” about could be preempted by others that we don’t know about. Mahowald, Shick and I were unable to rule out this possibility.
Going equivariant

If this approach is to succeed, we need some more structure in the localized Adams spectral sequence for $Y(n)$. 
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Recall that the construction of \( y(n) \) involved the diagram

\[
\begin{array}{c}
S^1 \xrightarrow{i} \Omega^2 S^3 \xrightarrow{g} BO \\
\uparrow \Omega J_{2n-1} S^2
\end{array}
\]
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\uparrow & & \uparrow \\
\Omega J_{2n-1} S^2 & \xrightarrow{g} & BO
\end{array}
$$

We can add another space and get

$$
\begin{array}{ccc}
S^1 & \xrightarrow{i} & \Omega^2 S^3 \\
\uparrow & & \uparrow \\
\Omega J_{2n-1} S^2 & \xrightarrow{g} & BO \\
\uparrow & & \uparrow \\
\Omega (SU(k+1)/SO(k+1)) & \text{for } k \gg 0.
\end{array}
$$
Going equivariant (continued)

\[ S^1 \xrightarrow{i} \Omega^2 S^3 \xrightarrow{g} BO \]

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$$H_* \Omega(SU(k + 1)/SO(k + 1)) = \mathbb{Z}/2[b_1, \ldots b_k].$$
Going equivariant (continued)

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and the loop map \( g_n \) exists for \( k \geq 2^n - 1 \).
Going equivariant (continued)

\[
\begin{array}{ccc}
S^1 & \overset{i}{\longrightarrow} & \Omega^2 S^3 \\
& \uparrow & \uparrow g \\
\Omega J_{2n-1} S^2 & \overset{g_n}{\longrightarrow} & \Omega(SU(k + 1)/SO(k + 1)) \\
& \uparrow a_k & \\
& \Omega^2 S^3 & \overset{g}{\longrightarrow} BO
\end{array}
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and the loop map \(g_n\) exists for \(k \geq 2^n - 1\). Thomifying the square on the right gives

\[
\begin{array}{ccc}
HZ/2 & \longrightarrow & MO \\
& \uparrow & \uparrow \\
y(n) & \longrightarrow & w(k),
\end{array}
\]
Going equivariant (continued)

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where \( w(k) \) is the Thom spectrum induced by the map \( a_k \).
Going equivariant (continued)

One can show that

\[ S^1 \xrightarrow{i} \Omega^2 S^3 \xrightarrow{g} BO \]

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Going equivariant (continued)

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& \uparrow & \\
\Omega J_{2^n-1} S^2 & \overset{g_n}{\longrightarrow} & \Omega(SU(k + 1)/SO(k + 1))
\end{array}
\]

is the fixed point set of the following diagram of $C_2$-spaces:

\[
\begin{array}{ccc}
S^\rho & \overset{i}{\longrightarrow} & \Omega^{1+\rho} S^{1+2\rho} \\
& \uparrow & \\
\Omega^\rho J_{2^n-1} S^{2\rho} & \overset{g_n}{\longrightarrow} & \Omega^\sigma SU(k + 1)_R
\end{array}
\]

where

\[
BU_R
\]
Going equivariant (continued)

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where

- \( BU_R \) and \( SU_R \) denote the spaces \( BU \) and \( SU \) equipped with a \( C_2 \)-action related to complex conjugation,
Going equivariant (continued)

One can show that

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Going equivariant (continued)

One can show that

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\[ \Omega^\rho J_{2^n-1} S^{2\rho} \xrightarrow{g_n} \Omega^\sigma SU(k+1)_R \]

where

- $BU_R$ and $SU_R$ denote the spaces $BU$ and $SU$ equipped with a $C_2$-action related to complex conjugation,
- $\sigma$ denotes the sign representation of $C_2$ and
- $\rho = 1 + \sigma$ denotes its regular representation.
Going equivariant (continued)

Here is our $C_2$-diagram again.
Going equivariant (continued)

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\[
\begin{align*}
S^\rho & \xrightarrow{i} \Omega^{1+\rho} S^{1+2\rho} \xrightarrow{g} BU_R & MU_R \\
\Omega^\rho J_{2^n-1} S^{2\rho} & \xrightarrow{g_n} \Omega^\sigma SU(k + 1)_R & X(k)_R
\end{align*}
\]
Going equivariant (continued)

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\[
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$$
\begin{array}{cccc}
S^\rho & \xrightarrow{i} & \Omega^{1+\rho}S^{1+2\rho} & \xrightarrow{g} & BU_R & \text{MU}_R \\
\uparrow & & \uparrow & & \uparrow i_k \\
\Omega^\rho J_{2^{n-1}}S^{2\rho} & \xrightarrow{g_n} & \Omega^\sigma SU(k + 1)_R & \xrightarrow{\Omega^\rho(SU/SU(k + 1))_R} \\
\end{array}
$$

with Thom spectra indicated on the right. Taking 2-local fibers of the vertical maps in the square gives

$$
\begin{array}{cccc}
\Omega^{1+\rho}S^{1+2\rho} & \xrightarrow{g} & BU_R & \text{MU}_R \\
\uparrow & & \uparrow a_k \\
\Omega^\rho J_{2^{n-1}}S^{2\rho} & \xrightarrow{g_n} & \Omega^\sigma SU(k + 1)_R \\
\uparrow & & \uparrow \\
\Omega^{2+\rho}S^{1+2^{n+1}\rho} & \xrightarrow{\Omega^\rho(SU/SU(k + 1))_R} \\
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The two fibers have the same connectivity when $k = 2^{n+1} - 2$. 
Going equivariant (continued)

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The two fibers have the same connectivity when $k = 2^{n+1} - 2$. 
Going equivariant (continued)

\[
\begin{align*}
&\Omega^{1+\rho} S^{1+2\rho} \xrightarrow{g} BU_R \\
&\Omega^\rho J_{2n-1} S^{2\rho} \xrightarrow{g_n} \Omega^\sigma SU(1 + |v_n|)_R \\
&\Omega^{2+\rho} S^{1+2^{n+1}\rho} \xrightarrow{} \Omega^\rho (SU/SU(1 + |v_n|))_R
\end{align*}
\]
Going equivariant (continued)

\[ \begin{align*}
\Omega^{1+\rho} S^{1+2\rho} & \xrightarrow{g} BU_R \\
\Omega^\rho J_{2^n-1} S^{2^\rho} & \xrightarrow{g_n} \Omega^\sigma SU(1 + |v_n|)_R \\
\Omega^{2+\rho} S^{1+2^{n+1}\rho} & \xrightarrow{} \Omega^\rho (SU/SU(1 + |v_n|))_R
\end{align*} \]

It follows that we have a map \( y(n) \rightarrow w(|v_n|) \) inducing a monomorphism in mod 2 homology,
Going equivariant (continued)

\[ \begin{align*} 
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\Omega^\rho J_{2n-1} S^{2\rho} & \xrightarrow{g_n} \Omega^\sigma SU(1 + |v_n|)_R \\
\Omega^{2+\rho} S^{1+2^{n+1}\rho} & \xrightarrow{} \Omega^\rho (SU/SU(1 + |v_n|))_R 
\end{align*} \]

It follows that we have a map \( y(n) \rightarrow w(|v_n|) \) inducing a monomorphism in mod 2 homology, and therefore maps

\[ S^{|v_n|} \rightarrow \Omega^3 S^{2^{n+1}+1} \rightarrow y(n) \rightarrow w(|v_n|), \]
Going equivariant (continued)

\[
\begin{align*}
\Omega^1 + \rho S^{1+2\rho} & \xrightarrow{g} BU_R \\
\Omega^\rho J_{2^n-1} S^{2\rho} & \xrightarrow{g_n} \Omega^\sigma SU(1 + |v_n|)_R \\
\Omega^{2+\rho} S^{1+2^{n+1}\rho} & \rightarrow \Omega^\rho (SU/SU(1 + |v_n|))_R
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\]

where \( w(k) \) is the geometric fixed point set of the Thom spectrum \( X(k)_R \).
Going equivariant (continued)

\[
\begin{array}{ccc}
\Omega^1 + \rho \, S^1 + 2^\rho & \xrightarrow{g} & BU_R \\
\uparrow & & \uparrow a_{|v_n|} \\
\Omega^\rho \, J_{2^{n-1}} \, S^{2^\rho} & \xrightarrow{g_n} & \Omega^\sigma \, SU(1 + |v_n|)_R \\
\uparrow & & \uparrow \\
\Omega^{2+\rho} \, S^1 + 2^{n+1}^\rho & \xrightarrow{\Omega^\rho (SU/SU(1 + |v_n|))_R} \\
\end{array}
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It follows that we have a map \( y(n) \to w(|v_n|) \) inducing a monomorphism in mod 2 homology, and therefore maps

\[
S^{|v_n|} \to \Omega^3 \, S^{2^{n+1} + 1} \to y(n) \to w(|v_n|),
\]

where \( w(k) \) is the geometric fixed point set of the Thom spectrum \( X(k)_R \). The above composite leads to a telescope \( W(|v_n|) \)
Going equivariant (continued)

\[
\begin{align*}
\Omega^{1+\rho} S^{1+2\rho} & \xrightarrow{g} BU_R \\
\Omega^{\rho} J_{2^{n-1}} S^{2\rho} & \xrightarrow{g_n} \Omega^{\sigma} SU(1 + |v_n|)_R \\
\Omega^{2+\rho} S^{1+2^{n+1}\rho} & \xrightarrow{} \Omega^{\rho}(SU/SU(1 + |v_n|))_R
\end{align*}
\]

It follows that we have a map \( y(n) \to w(|v_n|) \) inducing a monomorphism in mod 2 homology, and therefore maps

\[
S|v_n| \xrightarrow{} \Omega^3 S^{2^{n+1}+1} \xrightarrow{} y(n) \to w(|v_n|),
\]

where \( w(k) \) is the geometric fixed point set of the Thom spectrum \( X(k)_R \). The above composite leads to a telescope \( W(|v_n|) \) which is the geometric fixed point spectrum of the telescope for a map

\[
\Sigma^{(1+|v_n|)\rho-1} X(|v_n|)_R \to X(|v_n|)_R.
\]
It follows that we have a map $y(n) \rightarrow w(|v_n|)$ inducing a monomorphism in mod 2 homology, and therefore maps

$$S^{|v_n|} \rightarrow \Omega^3 S^{2n+1} \rightarrow y(n) \rightarrow w(|v_n|),$$

where $w(k)$ is the geometric fixed point set of the Thom spectrum $X(k)_R$. The above composite leads to a telescope $W(|v_n|)$ which is the geometric fixed point spectrum of the telescope for a map

$$\Sigma^{(1+|v_n|)\rho-1} X(|v_n|)_R \rightarrow X(|v_n|)_R.$$

The underlying spectrum of this telescope is contractible.
Going equivariant (continued)

\[
\begin{array}{cccc}
\Omega^1 + \rho S^{1+2\rho} & \xrightarrow{g} & BU_R \\
\uparrow & & \uparrow a|v_n| \\
\Omega^{\rho} J_{2^{n-1}} S^{2\rho} & \xrightarrow{g_n} & \Omega^\sigma SU(1 + |v_n|)_R \\
\uparrow & & \uparrow \\
\Omega^{2+\rho} S^{1+2^{n+1}\rho} & \xrightarrow{} & \Omega^\rho (SU/SU(1 + |v_n|))_R \\
\end{array}
\]

It follows that we have a map \(y(n) \rightarrow w(|v_n|)\) inducing a monomorphism in mod 2 homology, and therefore maps

\[
S^{|v_n|} \rightarrow \Omega^3 S^{2^{n+1}+1} \rightarrow y(n) \rightarrow w(|v_n|),
\]

where \(w(k)\) is the geometric fixed point set of the Thom spectrum \(X(k)_R\). The above composite leads to a telescope \(W(|v_n|)\) which is the geometric fixed point spectrum of the telescope for a map

\[
\Sigma^{1+|v_n|}\rho^{-1}X(|v_n|)_R \rightarrow X(|v_n|)_R.
\]

The underlying spectrum of this telescope is contractible because the underlying map is known to be nilpotent.