What is an $\infty$-category?

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Introduction

This is an expository talk on $\infty$-categories.
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The main references for this topic are two remarkable books by Jacob Lurie:

• Higher Topos Theory published in 2009 (949 pages), which we denote by [HTT].
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We will adhere to the following color convention:

• Ordinary categories will be written in green.
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For objects $W$, $X$ and $Y$ in an ordinary category $C$, one has a morphism sets $C(X, Y)$, $C(W, Y)$ and $C(W, X)$.
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$$C(X, Y) \times C(W, X) \longrightarrow C(W, Y)$$

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- For objects \(W, X\) and \(Y\) in an ordinary category \(\mathcal{C}\), one has a morphism sets \(\mathcal{C}(X, Y), \mathcal{C}(W, Y)\) and \(\mathcal{C}(W, X)\), with a composition map

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• In an $\infty$-category one need not worry about a model structure, but concepts of model category theory are needed to develop the theory of $\infty$-categories.
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The simplicial category $\Delta$ is that of finite ordered sets and order preserving maps.
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A simplicial set $X$ is a contravariant $\text{Set}$-valued functor on $\Delta$. Its value on $[k]$, its set of $k$-simplices, is denoted by $X_k$. $X$ comes equipped with families of maps $X_k \to X_{k-1}$ (called face maps) and $X_k \to X_{k+1}$ (degeneracies), each indexed by $i$ for $0 \leq i \leq k$. The $i$th such maps are induced by

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In the $i$th horn $\Lambda^i_n \subseteq \partial\Delta^n$ for $0 \leq i \leq n$, the set of $k$-simplices is the set of nonsurjective morphisms whose image does contain $i$. 

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In the **$i$th horn** $\Lambda^n_i \subseteq \partial\Delta^n$ for $0 \leq i \leq n$, the set of $k$-simplices is the set of nonsurjective morphisms whose image does contain $i$.

The **inner faces and horns** are those for which $0 < i < n$. 
Here are the three horns of a 2-simplex.
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\[
\begin{align*}
\Lambda^2_0 & \quad \Lambda^2_1 & \quad \Lambda^2_2 \\
0 & \rightarrow & 2 \\
& \downarrow & \\
1 & \rightarrow & 2 \\
0 & \rightarrow & 2
\end{align*}
\]
Review of simplicial sets (continued)

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In the \(i\)th horn, the missing face is opposite the \(i\)th vertex.
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\begin{array}{ccc}
\Lambda^2_0 & \Lambda^2_1 & \Lambda^2_2 \\
0 & 1 & 1 \\
\end{array}
\]

In the \(i\)th horn, the missing face is opposite the \(i\)th vertex.

A Kan complex is a simplicial set \(X\) for which every map from a horn \(\Lambda^n_i \to X\) extends to \(\Delta^n\).
The **topological** $n$-simplex $\Delta_{\text{top}}^n$ is the space
Review of simplicial sets (continued)

The topological \( n \)-simplex \( \Delta^n_{\text{top}} \) is the space

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\left\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0 \text{ and } \sum x_i = 1\right\}.
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The geometric realization $|X|$ of a simplicial set $X$ is the colimit of the $\text{Top}$-valued functor

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The geometric realization $|X|$ of a simplicial set $X$ is the colimit of the $\text{Top}$-valued functor

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This space turns out to be the union of geometric realizations of the nondegenerate topological simplices of $X$, meaning ones not in the image of any degeneracy map. In particular,

$$|\Delta^n| = \Delta_{\text{top}}^n \approx D^n,$$
The topological $n$-simplex $\Delta^n_{\text{top}}$ is the space

$$\left\{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0 \text{ and } \sum x_i = 1 \right\}.$$ 

The geometric realization $|X|$ of a simplicial set $X$ is the colimit of the $\text{Top}$-valued functor

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$$|\Delta^n| = \Delta^n_{\text{top}} \approx D^n, \quad |\partial \Delta^n| \approx S^{n-1},$$
The topological \( n \)-simplex \( \Delta^N_{\text{top}} \) is the space

\[
\left\{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0 \text{ and } \sum x_i = 1 \right\}.
\]

The geometric realization \( |X| \) of a simplicial set \( X \) is the colimit of the \( \text{Top} \)-valued functor

\[
[k] \mapsto X_k \times \Delta^k_{\text{top}}.
\]

This space turns out to be the union of geometric realizations of the nondegenerate topological simplices of \( X \), meaning ones not in the image of any degeneracy map. In particular,

\[
|\Delta^n| = \Delta^n_{\text{top}} \approx D^n, \quad |\partial\Delta^n| \approx S^{n-1}, \text{ and } |\Lambda_i^n| \approx D^{n-1}.
\]
Given simplicial sets $X$ and $Y$, one can define a simplicial set $X \times Y$ in which

$$(X \times Y)_n = \coprod_{0 \leq i \leq n} X_i \times Y_{n-i} \quad \text{and} \quad |X \times Y| = |X| \times |Y|.$$
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The category of simplicial sets is denoted by $\text{Set}_\Delta$. 

References
Review of simplicial sets (continued)

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Review of simplicial sets (continued)

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Hence $\text{Set}_\Delta$ is enriched over itself.
The nerve $\mathcal{N}C$ of a small category $C$ is the simplicial set

\[ \text{Of all the nerve!} \]

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The set of 4-simplices in $\mathcal{S}$

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Bousfield localization in $\infty$-categories

The $\infty$-category of spectra

References
The nerve $NC$ of a small category $C$ is the simplicial set in which the set of $n$-simplices $NC_n$ is the set of diagrams

$$X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n$$

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in $C$. Face and degeneracy maps are defined by composing adjacent morphisms and inserting identity maps.
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and define $NC_n$ to be the set of functors from $[n]$ to $C$. 

References
This simplicial set has the following property: Any simplicial map $\Lambda^n_i \to NC$ for $0 < i < n$ extends uniquely to $\Delta^n$. 
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Of all the nerve! (continued)
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It is known that the category $C$ is determined by its nerve, and that any simplicial set with property above is the nerve of some small category.
This simplicial set has the following property: Any simplicial map $\Lambda_i^n \to NC$ for $0 < i < n$ extends uniquely to $\Delta^n$.

\[
\begin{array}{ccc}
X_0 & \xrightarrow{g} & X_2 \\
\downarrow & & \downarrow \quad \text{or} \quad \text{extends uniquely}
\end{array}
\]

It is known that the category $C$ is determined by its nerve, and that any simplicial set with property above is the nerve of some small category.

A small category is thus equivalent to a simplicial set (its nerve) in which each map from an inner horn $\Lambda_i^n$ extends uniquely to a map from $\Delta^n$. 
Definition

An \(\infty\)-category (also called a quasicategory) \(\mathcal{C}\) is a simplicial set in which each simplicial map \(\Lambda^n_i \to \mathcal{C}\) for \(0 < i < n\) extends to some map \(\Delta^n \to \mathcal{C}\).
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There are several features of this definition worth noting.
The main definition

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There are several features of this definition worth noting.

- We are not requiring extensions of maps from \(\Lambda^n_0\) and \(\Lambda^n_n\) (known as the left and right outer horns) as in the definition of a Kan complex.
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An \( \infty \)-category (also called a quasicategory) \( \mathcal{C} \) is a simplicial set in which each simplicial map \( \Lambda^n_i \to \mathcal{C} \) for \( 0 < i < n \) extends to some map \( \Delta^n \to \mathcal{C} \). A functor \( F : \mathcal{C} \to \mathcal{C}' \) from one \( \infty \)-category to another is a simplicial map.

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The main definition

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An \( \infty \)-category (also called a quasicategory) \( C \) is a simplicial set in which each simplicial map \( \Lambda^i_n \rightarrow C \) for \( 0 < i < n \) extends to some map \( \Delta^n \rightarrow C \). A functor \( F : C \rightarrow C' \) from one \( \infty \)-category to another is a simplicial map.

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- The extension of each map from an inner horn is not required to be unique, as it is in the nerve of an ordinary category.

References
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- The extension of each map from an inner horn is not required to be unique, as it is in the nerve of an ordinary category. This means that this notion is more general than that of an ordinary category as seen through its nerve. Hence an ordinary category is a special case of an $\infty$-category.
The main definition (continued)

**Definition**

An $\infty$-category (also called a quasicategory) $C$ is a simplicial set in which each simplicial map $\Lambda^n_i \to C$ for $0 < i < n$ extends to some map $\Delta^n \to C$. A functor $F : C \to C'$ from one $\infty$-category to another is a simplicial map.
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An \(\infty\)-category (also called a quasicategory) \(C\) is a simplicial set in which each simplicial map \(\Lambda^n_i \to C\) for \(0 < i < n\) extends to some map \(\Delta^n \to C\). A functor \(F : C \to C'\) from one \(\infty\)-category to another is a simplicial map.

- Given such a simplicial set \(C\), we can think of elements of the sets \(C_0\) and \(C_1\) as objects and morphisms.
The main definition (continued)

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The main definition (continued)

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- Given such a simplicial set $C$, we can think of elements of the sets $C_0$ and $C_1$ as objects and morphisms. The two face maps $C_1 \rightrightarrows C_0$ define the source and target (aka domain and codomain) of each morphism. Elements in the sets $C_k$ for $k > 1$ can be thought of as higher morphisms in $C$. 
The main definition (continued)

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- A diagram

\[
\begin{array}{c}
X_0 \\
\downarrow f_{0,2} \\
X_1 & \xrightarrow{f_{0,1}} & X_2 \\
\downarrow f_{1,2} \\
\end{array}
\]

References
A diagram

```
X_0 \rightarrow X_1 \rightarrow X_2
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without the dashed arrow is equivalent to a map \( \Lambda^2_1 \rightarrow \mathcal{C} \).

### Definition

An \( \infty \)-category (also called a quasicategory) \( \mathcal{C} \) is a simplicial set in which each simplicial map \( \Lambda^i_n \rightarrow \mathcal{C} \) for \( 0 < i < n \) extends to some map \( \Delta^n \rightarrow \mathcal{C} \). A functor \( F : \mathcal{C} \rightarrow \mathcal{C}' \) from one \( \infty \)-category to another is a simplicial map.

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The set \( \mathcal{S}_{n+1} \) for \( n > 3 \)

A colimit in \( \mathcal{S} \)

Bousfield localization in \( \infty \)-categories

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**References**
The main definition (continued)

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- A diagram

\[
\begin{array}{ccc}
X_0 & - & - & - & - & - & - & \rightarrow & X_2 \\
| & f_{0,1} & \downarrow & f_{1,2} & \downarrow & \downarrow & f_{0,2} & \\
X_1 & & & & & & & \\
\end{array}
\]

without the dashed arrow is equivalent to a map $\Lambda_1^2 \to C$. Choosing a dashed arrow is equivalent to extending this map to $\partial\Delta^2$.  

References
The main definition (continued)

**Definition**

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- A diagram

\[
\begin{array}{c}
\quad X_1 \\
& f_{0,1} \quad f_{1,2} \\
X_0 \quad - \quad - \quad - \quad - \quad - \quad - \quad \Rightarrow \quad X_2 \\
& f_{0,2}
\end{array}
\]

without the dashed arrow is equivalent to a map \(\Lambda^2_1 \to C\). Choosing a dashed arrow is equivalent to extending this map to \(\partial \Delta^2\). Choosing a homotopy between \(f_{1,2}f_{0,1}\) and \(f_{0,2}\) is equivalent to extending this map to all of \(\Delta^2\).
The main definition (continued)

**Definition**

An \(\infty\)-category (also called a quasicategory) \(\mathcal{C}\) is a simplicial set in which each simplicial map \(\Lambda^n_i \to \mathcal{C}\) for \(0 < i < n\) extends to some map \(\Delta^n \to \mathcal{C}\). A functor \(F : \mathcal{C} \to \mathcal{C}'\) from one \(\infty\)-category to another is a simplicial map.

- A diagram

\[
\begin{array}{ccc}
\mathcal{X}_0 & \xrightarrow{f_{0,1}} & \mathcal{X}_1 \\
\mathcal{X}_0 & \xrightarrow{f_{0,2}} & \mathcal{X}_2 \\
\end{array}
\]

without the dashed arrow is equivalent to a map \(\Lambda^2_1 \to \mathcal{C}\). Choosing a dashed arrow is equivalent to extending this map to \(\partial\Delta^2\). Choosing a homotopy between \(f_{1,2}f_{0,1}\) and \(f_{0,2}\) is equivalent to extending this map to all of \(\Delta^2\). Such an extension is guaranteed to exist, but it is not unique.
The main definition (continued)

Definition

An $\infty$-category (also called a quasicategory) $\mathbb{C}$ is a simplicial set in which each simplicial map $\Lambda^i_n \to \mathbb{C}$ for $0 < i < n$ extends to some map $\Delta^n \to \mathbb{C}$. A functor $F : \mathbb{C} \to \mathbb{C}'$ from one $\infty$-category to another is a simplicial map.

- A diagram

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X_0 \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow X_2
f_{0,2} \quad f_{0,1} \quad f_{1,2}
```

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Definition

An $\infty$-category (also called a quasicategory) $\mathcal{C}$ is a simplicial set in which each simplicial map $\Lambda_i^n \to \mathcal{C}$ for $0 < i < n$ extends to some map $\Delta^n \to \mathcal{C}$. A functor $F : \mathcal{C} \to \mathcal{C}'$ from one $\infty$-category to another is a simplicial map.
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- The simplicial set \(\text{Set}_\Delta(K, D)\) of simplicial maps from a simplicial set \(K\) to an \(\infty\)-category \(D\) is itself an \(\infty\)-category.
The main definition (continued)

Definition

An \( \infty \)-category (also called a quasicategory) \( \mathcal{C} \) is a simplicial set in which each simplicial map \( \Lambda_i^n \to \mathcal{C} \) for \( 0 < i < n \) extends to some map \( \Delta^n \to \mathcal{C} \). A functor \( F : \mathcal{C} \to \mathcal{C}' \) from one \( \infty \)-category to another is a simplicial map.

- The simplicial set \( \text{Set}_\Delta(K, \mathcal{D}) \) of simplicial maps from a simplicial set \( K \) to an \( \infty \)-category \( \mathcal{D} \) is itself an \( \infty \)-category.
- \( K \) itself could be an \( \infty \)-category \( \mathcal{C} \),
The main definition (continued)

**Definition**

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- The simplicial set $\text{Set}_\Delta(K, \mathcal{D})$ of simplicial maps from a simplicial set $K$ to an $\infty$-category $\mathcal{D}$ is itself an $\infty$-category.
- $K$ itself could be an $\infty$-category $\mathcal{C}$, in particular it could be $N\mathcal{C}$ for an ordinary category $\mathcal{C}$. 

References
The main definition (continued)

Definition

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- The simplicial set $\text{Set}_\Delta(K, D)$ of simplicial maps from a simplicial set $K$ to an $\infty$-category $D$ is itself an $\infty$-category.
- $K$ itself could be an $\infty$-category $\mathcal{C}$, in particular it could be $N\mathcal{C}$ for an ordinary category $\mathcal{C}$. In other words, the collection of functors $\mathcal{C} \to \mathcal{D}$ is an $\infty$-category $\text{Fun}(\mathcal{C}, \mathcal{D})$. 
The main definition (continued)

Definition

An ∞-category (also called a quasicategory) \( \mathcal{C} \) is a simplicial set in which each simplicial map \( \Lambda^n_i \to \mathcal{C} \) for \( 0 < i < n \) extends to some map \( \Delta^n \to \mathcal{C} \). A functor \( F : \mathcal{C} \to \mathcal{C}' \) from one ∞-category to another is a simplicial map.

To a topological space \( X \) we can associate an ∞-category \( X \) (also known as \( \text{Sing} X \), the singular simplicial set of \( X \))
Definition

An ∞-category (also called a quasicategory) \( C \) is a simplicial set in which each simplicial map \( \Lambda^n_i \rightarrow C \) for \( 0 < i < n \) extends to some map \( \Delta^n \rightarrow C \). A functor \( F : C \rightarrow C' \) from one ∞-category to another is a simplicial map.

To a topological space \( X \) we can associate an ∞-category \( X \) (also known as \( \text{Sing} \) \( X \), the singular simplicial set of \( X \)) in which \( X_n \) is the set of continuous maps \( |\Delta^n| \rightarrow X \).
**Definition**

An ∞-category (also called a quasicategory) $\mathcal{C}$ is a simplicial set in which each simplicial map $\Lambda^n_i \to \mathcal{C}$ for $0 < i < n$ extends to some map $\Delta^n \to \mathcal{C}$. A functor $F : \mathcal{C} \to \mathcal{C}'$ from one ∞-category to another is a simplicial map.

To a topological space $X$ we can associate an ∞-category $X$ (also known as $\text{Sing} X$, the singular simplicial set of $X$) in which $X_n$ is the set of continuous maps $|\Delta^n| \to X$. $X$ is also a Kan complex since a map $|\Lambda^n_i| \to X$, for any horn $\Lambda^n_i$, extends to $|\Delta^n|$. 
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Such an $\infty$-category is called an $\infty$-groupoid.
Definition

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Such an \( \infty \)-category is called an \( \infty \)-groupoid because all morphisms, i.e., paths in \( X \), are invertible up to homotopy.
Let $\text{Top}$ denote the category of compactly generated weak Hausdorff spaces \textbf{with cardinality less than $\kappa$},
Let $\textbf{Top}$ denote the category of compactly generated weak Hausdorff spaces with cardinality less than $\kappa$, where $\kappa$ is a sufficiently large regular cardinal.
The $\infty$-category of topological spaces

Let $\mathbf{Top}$ denote the category of compactly generated weak Hausdorff spaces with cardinality less than $\kappa$, where $\kappa$ is a sufficiently large regular cardinal. This version of the category of topological spaces is small, so we could consider its nerve.
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There is another construction called the homotopy coherent nerve whose definition [HTT, Definition 1.1.5.5] baffled me for several years.
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Lurie’s $S$ is actually the homotopy coherent nerve of the category $\mathcal{Kan}$ of Kan complexes,
Let $\text{Top}$ denote the category of compactly generated weak Hausdorff spaces with cardinality less than $\kappa$, where $\kappa$ is a sufficiently large regular cardinal. This version of the category of topological spaces is small, so we could consider its nerve.

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Lurie’s $S$ is actually the homotopy coherent nerve of the category $\text{Kan}$ of Kan complexes, which is equivalent to the category of CW-complexes.
Let $\text{Top}$ denote the category of compactly generated weak Hausdorff spaces with cardinality less than $\kappa$, where $\kappa$ is a sufficiently large regular cardinal. This version of the category of topological spaces is small, so we could consider its nerve.

There is another construction called the homotopy coherent nerve whose definition [HTT, Definition 1.1.5.5] baffled me for several years. Rather than giving it here, I will describe the $\infty$-category $S$ (Lurie’s notation of [HTT, Definition 1.2.16.1]) one gets by applying it to $\text{Top}$. This is the $\infty$-category of topological spaces.

Lurie’s $S$ is actually the homotopy coherent nerve of the category $\mathcal{Kan}$ of Kan complexes, which is equivalent to the category of CW-complexes. The distinction between CW-complexes and more general spaces does not matter in what follows.
As in our main definition, $S$ is a simplicial set.
As in our main definition, $S$ is a simplicial set. Its vertices and edges are objects and morphisms in $\text{Top}$, meaning spaces and continuous maps.
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The set of 2-simplices is more interesting. In the subcategory $N\text{Top}$ (the ordinary nerve), it is the set of commutative diagrams of the form

$$
\begin{array}{ccc}
X_0 & \rightarrow & X_1 \\
\downarrow^{f_{1,2}} & & \downarrow^{f_{0,1}} \\
X_2 & \rightarrow & X_2.
\end{array}
$$

The top two edges can be viewed as a map $\Lambda^2 \rightarrow N\text{Top}$, with the full diagram being its unique extension to $\Delta^2$. 
The $\infty$-category of topological spaces (continued)

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The set of 2-simplices is more interesting. In the subcategory $\mathcal{N}\text{Top}$ (the ordinary nerve), it is the set of commutative diagrams of the form

```
\begin{tikzcd}
X_0 & X_1 & X_2 \\
& f_{0,1} & f_{1,2} \\
X_0 & f_{1,2}f_{0,1} & X_2.
\end{tikzcd}
```

The top two edges can be viewed as a map $\Lambda^1_2 \to \mathcal{N}\text{Top}$,
As in our main definition, $S$ is a simplicial set. Its vertices and edges are objects and morphisms in $\text{Top}$, meaning spaces and continuous maps.

The set of 2-simplices is more interesting. In the subcategory $N\text{Top}$ (the ordinary nerve), it is the set of commutative diagrams of the form

$$
\begin{array}{ccc}
X_0 & \xrightarrow{f_0,1} & X_1 & \xrightarrow{f_1,2} & X_2 \\
& & f_1,2f_0,1 & \nearrow \\
X_2 & \\
\end{array}
$$

The top two edges can be viewed as a map $\Lambda_2^1 \to N\text{Top}$, with the full diagram being its unique extension to $\Delta^2$. 
The $\infty$-category of topological spaces (continued)

$N\text{Top}_2$ is the set of commutative diagrams of the form

\begin{align*}
  X_0 \xrightarrow{f_{1,2}f_{0,1}} X_1 \xleftarrow{f_{0,1}} X_0 \xrightarrow{f_{1,2}} X_2.
\end{align*}

The set of 2-simplices $S_2$ consists of similar diagrams

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The $\infty$-category of topological spaces (continued)

$\mathcal{N}\text{Top}_2$ is the set of commutative diagrams of the form

$$
\begin{array}{ccc}
X_0 & \xymatrix{ & X_1 \ar[dl]_{f_{0,1}} \ar[dr]^{f_{1,2}} } & X_2. \\
& f_{1,2}f_{0,1} & \\
X_0 & \xymatrix{ & X_1 \ar[dl]_{f_{0,1}} \ar[dr]^{f_{1,2}} } & X_2. \\
& f_{1,2}f_{0,1} & \\
\end{array}
$$

The set of 2-simplices $S_2$ consists of similar diagrams in which the bottom arrow is replaced by any map $f_{0,2}$ homotopic to $f_{1,2}f_{0,1}$, with the homotopy $h_{0,2}$ being part of the datum.
The $\infty$-category of topological spaces (continued)

$\mathscr{N}Top_2$ is the set of commutative diagrams of the form

$$
\begin{array}{ccc}
X_0 & \longrightarrow & X_1 \\
| & \downarrow & | \\
| & f_{1,2} & f_{0,1} \\
\downarrow & \downarrow & \downarrow \\
X_0 & \longrightarrow & X_2.
\end{array}
$$

The set of 2-simplices $S_2$ consists of similar diagrams in which the bottom arrow is replaced by any map $f_{0,2}$ homotopic to $f_{1,2}f_{0,1}$, with the homotopy $h_{0,2}$ being part of the datum. Thus we have a diagram

$$
\begin{array}{ccc}
X_0 & \longrightarrow & X_1 \\
| & \downarrow & | \\
| & f_{0,2} & f_{0,1} \\
\downarrow & \downarrow & \downarrow \\
X_0 & \longrightarrow & X_2.
\end{array}
$$
What is an \( \infty \)-category?

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\( \mathcal{N} \text{Top}_{2} \) is the set of commutative diagrams of the form

\[
\begin{array}{ccc}
X_0 & \xrightarrow{f_{0,1}} & X_1 \\
\downarrow f_{1,2} & & \downarrow f_{1,2} \\
X_0 & \xrightarrow{f_{1,2}f_{0,1}} & X_2.
\end{array}
\]

The set of 2-simplices \( S_2 \) consists of similar diagrams in which the bottom arrow is replaced by any map \( f_{0,2} \) homotopic to \( f_{1,2}f_{0,1} \), with the homotopy \( h_{0,2} \) being part of the datum. Thus we have a diagram

\[
\begin{array}{ccc}
X_0 & \xrightarrow{f_{0,2}} & X_2 \\
\downarrow h_{0,2} & & \downarrow h_{0,2} \\
X_1 & \xrightarrow{f_{0,1}} & X_1 \\
\downarrow f_{1,2} & & \downarrow f_{1,2} \\
X_0 & \xrightarrow{f_{1,2}f_{0,1}} & X_2.
\end{array}
\]

The homotopy \( h_{0,2} \) is a map \( I \times X_0 \to X_2 \) with certain properties.
The $\infty$-category of topological spaces (continued)

The homotopy is a map

$$I \times X_0 \xrightarrow{h_{0,2}} X_2$$

with certain properties.
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The homotopy is a map

$$I \times X_0 \xrightarrow{h_{0,2}} X_2$$

with certain properties. It is adjoint to a path (which we denote by the same symbol)

$$I \xrightarrow{h_{0,2}} \text{Top}(X_0, X_2)$$
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The $\infty$-category of topological spaces (continued)

$X_0 \xrightarrow{f_0, 2} X_1 \xleftarrow{f_0, 1} X_2$

As in the ordinary case, the top two edges of the diagram can be viewed as a map $\Lambda_{2,1} \to S$. Now there is an extension of it to $\Delta_2$ for each path $h_{0,2}$ in $\text{Top}(X_0, X_2)$ starting at the point $f_{0,2} = f_{0,1}$. The space of such paths is contractible.
As in the ordinary case, the top two edges of the diagram can be viewed as a map $\Lambda^2 \rightarrow S$. 
As in the ordinary case, the top two edges of the diagram can be viewed as a map \( \Lambda^2_1 \to S \). Now there is an extension of it to \( \Delta^2 \) for each path \( h_{0,2} \) in \( \text{Top}(X_0, X_2) \) starting at the point \( f_{1,2}f_{0,1} \).
As in the ordinary case, the top two edges of the diagram can be viewed as a map $\Lambda^2_1 \to S$. Now there is an extension of it to $\Delta^2$ for each path $h_{0,2}$ in $\text{Top}(X_0, X_2)$ starting at the point $f_{1,2}f_{0,1}$. The space of such paths is contractible.
The following diagram shows four 2-simplices with their homotopies.
The set of 3-simplices in $S$

The following diagram shows four 2-simplices with their homotopies.

This is the boundary of a 3-simplex in $S$.

If there is a certain double homotopy $h_0,3: I^2 \to \text{Top}(X_0,X_3)$ shown on the next slide.
The set of 3-simplices in $S$

The following diagram shows four 2-simplices with their homotopies.

This is the boundary of a 3-simplex in $S$ iff there is a certain double homotopy
The set of 3-simplices in $S$

The following diagram shows four 2-simplices with their homotopies.

This is the boundary of a 3-simplex in $S$ iff there is a certain double homotopy adjoint to a map $h_{0,3} : I^2 \to \text{Top}(X_0, X_3)$ shown on the next slide.
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The diagram on the previous is the boundary of a 3-simplex in $S$ iff there a map $h_{0,3} : I^2 \to \text{Top}(X_0, X_3)$ of the form

$$f_{2,3}f_{1,2}f_{0,1} \quad \quad f_{2,3}h_{0,2} \quad \quad f_{2,3}f_{0,2}$$

$$h_{1,3}f_{0,1} \quad \quad h_{0,3}^1$$

$$f_{1,3}f_{0,1} \quad \quad h_{0,3}^2 \quad \quad f_{0,3}$$

This is a picture rather than a diagram. Each vertex of the square is a point in $\text{Top}(X_0, X_3)$, while the upper and left edges are the indicated composites. The other edges are the homotopies shown in the previous slide.
The set of 3-simplices in $S$ (continued)

The diagram on the previous is the boundary of a 3-simplex in $S$ iff there a map $h_{0,3} : I^2 \to \text{Top}(X_0, X_3)$ of the form

This is a picture rather than a diagram.
The diagram on the previous is the boundary of a 3-simplex in \( S \) iff there a map \( h_{0,3} : I^2 \to \text{Top}(X_0, X_3) \) of the form

\[
\begin{array}{c}
\bullet & - & \bullet \\
\downarrow & & \downarrow \\
\bullet & - & \bullet \\
\end{array}
\]

This is a picture rather than a diagram. Each vertex of the square is a point in \( \text{Top}(X_0, X_3) \), while the upper and left edges are the indicated composites.
The set of 3-simplices in $\mathcal{S}$ (continued)

The diagram on the previous is the boundary of a 3-simplex in $\mathcal{S}$ iff there a map $h_{0,3} : I^2 \to \text{Top}(X_0, X_3)$ of the form

$$f_{2,3}, f_{1,2}, f_{0,1} \quad \xrightarrow{f_{2,3}h_{0,2}} \quad f_{2,3}, f_{0,2}$$

$$h_{1,3}f_{0,1} \quad \quad \quad \quad \quad \quad \quad h_{1,3}^1$$

$$f_{1,3}f_{0,1} \quad \xrightarrow{h_{2,3}^2} \quad f_{0,3}$$

This is a picture rather than a diagram. Each vertex of the square is a point in $\text{Top}(X_0, X_3)$, while the upper and left edges are the indicated composites. The other edges are the homotopies shown in the previous slide.
The set of 4-simplices in $S$

For each 4-simplex, the additional datum is a map $h_{0,4} : I^3 \to \text{Top}(X_0, X_4)$ of the form...
The set of 4-simplices in $S$

For each 4-simplex, the additional datum is a map $h_{0,4} : I^3 \to \text{Top}(X_0, X_4)$ of the form

\[ h_{0,4} = h_{0,4}^{(0)} + h_{0,4}^{(1)} + h_{0,4}^{(2)} + h_{0,4}^{(3)} \]

\[ f_{3,4} f_{2,3} f_{1,2} f_{0,1} \rightarrow f_{3,4} f_{2,3} h_{0,2} \rightarrow f_{3,4} h_{1,3} f_{0,1} \rightarrow f_{3,4} h_{0,3} \rightarrow f_{3,4} f_{2,3} f_{0,2} \rightarrow f_{3,4} f_{0,3} \]

\[ f_{2,4} h_{0,2} \rightarrow h_{1,4} f_{0,1} \rightarrow f_{3,4} h_{1,3} f_{0,1} \rightarrow f_{3,4} h_{0,3} \rightarrow f_{2,4} f_{0,2} \rightarrow f_{0,4} \]

\[ h_{1,4} f_{0,1} \rightarrow f_{2,4} h_{0,2} \rightarrow f_{2,4} f_{0,2} \]

\[ f_{1,2} f_{0,1} \rightarrow f_{1,4} f_{0,1} \rightarrow f_{1,4} f_{0,1} \rightarrow f_{0,4} \]

The restrictions of $h_{0,4}$ to the left and top faces are the composite double homotopies indicated in green. The restrictions to the three faces abutting $f_{0,4}$ are double homotopies indicated in blue.
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The set of 4-simplices in $S$

For each 4-simplex, the additional datum is a map $h_{0,4} : l^3 \to \text{Top}(X_0, X_4)$ of the form

$$h_{0,4} = h_{0,4}^1 f_{3,4} + h_{0,4}^2 f_{2,4} + h_{0,4}^3 f_{1,4}$$

The restriction of $h_{0,4}$ to the left and top faces are the composite double homotopies indicated in green.
The set of 4-simplices in $S$

For each 4-simplex, the additional datum is a map $h_{0,4} : I^3 \to \text{Top}(X_0, X_4)$ of the form

$$h_{0,4} : f_{3,4}f_{2,3}f_{1,2}f_{0,1} \xrightarrow{f_{3,4}f_{2,3}h_{0,2}} f_{3,4}f_{2,3}f_{0,2} \xrightarrow{f_{3,4}h_{0,3}} f_{3,4}f_{0,3}$$

The restriction of $h_{0,4}$ to the left and top faces are the composite double homotopies indicated in green. The restrictions to the three faces abuting $f_{0,4}$ are double homotopies indicated in blue.

The set of 4-simplices in $S$

The restriction of $h_{0,4}$ to the left and top faces are the composite double homotopies indicated in green. The restrictions to the three faces abuting $f_{0,4}$ are double homotopies indicated in blue.
The set of 4-simplices in $S$ (continued)
The set of 4-simplices in $S$ (continued)

The restriction of $h_{0,4}$ to the back face (not labeled) is the composite

$$I \times I \xrightarrow{h_{2,4} \times h_{0,2}} \text{Top}(X_2, X_4) \times \text{Top}(X_0, X_2) \xrightarrow{\text{comp}} \text{Top}(X_0, X_4).$$
The set of 4-simplices in $S$ (continued)

The five labeled faces are associated with the five 3-dimensional faces of the corresponding 4-simplex in $S$.
The set of 4-simplices in $\mathcal{S}$ (continued)

The five labeled faces are associated with the five 3-dimensional faces of the corresponding 4-simplex in $\mathcal{S}$. 

The five labeled faces are associated with the five 3-dimensional faces of the corresponding 4-simplex in $\mathcal{S}$. 

The set of 4-simplices in $\mathcal{S}$ 

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The set $S_{n+1}$ for $n > 3$

For each $(n + 1)$-simplex there is a sequence of spaces and continuous maps.
The set $S_{n+1}$ for $n > 3$

For each $(n + 1)$-simplex there is a sequence of spaces and continuous maps

$$
X_0 \xrightarrow{f_{0,1}} X_1 \xrightarrow{f_{1,2}} \cdots \xrightarrow{f_{n,n+1}} X_{n+1}
$$

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The set $S_{n+1}$ for $n > 3$

For each $(n+1)$-simplex there is a sequence of spaces and continuous maps

\[
X_0 \xrightarrow{f_{0,1}} X_1 \xrightarrow{f_{1,2}} \cdots \xrightarrow{f_{n,n+1}} X_{n+1}
\]

and a map

\[
l^n \xrightarrow{h_{0,n}} \text{Top}(X_0, X_{n+1})
\]

\[
(0, \ldots, 0) \xrightarrow{} f_{n,n+1} \cdots f_{0,1}
\]

\[
(1, \ldots, 1) \xrightarrow{} f_{0,n+1}
\]
The set $S_{n+1}$ for $n > 3$

For each $(n + 1)$-simplex there is a sequence of spaces and continuous maps

$$X_0 \overset{f_{0,1}}{\longrightarrow} X_1 \overset{f_{1,2}}{\longrightarrow} \cdots \overset{f_{n,n+1}}{\longrightarrow} X_{n+1}$$

and a map

$$I^n \overset{h_{0,n}}{\longrightarrow} \text{Top}(X_0, X_{n+1})$$

$$(0, \ldots, 0) \overset{f_{n,n+1} \cdots f_{0,1}}{\longrightarrow}$$

$$(1, \ldots, 1) \overset{f_{0,n+1}}{\longrightarrow}$$

We refer to these two points as the left and right vertices of the $n$-cube,
The set $S_{n+1}$ for $n > 3$

For each $(n + 1)$-simplex there is a sequence of spaces and continuous maps

$$X_0 \xrightarrow{f_{0,1}} X_1 \xrightarrow{f_{1,2}} \cdots \xrightarrow{f_{n,n+1}} X_{n+1}$$

and a map

$$I^n \xrightarrow{h_{0,n}} \text{Top}(X_0, X_{n+1})$$

$$(0, \ldots, 0) \xrightarrow{f_{n,n+1} \cdots f_{0,1}} f_{0,n+1}$$

$$(1, \ldots, 1) \xleftarrow{f_{0,n+1} \cdots f_{n,n+1}}$$

We refer to these two points as the left and right vertices of the $n$-cube, and the $n$ faces meeting each of them as the left and right faces.
The set $S_{n+1}$ for $n > 3$

For each $(n+1)$-simplex there is a sequence of spaces and continuous maps

$$X_0 \xrightarrow{f_{0,1}} X_1 \xrightarrow{f_{1,2}} \cdots \xrightarrow{f_{n,n+1}} X_{n+1}$$

and a map

$$I^n \xrightarrow{h_{0,n}} \text{Top}(X_0, X_{n+1})$$

$$(0, \ldots, 0) \xrightarrow{f_{n,n+1} \cdots f_{0,1}} (1, \ldots, 1)$$

We refer to these two points as the left and right vertices of the $n$-cube, and the $n$ faces meeting each of them as the left and right faces.

The $n+2$ faces of the associated $(n+1)$-simplex correspond to the $n$ right faces of this cube,
The set $S_{n+1}$ for $n > 3$

For each $(n+1)$-simplex there is a sequence of spaces and continuous maps

$$X_0 \xrightarrow{f_{0,1}} X_1 \xrightarrow{f_{1,2}} \cdots \xrightarrow{f_{n,n+1}} X_{n+1}$$

and a map

$$I^n \xrightarrow{h_{0,n}} \text{Top}(X_0, X_{n+1})$$

$$(0, \ldots, 0) \mapsto f_{n,n+1} \cdots f_{0,1}$$

$$(1, \ldots, 1) \mapsto f_{0,n+1}$$

We refer to these two points as the left and right vertices of the $n$-cube, and the $n$ faces meeting each of them as the left and right faces.

The $n+2$ faces of the associated $(n+1)$-simplex correspond to the $n$ right faces of this cube, along with the two left faces.
The set $S_{n+1}$ for $n > 3$

For each $(n + 1)$-simplex there is a sequence of spaces and continuous maps

$$X_0 \xrightarrow{f_{0,1}} X_1 \xrightarrow{f_{1,2}} \cdots \xrightarrow{f_{n,n+1}} X_{n+1}$$

and a map

$$I^n \xrightarrow{h_{0,n}} \text{Top}(X_0, X_{n+1})$$

$$\begin{array}{c}
(0, \ldots, 0) \\
\downarrow \\
(1, \ldots, 1)
\end{array} \xrightarrow{f_{n,n+1} \cdots f_{0,1}} \xrightarrow{f_{0,n+1}}$$

We refer to these two points as the left and right vertices of the $n$-cube, and the $n$ faces meeting each of them as the left and right faces.

The $n + 2$ faces of the associated $(n + 1)$-simplex correspond to the $n$ right faces of this cube, along with the two left faces

$$\{(t_1, \ldots, t_{n-1}, 0)\} \quad \text{and} \quad \{(0, t_2, \ldots, t_n)\}.$$
To sum up, the $\infty$-category $S$ of topological spaces is a simplicial set in which
To sum up, the ∞-category $S$ of topological spaces is a simplicial set in which

- there is a vertex for each topological space in $\text{Top}$,
To sum up, the $\infty$-category $S$ of topological spaces is a simplicial set in which

- there is a vertex for each topological space in $\text{Top}$,
- there is an edge for each continuous map, and

This construction does not require any choices.
To sum up, the $\infty$-category $S$ of topological spaces is a simplicial set in which

- there is a vertex for each topological space in $\text{Top}$,
- there is an edge for each continuous map, and
- for $n > 0$, there is an $(n + 1)$-simplex for each sequence of spaces and continuous maps

\[ X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} X_{n+1} \quad \text{and} \]

The set $S_{n+1}$ for $n > 3$ (continued)
To sum up, the $\infty$-category $S$ of topological spaces is a simplicial set in which

- there is a vertex for each topological space in $\text{Top}$,
- there is an edge for each continuous map, and
- for $n > 0$, there is an $(n + 1)$-simplex for each sequence of spaces and continuous maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} X_{n+1}$$

and

- each map $h_n : I^n \rightarrow \text{Top}(X_0, X_{n+1})$ meeting certain boundary conditions described above.
To sum up, the ∞-category $S$ of topological spaces is a simplicial set in which

- there is a vertex for each topological space in $\text{Top}$,
- there is an edge for each continuous map, and
- for $n > 0$, there is an $(n + 1)$-simplex for each sequence of spaces and continuous maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} X_{n+1}$$

- each map $h_n : I^n \to \text{Top}(X_0, X_{n+1})$ meeting certain boundary conditions described above.

To repeat, there is an $(n + 1)$-simplex for every suitable datum.
To sum up, the $\infty$-category $S$ of topological spaces is a simplicial set in which

- there is a vertex for each topological space in $\text{Top}$,
- there is an edge for each continuous map, and
- for $n > 0$, there is an $(n + 1)$-simplex for each sequence of spaces and continuous maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} X_{n+1}$$

and

- each map $h_n : I^n \rightarrow \text{Top}(X_0, X_{n+1})$ meeting certain boundary conditions described above.

To repeat, there is an $(n + 1)$-simplex for every suitable datum. This construction does not require any choices.
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\begin{array}{ccc}
S^{n-1} & \rightarrow & D^n \\
\downarrow & & \downarrow \\
D^n & \rightarrow & * \\
\end{array}
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\downarrow & \downarrow & \\
* & & 
\end{array}
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$$
\begin{array}{ccc}
S^{n-1} & \rightarrow & D^n \\
\downarrow & & \downarrow \\
D^n & & \ast
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
S^{n-1} & \rightarrow & \ast \\
\downarrow & & \downarrow \\
\ast & & \ast
\end{array}
$$

where the maps in the left diagram are inclusions of the boundary.
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A colimit in $S$
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$$S^{n-1} \to D^n \quad \text{and} \quad S^{n-1} \to \ast$$

where the maps in the left diagram are inclusions of the boundary. The two diagrams are homotopy equivalent but have distinct pushouts, namely $S^n$ and $\ast$. What to do?
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A colimit in $S$ (continued)

$S^{n-1} \longrightarrow D^n$ and $S^{n-1} \longrightarrow *$

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One solution is to define a model structure on the category of pushout diagrams in $\text{Top}$,
A colimit in $S$ (continued)

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One solution is to define a model structure on the category of pushout diagrams in Top, in which equivalences and fibrations are levelwise equivalences and fibrations, and cofibrations are defined in terms of lifting properties.
A colimit in $S$ (continued)

\[
\begin{array}{ccc}
S^{n-1} & \rightarrow & D^n \\
\downarrow & & \downarrow \\
D^n & & * \\
\end{array}
\]

and

\[
\begin{array}{ccc}
S^{n-1} & \rightarrow & * \\
\downarrow & & \\
* & & \\
\end{array}
\]

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A colimit in $S$ (continued)

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The two diagrams are homotopy equivalent but have distinct pushouts, namely $S^n$ and $\ast$. What to do?

Another solution is to develop the theory of homotopy limits and colimits as in the yellow monster of Bousfield-Kan [BK72]. It turns out that the homotopy colimit of each diagram above is $S^n$. 
A colimit in $\mathcal{S}$ (continued)

In an ordinary category $\mathcal{C}$, the colimit of a diagram $\rho$ is an initial object in the category of objects equipped with compatible maps from all the objects in $\rho$,
A colimit in $S$ (continued)

In an ordinary category $C$, the colimit of a diagram $\rho$ is an initial object in the category of objects equipped with compatible maps from all the objects in $\rho$, which we denote by $C_\rho/$, the category of objects under $\rho$. 
In an ordinary category $\mathcal{C}$, the colimit of a diagram $p$ is an initial object in the category of objects equipped with compatible maps from all the objects in $p$, which we denote by $\mathcal{C}_p/$, the category of objects under $p$.

In an $\infty$-category $\mathcal{C}$, an initial object $X$ is one for which the mapping space $\mathcal{C}(X, Y)$ is contractible for each object $Y$. 
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  ● ← ● → ●.
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References
A colimit in $\mathcal{S}$ (continued)

$$\begin{align*}
S^{n-1} & \longrightarrow D^n & \text{and} & & S^{n-1} & \longrightarrow * \\
\downarrow & & & & \downarrow \\
D^n & & & & * 
\end{align*}$$

Let $p$ be the diagram on the right. We are looking for an initial object in $\mathcal{S}/p$. An object in $\mathcal{S}/p$ is a diagram $\ast \& \& \& \& S^{n-1}/o/o/o Y$, which is a pair of 2-simplices in $\mathcal{S}$. This amounts to a map $f: S^{n-1} \to Y$ equipped with a pair of null homotopies. These define extensions of $f$ to the northern and southern hemispheres of $S^n$. It follows that $S^n$, which is the homotopy colimit of $p$ in Top, is the ordinary colimit of $p$ in $\mathcal{S}$.

More details can be found in [HTT, 4.2.4].
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A colimit in \( \mathcal{S} \) (continued)

\[
\begin{array}{ccc}
S^{n-1} & \longrightarrow & D^n \\
\downarrow & & \downarrow \\
D^n & & * \\
\end{array}
\]

and

\[
\begin{array}{ccc}
S^{n-1} & \longrightarrow & * \\
\downarrow & & \\
* & & \\
\end{array}
\]

Let \( p \) be the diagram on the right.
A colimit in $S$ (continued)

Let $p$ be the diagram on the right. We are looking for an initial object in $S/p$. 

$$
\begin{align*}
S^{n-1} & \longrightarrow D^n & \text{and} & \quad S^{n-1} & \longrightarrow * \\
\downarrow & \quad & \downarrow & \quad & \downarrow \\
D^n & \quad & * 
\end{align*}
$$
A colimit in $S$ (continued)

Let $p$ be the diagram on the right. We are looking for an initial object in $S/p$. An object in $S/p$ is a diagram

\[ S^{n-1} \xrightarrow{f} Y \xleftarrow{} \ast \]

\[ D^n \xrightarrow{} \ast \]

\[ S^n \xrightarrow{f} \ast \]

\[ D^n \]

\[ \ast \]

\[ Y, \]

This amounts to a map $f: S^{n-1} \to Y$ equipped with a pair of null homotopies. These define extensions of $f$ to the northern and southern hemispheres of $S^n$, which is the homotopy colimit of $p$ in $\text{Top}$, is the ordinary colimit of $p$ in $S$. More details can be found in [HTT, 4.2.4].
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$$
\begin{array}{ccc}
\ast & \rightarrow & \ast \\
& \downarrow f & \\
& Y, & \\
\end{array}
$$

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A colimit in $S$ (continued)

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S^{n-1} \longrightarrow D^n \\
\downarrow \\
D^n
\end{array}
\quad \text{and} \quad
\begin{array}{c}
S^{n-1} \longrightarrow * \\
\downarrow \\
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Y
\end{array}
$$

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\[\downarrow \quad \quad \downarrow \quad \quad \downarrow \]

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When we enlarge the class of weak equivalences (in the category of spaces or spectra) to those maps inducing an isomorphism in Morava \(E\)-theory (or Morava \(K\)-theory) for a fixed prime \(p\) and height \(n\), **this fibrant replacement functor is the \(L_n\) (or \(L_{K(n)}\)) of chromatic homotopy theory.**
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[HTT, Proposition 5.5.4.15] is statement about an analog of Bousfield localization. The input is a presentable $\infty$-category $C$ with a set of morphisms $S$ that are meant to be made into weak equivalences.
What is an $\infty$-category? (continued)

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In [HTT, Definition 5.5.4.1] an object $Z$ is said to be $S$-local if each morphism $s : X \to Y$ in $S$ induces a homotopy equivalence $C(Y, Z) \to C(X, Z)$.
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**Bousfield localization in $\infty$-categories (continued)**

[HTT, Proposition 5.5.4.15] is statement about an analog of Bousfield localization. The input is a presentable $\infty$-category $C$ with a set of morphisms $S$ that are meant to be made into weak equivalences. **Presentable** means that $C$ has small colimits and every object is a colimit of small objects.

In [HTT, Definition 5.5.4.1] an object $Z$ is said to be $S$-local if each morphism $s : X \to Y$ in $S$ induces a homotopy equivalence $C(Y, Z) \to C(X, Z)$. A morphism $s : A \to B$ is an $S$-equivalence if it induces a homotopy equivalence $C(B, Z) \to C(A, Z)$ for each $S$-local object $Z$. 

---
Let $\bar{S}$ be the set of all $S$-equivalences.
Let $\tilde{S}$ be the set of all $S$-equivalences. It can be explicitly constructed from $S$. 
Let $\overline{S}$ be the set of all $S$-equivalences. It can be explicitly constructed from $S$. Let $C'$ be the full subcategory of $S$-local objects.
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1. For each object $X \in \mathcal{C}$, there exists a morphism $s : X \to X'$ such that $X'$ is $S$-local and $s$ belongs to $\mathcal{S}$.
2. The $\infty$-category $\mathcal{C}'$ is presentable.
Let $\overline{S}$ be the set of all $S$-equivalences. It can be explicitly constructed from $S$. Let $C'$ be the full subcategory of $S$-local objects. Then

1. For each object $X \in C$, there exists a morphism $s : X \to X'$ such that $X'$ is $S$-local and $s$ belongs to $\overline{S}$.
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3. The inclusion functor $C' \subseteq C$ has a left adjoint $L$. 

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2. The $\infty$-category $C'$ is presentable.

3. The inclusion functor $C' \subseteq C$ has a left adjoint $L$. This is the analog of Bousfield’s fibrant replacement functor in model category theory.
The passage from $S$, the $\infty$-category of spaces, to $Sp$, the $\infty$-category of spectra, is described by Lurie in [HA, 1.4].
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of $\infty$-categories and functors.
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$\text{Sp}$ is the homotopy limit of the tower

$$
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$$

$X_2 \quad X_1 \quad X_0$
The \(∞\)-category of spectra (continued)

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To unpack this definition, note that a vertex in this homotopy limit (meaning an object in the \(∞\)-category \(Sp\)) consists of a sequence of pointed spaces \(X_0, X_1, X_2, \ldots\),
The $\infty$-category of spectra (continued)

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To unpack this definition, note that a vertex in this homotopy limit (meaning an object in the $\infty$-category $\mathbf{Sp}$) consists of a sequence of pointed spaces $X_0, X_1, X_2, \ldots$, along with weak equivalences $X_i \to \Omega X_{i+1}$.
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The $\infty$-category of spectra (continued)

The $\infty$-category $\mathcal{Sp}$ satisfies the following, which is [HA, Definition 1.1.1.9].
The $\infty$-category of spectra (continued)

The $\infty$-category $\text{Sp}$ satisfies the following, which is [HA, Definition 1.1.1.9].

**Definition**

An $\infty$-category $\mathcal{C}$ is **stable** if

1. It is pointed.
2. For each morphism $f: X \to Y$ there are pullback and pushout diagrams $W / f / X$ and $X f / / Y$ and $X f / / Z$, the fiber and cofiber sequences of $f$.
3. A diagram of the above form is a pushout if and only if it is a pullback, i.e., fiber sequences and cofiber sequences are the same.
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Thank you and Happy Birthday Andy!
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