Hiking in the Alps: $C_p$-fixed points of Lubin-Tate spectra

Doug Ravenel
University of Rochester

March 23, 2023
This is joint work with Mike Hill and Mike Hopkins.
This is joint work with Mike Hill and Mike Hopkins.

We were thinking about this problem in 2007-8,
This is joint work with Mike Hill and Mike Hopkins.

We were thinking about this problem in 2007-8, but we got distracted by the Kervaire invariant.
This is joint work with Mike Hill and Mike Hopkins.

We were thinking about this problem in 2007-8, but we got distracted by the Kervaire invariant.

For several years after that we could not remember what we had proved about $C_p$ fixed points.
Historical introduction

This is joint work with Mike Hill and Mike Hopkins.

We were thinking about this problem in 2007-8, but we got distracted by the Kervaire invariant.

For several years after that we could not remember what we had proved about $C_p$ fixed points.

Fortunately Mark Behrens took some careful notes for us.
A central object of study in chromatic homotopy theory is $S^0_{K(n)}$, the Bousfield localization of the sphere spectrum $S^0$ with respect to the $n$th Morava K-theory $K(n)$. A theorem of Goerss-Hopkins-Miller identifies it as $E_{hG_n}$, the homotopy fixed point set of the action of the $n$th extended Morava stabilizer group $G_n$ on the $n$th Lubin-Tate spectrum $E_n$, also known as Morava E-theory. For any closed subgroup $H \subseteq G_n$, one also has a homotopy fixed point spectrum $E_{hH}$ under $S^0_{K(n)}$. $G_n$ is known to have a subgroup of order $p$ when $p - 1$ divides $n$. Our goal is to study $E_{hC_p}(p - 1)^f$ for positive integers $f$. 
A central object of study in chromatic homotopy theory is $S^0_{K(n)}$, the Bousfield localization of the sphere spectrum $S^0$ with respect to the $n$th Morava K-theory $K(n)$. 
A central object of study in chromatic homotopy theory is $S^0_{K(n)}$, the Bousfield localization of the sphere spectrum $S^0$ with respect to the $n$th Morava K-theory $K(n)$.

A theorem of Goerss-Hopkins-Miller identifies it as $E^{hG_n}_n$, where $G_n$ is known to have a subgroup of order $p$ when $p-1$ divides $n$. Our goal is to study $E^{hC_p}_{p^{p-1}}$ for positive integers $f$. 
A central object of study in chromatic homotopy theory is $S^0_{K(n)}$, the Bousfield localization of the sphere spectrum $S^0$ with respect to the $n$th Morava K-theory $K(n)$.

A theorem of Goerss-Hopkins-Miller identifies it as $E_{hG_n}^n$, the homotopy fixed point set of the action of the $n$th extended Morava stabilizer group $G_n$. 


A central object of study in chromatic homotopy theory is $S^0_{K(n)}$, the Bousfield localization of the sphere spectrum $S^0$ with respect to the $n$th Morava K-theory $K(n)$.

A theorem of Goerss-Hopkins-Miller identifies it as $E_n^{hG_n}$, the homotopy fixed point set of the action of the $n$th extended Morava stabilizer group $G_n$ on the $n$th Lubin-Tate spectrum $E_n$. 
A central object of study in chromatic homotopy theory is $S^0_{K(n)}$, the Bousfield localization of the sphere spectrum $S^0$ with respect to the $n$th Morava K-theory $K(n)$.

A theorem of Goerss-Hopkins-Miller identifies it as $E^n_{h\mathbb{G}_n}$, the homotopy fixed point set of the action of the $n$th extended Morava stabilizer group $\mathbb{G}_n$ on the $n$th Lubin-Tate spectrum $E_n$, also known as Morava E-theory.
A central object of study in chromatic homotopy theory is $S^0_{K(n)}$, the Bousfield localization of the sphere spectrum $S^0$ with respect to the $n$th Morava K-theory $K(n)$.

A theorem of Goerss-Hopkins-Miller identifies it as $E_n^{hG_n}$, the homotopy fixed point set of the action of the $n$th extended Morava stabilizer group $G_n$ on the $n$th Lubin-Tate spectrum $E_n$, also known as Morava E-theory.

For any closed subgroup $H \subseteq G_n$, one also has a homotopy fixed point spectrum $E_n^{hH}$ under $S^0_{K(n)}$. 
A central object of study in chromatic homotopy theory is $S^0_{K(n)}$, the Bousfield localization of the sphere spectrum $S^0$ with respect to the $n$th Morava K-theory $K(n)$.

A theorem of Goerss-Hopkins-Miller identifies it as $E^n_{hG_n}$, the homotopy fixed point set of the action of the $n$th extended Morava stabilizer group $G_n$ on the $n$th Lubin-Tate spectrum $E_n$, also known as Morava E-theory.

For any closed subgroup $H \subseteq G_n$, one also has a homotopy fixed point spectrum $E^n_{hH}$ under $S^0_{K(n)}$. $G_n$ is known to have a subgroup of order $p$ when $p - 1$ divides $n$. 
A central object of study in chromatic homotopy theory is $S^0_{K(n)}$, the Bousfield localization of the sphere spectrum $S^0$ with respect to the $n$th Morava K-theory $K(n)$.

A theorem of Goerss-Hopkins-Miller identifies it as $E_n^{hG_n}$, the homotopy fixed point set of the action of the $n$th extended Morava stabilizer group $G_n$ on the $n$th Lubin-Tate spectrum $E_n$, also known as Morava E-theory.

For any closed subgroup $H \subseteq G_n$, one also has a homotopy fixed point spectrum $E_n^{hH}$ under $S^0_{K(n)}$. $G_n$ is known to have a subgroup of order $p$ when $p - 1$ divides $n$. Our goal is to study $E_n^{hC_{p}^{f}}$ for positive integers $f$. 

$K(n)$ localization
Properties of $E_n$ and $G_n$

$E_n$ is a complex oriented 2-periodic $E_\infty$ (meaning strictly commutative) ring spectrum.
Properties of $E_n$ and $G_n$

$E_n$ is a complex oriented 2-periodic $E_\infty$ (meaning strictly commutative) ring spectrum. Its homotopy groups comprise the graded ring

$$\pi_* E_n = W[[u_1, \ldots u_{n-1}]][u^{-1}]^\wedge$$
Properties of $E_n$ and $G_n$

$E_n$ is a complex oriented 2-periodic $E_\infty$ (meaning strictly commutative) ring spectrum. Its homotopy groups comprise the graded ring

$$\pi_* E_n = W[[u_1, \ldots u_{n-1}][u^{\pm 1}]^\wedge$$

where

- $W$ denotes the Witt ring $W(\mathbb{F}_{p^n})$ of the field with $p^n$ elements.
Properties of $E_n$ and $G_n$

$E_n$ is a complex oriented 2-periodic $E_\infty$ (meaning strictly commutative) ring spectrum. Its homotopy groups comprise the graded ring

$$\pi_* E_n = W[[u_1, \ldots, u_{n-1}]] [u^{\pm 1}]^\wedge$$

where

- $W$ denotes the Witt ring $W(\mathbb{F}_p^n)$ of the field with $p^n$ elements. This is a degree $n$ extension of the ring $\mathbb{Z}_p$ of $p$-adic integers

- $\wedge$ denotes completion with respect to the maximal ideal $I_n = (p, u_1, \ldots, u_{n-1})$. 
Properties of $E_n$ and $G_n$

$E_n$ is a complex oriented 2-periodic $E_\infty$ (meaning strictly commutative) ring spectrum. Its homotopy groups comprise the graded ring

$$\pi_* E_n = W[u_1, \ldots, u_{n-1}][u^{\pm 1}]^\wedge$$

where

- $W$ denotes the Witt ring $W(\mathbb{F}_{p^n})$ of the field with $p^n$ elements. This is a degree $n$ extension of the ring $\mathbb{Z}_p$ of $p$-adic integers that lifts $\mathbb{F}_{p^n}$ as a degree $n$ extension of the prime field $\mathbb{F}_p$. 


Properties of $E_n$ and $G_n$

$E_n$ is a complex oriented 2-periodic $E_\infty$ (meaning strictly commutative) ring spectrum. Its homotopy groups comprise the graded ring

$$\pi_* E_n = W[[u_1, \ldots, u_{n-1}]] [u^{\pm 1}]^\wedge$$

where

- $W$ denotes the Witt ring $W(F_p^n)$ of the field with $p^n$ elements. This is a degree $n$ extension of the ring $\mathbb{Z}_p$ of $p$-adic integers that lifts $F_p^n$ as a degree $n$ extension of the prime field $F_p$.
- The power series variables $u_i$ each have degree 0.
Properties of $E_n$ and $G_n$

$E_n$ is a complex oriented 2-periodic $E_\infty$ (meaning strictly commutative) ring spectrum. Its homotopy groups comprise the graded ring

$$\pi_* E_n = W[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]^\wedge$$

where

- $W$ denotes the Witt ring $W(\mathbb{F}_{p^n})$ of the field with $p^n$ elements. This is a degree $n$ extension of the ring $\mathbb{Z}_p$ of $p$-adic integers that lifts $\mathbb{F}_{p^n}$ as a degree $n$ extension of the prime field $\mathbb{F}_p$.
- The power series variables $u_i$ each have degree 0.
- The invertible variable $u$ has degree $-2$. 
Properties of $E_n$ and $G_n$

$E_n$ is a complex oriented 2-periodic $E_\infty$ (meaning strictly commutative) ring spectrum. Its homotopy groups comprise the graded ring

$$\pi_* E_n = W[u_1, \ldots, u_{n-1}][u^{\pm 1}]^\wedge$$

where

- $W$ denotes the Witt ring $W(\mathbb{F}_{p^n})$ of the field with $p^n$ elements. This is a degree $n$ extension of the ring $\mathbb{Z}_p$ of $p$-adic integers that lifts $\mathbb{F}_{p^n}$ as a degree $n$ extension of the prime field $\mathbb{F}_p$.
- The power series variables $u_i$ each have degree 0.
- The invertible variable $u$ has degree $-2$.
- The symbol $^\wedge$ at the end denotes completion with respect to the maximal ideal $I_n = (p, u_1, \ldots, u_{n-1})$. 
Properties of $E_n$ and $G_n$ (continued)

$$\pi_* E_n = W[ u_1, \ldots, u_{n-1} ][ u^{\pm 1} ]^\wedge$$
Here is an alternate description of this ring as a completed localization of a graded polynomial ring.

\[ \pi_* E_n = W[ u_1, \ldots, u_{n-1} ][ u^{\pm 1} ]^\wedge \]
Properties of $E_n$ and $G_n$ (continued)

\[
\pi_* E_n = W[u_1, \ldots, u_{n-1}] [u^{\pm 1}]^\wedge
\]

Here is an alternate description of this ring as a completed localization of a graded polynomial ring.

- Let $R_n = W[x_0, \ldots, x_{n-1}]$ with $|x_i| = -2$. 
Properties of $E_n$ and $G_n$ (continued)

\[ \pi_* E_n = W[u_1, \ldots, u_{n-1}][u^\pm 1] \]

Here is an alternate description of this ring as a completed localization of a graded polynomial ring.

- Let $R_n = W[x_0, \ldots, x_{n-1}]$ with $|x_i| = -2$.
- Invert $\Phi := x_0 \cdot \cdots \cdot x_{n-1}$.

In short, we start with a graded polynomial local ring, invert each of its specified generators, and then complete at its graded maximal ideal. We will come back to this later.
Properties of $E_n$ and $\mathbb{G}_n$ (continued)

$$\pi_* E_n = W[u_1, \ldots, u_{n-1}][u^\pm 1]^\wedge$$

Here is an alternate description of this ring as a completed localization of a graded polynomial ring.

- Let $R_n = W[x_0, \ldots, x_{n-1}]$ with $|x_i| = -2$.
- Invert $\Phi := x_0 \cdots x_{n-1}$, define $u_i := (x_0/x_i) - 1$ for $1 \leq i \leq n - 1$, ...
Properties of $E_n$ and $G_n$ (continued)

$$\pi_* E_n = W[u_1, \ldots, u_{n-1}][u^{\pm 1}]^\wedge$$

Here is an alternate description of this ring as a completed localization of a graded polynomial ring.

- Let $R_n = W[x_0, \ldots, x_{n-1}]$ with $|x_i| = -2$.
- Invert $\Phi := x_0 \cdots x_{n-1}$, define $u_i := (x_0/x_i) - 1$ for $1 \leq i \leq n - 1$, and $u := x_0^n/(x_1 \cdots x_{n-1})$. 
Properties of $E_n$ and $G_n$ (continued)

$$\pi_* E_n = W[ u_1, \ldots, u_{n-1} ][u^{\pm 1}]$$

Here is an alternate description of this ring as a completed localization of a graded polynomial ring.

- Let $R_n = W[x_0, \ldots, x_{n-1}]$ with $|x_i| = -2$.
- Invert $\Phi := x_0 \cdots x_{n-1}$, define $u_i := (x_0/x_i) - 1$ for $1 \leq i \leq n - 1$, and $u := x_0^n/(x_1 \cdots x_{n-1})$. Then we have

$$R_n[\Phi^{\pm 1}] = W[u_1, \ldots, u_{n-1}][u^{\pm 1}]$$
Properties of $E_n$ and $G_n$ (continued)

$$\pi_* E_n = \mathcal{W}[u_1, \ldots, u_{n-1}][u^{\pm 1}]$$

Here is an alternate description of this ring as a completed localization of a graded polynomial ring.

- Let $R_n = \mathcal{W}[x_0, \ldots, x_{n-1}]$ with $|x_i| = -2$.
- Invert $\Phi := x_0 \cdots x_{n-1}$, define $u_i := (x_0/x_i) - 1$ for $1 \leq i \leq n - 1$, and $u := x_0^n/(x_1 \cdots x_{n-1})$. Then we have

$$R_n[\Phi^{\pm 1}] = \mathcal{W}[u_1, \ldots, u_{n-1}][u^{\pm 1}].$$

- Let $m$ be the kernel of the map $R_n[\Phi^{\pm 1}] \to \mathbb{F}_p [u^{\pm 1}]$ sending each $x_i$ to $u$. 

Doug Ravenel

Hiking in the Alps: $C_p$ fixed points of Lubin-Tate spectra

Historical introduction

$K(n)$ localization

Properties of $E_n$ and $G_n$

Finding a root of unity

Group cohomology

The main theorem

A classical example

$\text{TMF} \text{ at } p = 3$

Larger primes
Properties of $E_n$ and $G_n$ (continued)

$$\pi_* E_n = W[u_1, \ldots, u_{n-1}][u^{\pm 1}]^\wedge$$

Here is an alternate description of this ring as a **completed localization** of a graded polynomial ring.

- Let $R_n = W[x_0, \ldots, x_{n-1}]$ with $|x_i| = -2$.
- Invert $\Phi := x_0 \cdots x_{n-1}$, define $u_i := (x_0/x_i) - 1$ for $1 \leq i \leq n - 1$, and $u := x_0^n/(x_1 \cdots x_{n-1})$. Then we have

$$R_n[\Phi^{\pm 1}] = W[u_1, \ldots, u_{n-1}][u^{\pm 1}].$$

- Let $\mathfrak{m}$ be the kernel of the map $R_n[\Phi^{\pm 1}] \to \mathbb{F}_{p^n}[u^{\pm 1}]$ sending each $x_i$ to $u$. 

Doug Ravenel

Hiking in the Alps: $G_p$ fixed points of Lubin-Tate spectra

Historical introduction

$K(n)$ localization

Properties of $E_n$ and $G_n$

Finding a root of unity

Group cohomology

The main theorem

A classical example

$TMF$ at $p = 3$

Larger primes
Properties of $E_n$ and $G_n$ (continued)

\[ \pi_* E_n = W[u_1, \ldots, u_{n-1}] [u^{\pm 1}]^\wedge \]

Here is an alternate description of this ring as a completed localization of a graded polynomial ring.

- Let $R_n = W[x_0, \ldots, x_{n-1}]$ with $|x_i| = -2$.
- Invert $\Phi := x_0 \cdots x_{n-1}$, define $u_i := (x_0/x_i)^{-1}$ for $1 \leq i \leq n-1$, and $u := x_0^n/(x_1 \ldots x_{n-1})$. Then we have

\[ R_n[\Phi^{\pm 1}] = W[u_1, \ldots, u_{n-1}] [u^{\pm 1}] \]

- Let $m$ be the kernel of the map $R_n[\Phi^{\pm 1}] \to \mathbb{F}_{p^n}[u^{\pm 1}]$ sending each $x_i$ to $u$. Then complete with respect to $m$. 
Properties of $E_n$ and $G_n$ (continued)

$$\pi_* E_n = W[u_1, \ldots, u_{n-1}][u^{\pm 1}]^\wedge$$

Here is an alternate description of this ring as a completed localization of a graded polynomial ring.

- Let $R_n = W[x_0, \ldots, x_{n-1}]$ with $|x_i| = -2$.
- Invert $\Phi := x_0 \cdots x_{n-1}$, define $u_i := (x_0/x_i)^{-1}$ for $1 \leq i \leq n-1$, and $u := x_0^n/(x_1 \cdots x_{n-1})$. Then we have

$$R_n[\Phi^{\pm 1}] = W[u_1, \ldots, u_{n-1}][u^{\pm 1}].$$

- Let $m$ be the kernel of the map $R_n[\Phi^{\pm 1}] \to \mathbb{F}_p[u^{\pm 1}]$ sending each $x_i$ to $u$. Then complete with respect to $m$. The result is isomorphic to $\pi_* E_n$. 

In short, we start with a graded polynomial local ring, invert each of its specified generators, and then complete at its graded maximal ideal. We will come back to this later.
Properties of $E_n$ and $G_n$ (continued)

\[ \pi_* E_n = W[u_1, \ldots, u_{n-1}][u^{\pm 1}]^{\wedge} \]

Here is an alternate description of this ring as a completed localization of a graded polynomial ring.

- Let $R_n = W[x_0, \ldots, x_{n-1}]$ with $|x_i| = -2$.
- Invert $\Phi := x_0 \cdots x_{n-1}$, define $u_i := (x_0/x_i) - 1$ for $1 \leq i \leq n - 1$, and $u := x_0^n/(x_1 \cdots x_{n-1})$. Then we have

\[ R_n[\Phi^{\pm 1}] = W[u_1, \ldots, u_{n-1}][u^{\pm 1}]. \]

- Let $m$ be the kernel of the map $R_n[\Phi^{\pm 1}] \to \mathbb{F}_p^n[u^{\pm 1}]$ sending each $x_i$ to $u$. Then complete with respect to $m$. The result is isomorphic to $\pi_* E_n$.

In short, we start with a graded polynomial local ring,

\[ \pi_* E_n = W[u_1, \ldots, u_{n-1}][u^{\pm 1}]^{\wedge} \]
Properties of $E_n$ and $G_n$ (continued)

\[ \pi_* E_n = W[u_1, \ldots, u_{n-1}][u^{\pm 1}]^\wedge \]

Here is an alternate description of this ring as a completed localization of a graded polynomial ring.

- Let $R_n = W[x_0, \ldots, x_{n-1}]$ with $|x_i| = -2$.
- Invert $\Phi := x_0 \cdots x_{n-1}$, define $u_i := (x_0/x_i) - 1$ for $1 \leq i \leq n - 1$, and $u := x_0^n/(x_1 \cdots x_{n-1})$. Then we have

\[ R_n[\Phi^{\pm 1}] = W[u_1, \ldots, u_{n-1}][u^{\pm 1}] \]

- Let $m$ be the kernel of the map $R_n[\Phi^{\pm 1}] \to \mathbb{F}_p^n[u^{\pm 1}]$ sending each $x_i$ to $u$. Then complete with respect to $m$. The result is isomorphic to $\pi_* E_n$.

In short, we start with a graded polynomial local ring, invert each of its specified generators,
Properties of $E_n$ and $G_n$ (continued)

\[
\pi_* E_n = W[u_1, \ldots, u_{n-1}]|u^{\pm 1}|
\]

Here is an alternate description of this ring as a completed localization of a graded polynomial ring.

- Let $R_n = W[x_0, \ldots, x_{n-1}]$ with $|x_i| = -2$.
- Invert $\Phi := x_0 \cdots x_{n-1}$, define $u_i := (x_0/x_i) - 1$ for $1 \leq i \leq n-1$, and $u := x_0^n/(x_1 \cdots x_{n-1})$. Then we have

\[
R_n[\Phi^{\pm 1}] = W[u_1, \ldots, u_{n-1}]|u^{\pm 1}|.
\]

- Let $m$ be the kernel of the map $R_n[\Phi^{\pm 1}] \to \mathbb{F}_p^n[u^{\pm 1}]$ sending each $x_i$ to $u$. Then complete with respect to $m$. The result is isomorphic to $\pi_* E_n$.

In short, we start with a graded polynomial local ring, invert each of its specified generators, and then complete at its graded maximal ideal.
Properties of $E_n$ and $G_n$ (continued)

\[ \pi_* E_n = W[u_1, \ldots, u_{n-1}][u^{\pm 1}] \]

Here is an alternate description of this ring as a completed localization of a graded polynomial ring.

- Let $R_n = W[x_0, \ldots, x_{n-1}]$ with $|x_i| = -2$.
- Invert $\Phi := x_0 \cdots x_{n-1}$, define $u_i := (x_0/x_i) - 1$ for $1 \leq i \leq n - 1$, and $u := x_0^n/(x_1 \ldots x_{n-1})$. Then we have
  \[ R_n[\Phi^{\pm 1}] = W[u_1, \ldots, u_{n-1}][u^{\pm 1}] \]

- Let $m$ be the kernel of the map $R_n[\Phi^{\pm 1}] \to \mathbb{F}_p^n[u^{\pm 1}]$ sending each $x_i$ to $u$. Then complete with respect to $m$. The result is isomorphic to $\pi_* E_n$.

In short, we start with a graded polynomial local ring, invert each of its specified generators, and then complete at its graded maximal ideal. We will come back to this later.
The extended Morava stabilizer group $\mathbb{G}_n$ is related to the automorphism group $\mathbb{S}_n$ of the Honda height $n$ formal group law $F_n$ over $\mathbb{F}_{p^n}$. 

It is known that this group does change if we enlarge the field over which $F_n$ is defined. To describe $\mathbb{G}_n$, we describe the endomorphism ring of $F_n$, $\text{End}(F_n)$.

The Frobenius automorphism, the $p$th power map of $\mathbb{F}_{p^n}$, lifts to an ring automorphism of $W$ which we denote by $w \mapsto w \sigma$.

Theorem $\text{End}(F_n)$ is the algebra obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F_n = p$ and $Fw = w \sigma F$ for $w \in W$. 

Properties of $E_n$ and $G_n$ (continued)
The extended Morava stabilizer group $\mathbb{G}_n$ is related to the automorphism group $\mathfrak{S}_n$ of the Honda height $n$ formal group law $F_n$ over $\mathbb{F}_{p^n}$. It is known that this group does change if we enlarge the field over which $F_n$ is defined.
Properties of $E_n$ and $G_n$ (continued)

The extended Morava stabilizer group $G_n$ is related to the automorphism group $S_n$ of the Honda height $n$ formal group law $F_n$ over $\mathbb{F}_p^n$. It is known that this group does change if we enlarge the field over which $F_n$ is defined.

To describe $G_n$, we describe the endomorphism ring of $F_n$, $\text{End}(F_n)$.
The extended Morava stabilizer group $\mathbb{G}_n$ is related to the automorphism group $\mathfrak{S}_n$ of the Honda height $n$ formal group law $F_n$ over $\mathbb{F}_p^n$. It is known that this group does change if we enlarge the field over which $F_n$ is defined.

To describe $\mathbb{G}_n$, we describe the endomorphism ring of $F_n$, $\text{End}(F_n)$. The Frobenius automorphism, the $p$th power map of $\mathbb{F}_p^n$, 

$$
\text{End}(F_n) = \text{End}(\mathbb{F}_p^n)
$$
The extended Morava stabilizer group $\mathbb{G}_n$ is related to the automorphism group $\mathbb{S}_n$ of the Honda height $n$ formal group law $F_n$ over $\mathbb{F}_{p^n}$. It is known that this group does change if we enlarge the field over which $F_n$ is defined.

To describe $\mathbb{G}_n$, we describe the endomorphism ring of $F_n$, $\text{End}(F_n)$. The Frobenius automorphism, the $p$th power map of $\mathbb{F}_{p^n}$, lifts to an automorphism of $W$ which we denote by $w \mapsto w^\sigma$. 
The extended Morava stabilizer group $\mathbb{G}_n$ is related to the automorphism group $S_n$ of the Honda height $n$ formal group law $F_n$ over $\mathbb{F}_{p^n}$. It is known that this group does change if we enlarge the field over which $F_n$ is defined.

To describe $\mathbb{G}_n$, we describe the endomorphism ring of $F_n$, $\text{End}(F_n)$. The Frobenius automorphism, the $p$th power map of $\mathbb{F}_{p^n}$, lifts to an endomorphism of $W$ which we denote by $w \mapsto w^\sigma$.

**Theorem**

$\text{End}(F_n)$ is the algebra obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$. 
Properties of $E_n$ and $G_n$ (continued)

Theorem

$\text{End}(F_n)$ is the algebra $W\langle\langle F \rangle\rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$. 

This algebra is a free module over $W$ of rank $n$, and hence a free module over $\mathbb{Z}_p$ of rank $n^2$. An element of the form $e = e_0 + e_1 F + \cdots + e_{n-1} F^{n-1}$ with $e_i \in W$ is invertible if $e_0$ is a unit in $W$. They form a group under multiplication. This is the automorphism group $\text{Aut}(F_n)$ of $F_n$, commonly known as the $n$th Morava stabilizer group $S_n$. $G_n$ is its extension by the Galois group $\text{Gal}(F_p^n, F_p) \cong \text{Gal}(W, \mathbb{Z}_p) \cong C_n$. 

Historical introduction

$K(n)$ localization

Finding a root of unity

Group cohomology

The main theorem

A classical example

$TMF$ at $p = 3$

Larger primes
Theorem

\(\text{End}(F_n)\) is the algebra \(W\langle\langle F\rangle\rangle\) obtained from \(W\) by adjoining a noncommuting indeterminate \(F\) with \(F^n = p\) and \(Fw = w^\sigma F\) for \(w \in W\).

This algebra is a free module over \(W\) of rank \(n\),
Theorem

End($F_n$) is the algebra $W \langle \langle F \rangle \rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.

This algebra is a free module over $W$ of rank $n$, and hence a free module over $\mathbb{Z}_p$ of rank $n^2$. 
Properties of $E_n$ and $G_n$ (continued)

Theorem

End($F_n$) is the algebra $W\langle\langle F \rangle\rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.

This algebra is a free module over $W$ of rank $n$, and hence a free module over $\mathbb{Z}_p$ of rank $n^2$. An element of the form

$$e = e_0 + e_1 F + \cdots + e_{n-1} F^{n-1}$$

with $e_i \in W$
Properties of $E_n$ and $G_n$ (continued)

**Theorem**

\[ \text{End}(F_n) \text{ is the algebra } W \langle \langle F \rangle \rangle \text{ obtained from } W \text{ by adjoining a noncommuting indeterminate } F \text{ with } F^n = p \text{ and } Fw = w^\sigma F \text{ for } w \in W. \]

This algebra is a free module over $W$ of rank $n$, and hence a free module over $\mathbb{Z}_p$ of rank $n^2$. An element of the form

\[ e = e_0 + e_1 F + \cdots + e_{n-1} F^{n-1} \]

with $e_i \in W$ is invertible if $e_0$ is a unit in $W$. 

Properties of $E_n$ and $G_n$ (continued)

**Theorem**

End($F_n$) is the algebra $W\langle\langle F\rangle\rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.

This algebra is a free module over $W$ of rank $n$, and hence a free module over $\mathbb{Z}_p$ of rank $n^2$. An element of the form

$$e = e_0 + e_1 F + \cdots + e_{n-1} F^{n-1}$$

with $e_i \in W$

is invertible if $e_0$ is a unit in $W$. They form a group under multiplication. This is the automorphism group $\text{Aut}(F_n)$ of $F_n$, commonly known as the $n$th Morava stabilizer group $S_n$. 
**Properties of \( E_n \) and \( G_n \) (continued)**

**Theorem**

\[
\text{End}(F_n) \text{ is the algebra } W\langle\langle F \rangle\rangle \text{ obtained from } W \text{ by adjoining a noncommuting indeterminate } F \text{ with } F^n = p \text{ and } Fw = w^\sigma F \text{ for } w \in W.
\]

This algebra is a free module over \( W \) of rank \( n \), and hence a free module over \( \mathbb{Z}_p \) of rank \( n^2 \). An element of the form

\[
e = e_0 + e_1 F + \cdots + e_{n-1} F^{n-1}
\]

is invertible if \( e_0 \) is a unit in \( W \). They form a group under multiplication. This is the automorphism group \( \text{Aut}(F_n) \) of \( F_n \), commonly known as the \( n \)th Morava stabilizer group \( S_n \). \( G_n \) is its extension by the Galois group.
**Theorem**

\[ \text{End}(F_n) \text{ is the algebra } W\langle\langle F \rangle\rangle \text{ obtained from } W \text{ by adjoining a noncommuting indeterminate } F \text{ with } F^n = p \text{ and } Fw = w^\sigma F \text{ for } w \in W. \]

This algebra is a free module over \( W \) of rank \( n \), and hence a free module over \( \mathbb{Z}_p \) of rank \( n^2 \). An element of the form

\[ e = e_0 + e_1 F + \cdots + e_{n-1} F^{n-1} \]

with \( e_i \in W \)

is invertible if \( e_0 \) is a unit in \( W \). They form a group under multiplication. This is the automorphism group \( \text{Aut}(F_n) \) of \( F_n \), commonly known as the \( n \)th Morava stabilizer group \( S_n \). \( G_n \) is its extension by the Galois group

\[ \text{Gal}(\mathbb{F}_{p^n}, \mathbb{F}_p) \cong \text{Gal}(W, \mathbb{Z}_p) \cong C_n. \]
Properties of $E_n$ and $G_n$ (continued)

**Theorem**

$\text{End}(F_n)$ is the algebra $W\langle\langle F\rangle\rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$. 

Let $\omega \in W$ be a primitive $(p^n-1)$th root of unity, and let $\bar{\omega} \in F_p$ be its mod $p$ reduction. Then the elements $\omega$ and $F$ in $\text{End}(F_n)$ correspond to the endomorphisms $x \mapsto \omega x$ and $x \mapsto x^p$ of $F_n$. 

Our algebra $\text{End}(F_n)$ is a complete discrete valuation ring in which the valuation of $F$ is $1/n$. This valuation extends the usual one on $W$, in which the valuation of $p$ is $1$. 
Properties of $E_n$ and $G_n$ (continued)

Theorem

$\text{End}(F_n)$ is the algebra $W\langle\langle F \rangle\rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.

Let $\omega \in W$ be a primitive $(p^n - 1)$th root of unity, and let $\overline{\omega} \in \mathbb{F}_{p^n}$ be its mod $p$ reduction.
Theorem

\[ \text{End}(F_n) \text{ is the algebra } W\langle\langle F \rangle\rangle \text{ obtained from } W \text{ by adjoining a noncommuting indeterminate } F \text{ with } F^n = p \text{ and } Fw = w^\sigma F \text{ for } w \in W. \]

Let \( \omega \in W \) be a primitive \((p^n - 1)\)th root of unity, and let \( \bar{\omega} \in \mathbb{F}_p^n \) be its mod \( p \) reduction. Then the elements \( \omega \) and \( F \) in \( \text{End}(F_n) \) correspond to the endomorphisms
**Theorem**

\( \text{End}(F_n) \) is the algebra \( W\langle \langle F \rangle \rangle \) obtained from \( W \) by adjoining a noncommuting indeterminate \( F \) with \( F^n = p \) and \( Fw = w^\sigma F \) for \( w \in W \).

Let \( \omega \in W \) be a primitive \((p^n - 1)\)th root of unity, and let \( \overline{\omega} \in \mathbb{F}_{p^n} \) be its mod \( p \) reduction. Then the elements \( \omega \) and \( F \) in \( \text{End}(F_n) \) correspond to the endomorphisms

\[
\begin{align*}
x & \mapsto \overline{\omega}x \\
& \text{ and } \\
x & \mapsto x^p
\end{align*}
\]

of \( F_n \).
Theorem

End($F_n$) is the algebra $W\langle\langle F \rangle\rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.

Let $\omega \in W$ be a primitive $(p^n - 1)$th root of unity, and let $\overline{\omega} \in \mathbb{F}_{p^n}$ be its mod $p$ reduction. Then the elements $\omega$ and $F$ in $\text{End}(F_n)$ correspond to the endomorphisms

$$x \mapsto \overline{\omega}x \quad \text{and} \quad x \mapsto x^p$$

of $F_n$.

Our algebra $\text{End}(F_n)$ is a complete discrete valuation ring in which the valuation of $F$ is $1/n$. 
**Theorem**

End($F_n$) is the algebra $W\langle\langle F \rangle\rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.

Let $\omega \in W$ be a primitive $(p^n - 1)$th root of unity, and let $\overline{\omega} \in \mathbb{F}_{p^n}$ be its mod $p$ reduction. Then the elements $\omega$ and $F$ in $\text{End}(F_n)$ correspond to the endomorphisms

$$x \mapsto \overline{\omega}x \quad \text{and} \quad x \mapsto x^p$$

of $F_n$.

Our algebra $\text{End}(F_n)$ is a complete discrete valuation ring in which the valuation of $F$ is $1/n$. This valuation extends the usual one on $W$, in which the valuation of $p$ is $1$. 
Finding a $p$th root of unity

**Theorem**

$\text{End}(F_n)$ is the algebra $W\langle \langle F \rangle \rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$. 
Finding a $p$th root of unity

**Theorem**

$\text{End}(F_n)$ is the algebra $W\langle\langle F\rangle\rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.

Finding an element of order $p$ in $S_n$, 

• $\text{End}(F_n) \otimes \mathbb{Q}_p$ is a division algebra $D_n$ with center $\mathbb{Q}_p$.
• $D_n$ is known to contain every field $K$ that is a finite extension of $\mathbb{Q}_p$ whose degree divides $n$.
• The valuation we have defined on $D_n$ restricts to the usual one on each such $K$.
• The field $L = \mathbb{Q}_p \left[\sqrt[p]{1}\right]$ has degree $p-1$, and is thus contained in $D_n$ iff $p-1$ divides $n$.
• Its maximal ideal is generated by an element $\pi$ with valuation $1/(p-1)$.
Finding a $p$th root of unity

**Theorem**

$\text{End}(F_n)$ is the algebra $W\langle\langle F\rangle\rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.

Finding an element of order $p$ in $S_n$, is equivalent to finding a $p$th root of unity in $\text{End}(F_n)$. For this we will use the following facts about it.
Finding a pth root of unity

**Theorem**

\( \text{End}(F_n) \) is the algebra \( W \langle \langle F \rangle \rangle \) obtained from \( W \) by adjoining a noncommuting indeterminate \( F \) with \( F^n = p \) and \( Fw = w^\sigma F \) for \( w \in W \).

Finding an element of order \( p \) in \( S_n \), is equivalent to finding a pth root of unity in \( \text{End}(F_n) \). For this we will use the following facts about it.

- \( \text{End}(F_n) \otimes \mathbb{Q}_p \) is a division algebra \( D_n \) with center \( \mathbb{Q}_p \).
Finding a $p$th root of unity

**Theorem**

$\text{End}(F_n)$ is the algebra $W \langle \langle F \rangle \rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.

Finding an element of order $p$ in $\mathbb{S}_n$, is equivalent to finding a $p$th root of unity in $\text{End}(F_n)$. For this we will use the following facts about it.

- $\text{End}(F_n) \otimes \mathbb{Q}_p$ is a division algebra $D_n$ with center $\mathbb{Q}_p$.
- $D_n$ is known to contain every field $K$ that is a finite extension of $\mathbb{Q}_p$ whose degree divides $n$. 

Finding a $p$th root of unity

**Theorem**

$\text{End}(F_n)$ is the algebra $W \langle \langle F \rangle \rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.

Finding an element of order $p$ in $S_n$, is equivalent to finding a $p$th root of unity in $\text{End}(F_n)$. For this we will use the following facts about it.

- $\text{End}(F_n) \otimes \mathbb{Q}_p$ is a division algebra $D_n$ with center $\mathbb{Q}_p$.
- $D_n$ is known to contain every field $K$ that is a finite extension of $\mathbb{Q}_p$ whose degree divides $n$. The valuation we have defined on $D_n$ restricts to the usual one on each such $K$. 
Finding a $p$th root of unity

**Theorem**

$\text{End}(F_n)$ is the algebra $W\langle\langle F\rangle\rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.

Finding an element of order $p$ in $S_n$, is equivalent to finding a $p$th root of unity in $\text{End}(F_n)$. For this we will use the following facts about it.

- $\text{End}(F_n) \otimes \mathbb{Q}_p$ is a division algebra $D_n$ with center $\mathbb{Q}_p$.
- $D_n$ is known to contain every field $K$ that is a finite extension of $\mathbb{Q}_p$ whose degree divides $n$. The valuation we have defined on $D_n$ restricts to the usual one on each such $K$.
- The field $L = \mathbb{Q}_p[\sqrt[1]{1}]$ has degree $p - 1$,
Finding a $p$th root of unity

**Theorem**

\[ \text{End}(F_n) \text{ is the algebra } W\langle\langle F \rangle\rangle \text{ obtained from } W \text{ by adjoining a noncommuting indeterminate } F \text{ with } F^n = p \text{ and } Fw = w^\sigma F \text{ for } w \in W. \]

Finding an element of order $p$ in $S_n$, is equivalent to finding a $p$th root of unity in $\text{End}(F_n)$. For this we will use the following facts about it.

- $\text{End}(F_n) \otimes \mathbb{Q}_p$ is a division algebra $D_n$ with center $\mathbb{Q}_p$.
- $D_n$ is known to contain every field $K$ that is a finite extension of $\mathbb{Q}_p$ whose degree divides $n$. The valuation we have defined on $D_n$ restricts to the usual one on each such $K$.
- The field $L = \mathbb{Q}_p[\sqrt[p-1]{1}]$ has degree $p - 1$, and is thus contained in $D_n$ iff $p - 1$ divides $n$. 
Finding a $p$th root of unity

**Theorem**

$\text{End}(F_n)$ is the algebra $W\langle\langle F \rangle\rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.

Finding an element of order $p$ in $S_n$, is equivalent to finding a $p$th root of unity in $\text{End}(F_n)$. For this we will use the following facts about it.

- $\text{End}(F_n) \otimes \mathbb{Q}_p$ is a division algebra $D_n$ with center $\mathbb{Q}_p$.
- $D_n$ is known to contain every field $K$ that is a finite extension of $\mathbb{Q}_p$ whose degree divides $n$. The valuation we have defined on $D_n$ restricts to the usual one on each such $K$.
- The field $L = \mathbb{Q}_p[\sqrt[\varphi]{1}]$ has degree $p - 1$, and is thus contained in $D_n$ iff $p - 1$ divides $n$. Its maximal ideal is generated by an element $\pi$ with valuation $1/(p - 1)$.
Finding a pth root of unity (continued)

**Theorem**

\[ \text{End}(F_n) \text{ is the algebra } W \langle \langle F \rangle \rangle \text{ obtained from } W \text{ by adjoining a noncommuting indeterminate } F \text{ with } F^n = p \text{ and } Fw = w^\sigma F \text{ for } w \in W. \]
Finding a $p$th root of unity (continued)

**Theorem**

$\text{End}(F_n)$ is the algebra $W\langle\langle F \rangle\rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.

The above discussion implies that for $n = (p - 1)f$ for a positive integer $f$,
Theorem

\[ \text{End}(F_n) \text{ is the algebra } W\langle F \rangle \text{ obtained from } W \text{ by adjoining a noncommuting indeterminate } F \text{ with } F^n = p \text{ and } Fw = w^{\sigma} F \text{ for } w \in W. \]

The above discussion implies that for \( n = (p - 1)f \) for a positive integer \( f \), a primitive \( p \)th root of unity exists in the sub \( W \)-algebra of \( \text{End}(F_n) \) generated by \( F^f \).
**Finding a pth root of unity (continued)**

**Theorem**

End($F_n$) is the algebra $W\langle\langle F\rangle\rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.

The above discussion implies that for $n = (p - 1)f$ for a positive integer $f$, a primitive $p$th root of unity exists in the sub $W$-algebra of End($F_n$) generated by $F^f$. It thus has the form

$$\zeta = 1 + z_1 F^f + \cdots + z_{p-2} F^{(p-2)f} + pz_{p-1}$$

with $z_i \in W$. 

Recall that $F^{(p - 1)f} = p$. There are many such elements $\zeta$. 

---

**Hiking in the Alps: $C_p$ fixed points of Lubin-Tate spectra**

Doug Ravenel

Historical introduction

$K(n)$ localization

Properties of $E_n$ and $G_n$

Finding a root of unity

Group cohomology

The main theorem

A classical example

$TMF$ at $p = 3$

Larger primes
Theorem

End\( \left( F_n \right) \) is the algebra \( W \langle \langle F \rangle \rangle \) obtained from \( W \) by adjoining a noncommuting indeterminate \( F \) with \( F^n = p \) and \( Fw = w^\sigma F \) for \( w \in W \).

The above discussion implies that for \( n = (p - 1)f \) for a positive integer \( f \), a primitive \( p \)th root of unity exists in the sub \( W \)-algebra of \( \text{End}(F_n) \) generated by \( F^f \). It thus has the form

\[
\zeta = 1 + z_1 F^f + \cdots + z_{p-2} F^{(p-2)f} + p z_{p-1}
\]

with \( z_i \in W \), where \( z_1 \) is a unit.
Finding a $p$th root of unity (continued)

Theorem

$\text{End}(F_n)$ is the algebra $W\langle\langle F\rangle\rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.

The above discussion implies that for $n = (p - 1)f$ for a positive integer $f$, a primitive $p$th root of unity exists in the sub $W$-algebra of $\text{End}(F_n)$ generated by $F^f$. It thus has the form

$$
\zeta = 1 + z_1 F^f + \cdots + z_{p-2} F^{(p-2)f} + pz_{p-1}
$$

with $z_i \in W$, where $z_1$ is a unit. Recall that $F^{(p-1)f} = p$. 
Finding a $p$th root of unity (continued)

**Theorem**

$\text{End}(F_n)$ is the algebra $W\langle\langle F \rangle\rangle$ obtained from $W$ by adjoining a noncommuting indeterminate $F$ with $F^n = p$ and $Fw = w^\sigma F$ for $w \in W$.

The above discussion implies that for $n = (p - 1)f$ for a positive integer $f$, a primitive $p$th root of unity exists in the sub $W$-algebra of $\text{End}(F_n)$ generated by $F^f$. It thus has the form

$$\zeta = 1 + z_1 F^f + \cdots + z_{p-2} F^{(p-2)f} + pz_{p-1}$$

with $z_i \in W$, where $z_1$ is a unit. Recall that $F^{(p-1)f} = p$. There are many such elements $\zeta$. 

The main tool for computing the homotopy groups of the homotopy fixed point spectrum of $E^hG$ for a group $G$ acting on a spectrum $E$ is the homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s(G; \pi_t E) \Rightarrow \pi_{t-s} E^hG$$
The main tool for computing the homotopy groups of the homotopy fixed point spectrum of $E^{hG}$ for a group $G$ acting on a spectrum $E$ is the homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s(G; \pi_t E) \Rightarrow \pi_{t-s}E^{hG}$$

Its use requires knowledge of the action of $G$ on $\pi_* E$. 
The main tool for computing the homotopy groups of the homotopy fixed point spectrum of $E^{hG}$ for a group $G$ acting on a spectrum $E$ is the homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s(G; \pi_t E) \Rightarrow \pi_{t-s} E^{hG}$$

Its use requires knowledge of the action of $G$ on $\pi_* E$. In the case of $\mathbb{G}_t$ acting on $\pi_* E_n$ this is far from easy,
Group cohomology

The main tool for computing the homotopy groups of the homotopy fixed point spectrum of $E^{hG}$ for a group $G$ acting on a spectrum $E$ is the homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s(G; \pi_t E) \implies \pi_{t-s} E^{hG}$$

Its use requires knowledge of the action of $G$ on $\pi_* E$. In the case of $G$ acting on $\pi_* E_n$ this is far from easy, despite the identification of the above with the $E_2$-term of the Adams-Novikov spectral sequence.
The main tool for computing the homotopy groups of the homotopy fixed point spectrum of $E^{hG}$ for a group $G$ acting on a spectrum $E$ is the homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s(G; \pi_t E) \implies \pi_{t-s} E^{hG}$$

Its use requires knowledge of the action of $G$ on $\pi_* E$. In the case of $G$ acting on $\pi_* E_n$ this is far from easy, despite the identification of the above with the $E_2$-term of the Adams-Novikov spectral sequence. It is more manageable when we replace $G$ by a subgroup of order $p$. 

Group cohomology
Group cohomology (continued)

We recall some facts about group cohomology for $G = C_p$. 

We recall some facts about group cohomology for $G = C_p$. 

For a generator $\gamma \in C_p$, the integral group ring $\mathbb{Z}C_p$ is $\mathbb{Z}[\gamma]/(\gamma^p - 1)$. The following is a minimal free $\mathbb{Z}C_p$-resolution of $\mathbb{Z}$ with the trivial $C_p$-action.

\[
\begin{array}{cccccccc}
0 & 1 & 2 & \cdots \\
0 & \mathbb{Z} & 0 & 0 & \cdots \\
\mathbb{Z}C_p & \nabla & 0 & 0 & \cdots \\
\mathbb{Z}C_p & T & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

where $\nabla$ is the augmentation defined by $\nabla(\gamma^i) = 1$, and $T = 1 + \gamma + \cdots + \gamma^p - 1$ is the trace. Applying the functor $\text{Hom}_{\mathbb{Z}C_p}(-, \mathbb{Z}_p)$ to this chain complex gives the cochain complex $\mathbb{Z}_p^i/C_p; \mathbb{Z}_p^i$ leading to the expected $H^i(C_p; \mathbb{Z}_p) = \begin{cases} 
\mathbb{Z}_p & \text{for } i = 0 \\
\mathbb{Z}/p & \text{for } i > 0 \text{ even} \\
0 & \text{otherwise}.
\end{cases}$
We recall some facts about group cohomology for $G = C_p$. For a generator $\gamma \in C_p$, the integral group ring $\mathbb{Z}C_p$ is $\mathbb{Z}[\gamma]/(\gamma^p - 1)$. 

---

$\textbf{Group cohomology (continued)}$
Group cohomology (continued)

We recall some facts about group cohomology for $G = C_p$. For a generator $\gamma \in C_p$, the integral group ring $\mathbb{Z}C_p$ is $\mathbb{Z}[\gamma]/(\gamma^p - 1)$. The following is a minimal free $\mathbb{Z}C_p$-resolution of $\mathbb{Z}$ with the trivial $C_p$-action.

\[
\begin{array}{c}
0 & 1 & 2 \\
0 & \mathbb{Z} & \mathbb{Z}C_p \\
\mathbb{Z}C_p & \mathbb{Z}C_p & \mathbb{Z}C_p \\
\mathbb{Z}C_p & \mathbb{Z}C_p & \mathbb{Z}C_p \\
\cdot & \cdot & \cdot \\
\end{array}
\]

where $\nabla$ is the augmentation defined by $\nabla(\gamma^i) = 1$, and $T = 1 + \gamma + \cdots + \gamma^{p-1}$ is the trace. Applying the functor $\text{Hom}_{\mathbb{Z}C_p}(-, \mathbb{Z}_p)$ to this chain complex gives the cochain complex $\mathbb{Z}_p$ leading to the expected $H^i(C_p; \mathbb{Z}_p) = \begin{cases} 
\mathbb{Z}_p & \text{for } i = 0 \\
\mathbb{Z}_p/\mathbb{Z} & \text{for } i > 0 \text{ even} \\
0 & \text{otherwise.}
\end{cases}$
We recall some facts about group cohomology for $G = C_p$. For a generator $\gamma \in C_p$, the integral group ring $\mathbb{Z}C_p$ is $\mathbb{Z}[\gamma]/(\gamma^p - 1)$. The following is a minimal free $\mathbb{Z}C_p$-resolution of $\mathbb{Z}$ with the trivial $C_p$-action.

$$
\begin{array}{cccc}
0 & 1 & 2 \\
0 & \mathbb{Z} & \mathbb{Z}C_p & \mathbb{Z}C_p & \mathbb{Z}C_p & \ldots
\end{array}
$$

The following is a minimal free $\mathbb{Z}C_p$-resolution of $\mathbb{Z}$ with the trivial $C_p$-action.

$$
\begin{array}{cccc}
0 & 1 & 2 \\
0 & \mathbb{Z} & \mathbb{Z}C_p & \mathbb{Z}C_p & \mathbb{Z}C_p & \ldots
\end{array}
$$

where $\nabla$ is the augmentation defined by $\nabla(\gamma^i) = 1$, and $T = 1 + \gamma + \cdots + \gamma^{p-1}$ is the trace. Applying the functor $\text{Hom}_{\mathbb{Z}C_p}(\mathbb{Z}, \mathbb{Z}_p)$ to this chain complex gives the cochain complex

$$
\begin{array}{cccc}
0 & 1 & 2 \\
0 & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \cdots
\end{array}
$$

leading to the expected $H^i(C_p; \mathbb{Z}_p) =$\begin{cases} \mathbb{Z}_p & \text{for } i = 0 \\ \mathbb{Z}_p/\mathbb{Z} & \text{for } i > 0 \text{ even} \\ 0 & \text{otherwise} \end{cases}$
Group cohomology (continued)

We recall some facts about group cohomology for $G = C_p$. For a generator $\gamma \in C_p$, the integral group ring $\mathbb{Z}C_p$ is $\mathbb{Z}[\gamma]/(\gamma^p - 1)$. The following is a minimal free $\mathbb{Z}C_p$-resolution of $\mathbb{Z}$ with the trivial $C_p$-action.

$$
\begin{array}{cccccc}
0 & 1 & 2 & \ldots \\
\mathbb{Z} & \mathbb{Z}C_p & \mathbb{Z}C_p & \mathbb{Z}C_p & \mathbb{Z}C_p & \ldots \\
\xleftarrow{\nabla} & \xleftarrow{1-\gamma} & \xleftarrow{T} & & & \\
\end{array}
$$

where $\nabla$ is the augmentation defined by $\nabla(\gamma^i) = 1$, and $T = 1 + \gamma + \cdots + \gamma^{p-1}$ is the trace.
Group cohomology (continued)

We recall some facts about group cohomology for $G = C_p$. For a generator $\gamma \in C_p$, the integral group ring $\mathbb{Z}C_p$ is $\mathbb{Z}[\gamma]/(\gamma^p - 1)$. The following is a minimal free $\mathbb{Z}C_p$-resolution of $\mathbb{Z}$ with the trivial $C_p$-action.

\[
\begin{array}{cccc}
0 & 1 & 2 & \\
\mathbb{Z} & \mathbb{Z}C_p & \mathbb{Z}C_p & \\
\bigtriangleup & 1-\gamma & T & \\
\end{array}
\]

where $\bigtriangleup$ is the augmentation defined by $\bigtriangleup(\gamma^i) = 1$, and $T = 1 + \gamma + \cdots + \gamma^{p-1}$ is the trace.
Group cohomology (continued)

We recall some facts about group cohomology for $G = C_p$. For a generator $\gamma \in C_p$, the integral group ring $\mathbb{Z} C_p$ is $\mathbb{Z}[\gamma]/(\gamma^p - 1)$. The following is a minimal free $\mathbb{Z} C_p$-resolution of $\mathbb{Z}$ with the trivial $C_p$-action.

\[
\begin{array}{ccc}
0 & \xleftarrow{\nabla} & 1 & \xleftarrow{1 - \gamma} & 2 \\
0 & \xleftarrow{\nabla} & \mathbb{Z} C_p & \xleftarrow{1 - \gamma} & \mathbb{Z} C_p & \xleftarrow{T} & \mathbb{Z} C_p & \xleftarrow{T} & \ldots
\end{array}
\]

where $\nabla$ is the augmentation defined by $\nabla(\gamma^i) = 1$, and $T = 1 + \gamma + \cdots + \gamma^{p-1}$ is the trace.

Applying the functor $\text{Hom}_{\mathbb{Z} C_p}(-, \mathbb{Z}_p)$ to this chain complex gives the cochain complex

\[
\begin{array}{ccc}
\mathbb{Z}_p & \xrightarrow{0} & \mathbb{Z}_p & \xrightarrow{p} & \mathbb{Z}_p & \xrightarrow{0} & \ldots
\end{array}
\]
We recall some facts about group cohomology for $G = C_p$. For a generator $\gamma \in C_p$, the integral group ring $\mathbb{Z}C_p$ is $\mathbb{Z}[\gamma]/(\gamma^p - 1)$. The following is a minimal free $\mathbb{Z}C_p$-resolution of $\mathbb{Z}$ with the trivial $C_p$-action.

\[
\begin{array}{cccccc}
0 & & 1 & & 2 & \\
& \mathbb{Z} & \xleftarrow{\nabla} & \mathbb{Z}C_p & \xleftarrow{1-\gamma} & \mathbb{Z}C_p & \xleftarrow{T} & \mathbb{Z}C_p & \xleftarrow{} & \ldots \\
\end{array}
\]

where $\nabla$ is the augmentation defined by $\nabla(\gamma^i) = 1$, and $T = 1 + \gamma + \cdots + \gamma^{p-1}$ is the trace.

Applying the functor $\text{Hom}_{\mathbb{Z}C_p}(-, \mathbb{Z}_p)$ to this chain complex gives the cochain complex

\[
\begin{array}{cccc}
\mathbb{Z}_p & \xrightarrow{0} & \mathbb{Z}_p & \xrightarrow{p} & \mathbb{Z}_p & \xrightarrow{0} & \ldots \\
\end{array}
\]

leading to the expected

\[
H^i(C_p; \mathbb{Z}_p) = \begin{cases} 
\mathbb{Z}_p & \text{for } i = 0 \\
\mathbb{Z}/p & \text{for } i > 0 \text{ even} \\
0 & \text{otherwise.}
\end{cases}
\]
Group cohomology (continued)

\[
0 \leftarrow \mathbb{Z} \leftarrow \nabla \mathbb{Z}C_p \leftarrow^{1-\gamma} \mathbb{Z}C_p \leftarrow^{T} \mathbb{Z}C_p \leftarrow \ldots
\]
Group cohomology (continued)

The cokernel of $T$, also the kernel of $\nabla$, is the reduced regular representation $\bar{\rho}$.
Group cohomology (continued)

The cokernel of $T$, also the kernel of $\nabla$, is the reduced regular representation $\bar{\rho}$.

Similar computations give

$$H^i(C_p; \bar{\rho}) = \begin{cases} 
0 & \text{for } i = 0 \\
\mathbb{Z}/p & \text{for } i \text{ odd} \\
0 & \text{otherwise.}
\end{cases}$$
Group cohomology (continued)

The cokernel of $T$, also the kernel of $\nabla$, is the reduced regular representation $\bar{\rho}$.

Similar computations give

$$H^i(C_p; \bar{\rho}) = \begin{cases} 
0 & \text{for } i = 0 \\
\mathbb{Z}/p & \text{for } i \text{ odd} \\
0 & \text{otherwise}
\end{cases}$$

and

$$H^i(C_p; \mathbb{Z}C_p) = \begin{cases} 
\mathbb{Z} & \text{for } i = 0 \\
0 & \text{otherwise}
\end{cases}$$
The main theorem

We will now describe $\pi_* E_n$ for $n = (p - 1)f$ as a module over the group ring $WC_p$, where $W = W(\mathbb{F}_p^n)$.
We will now describe $\pi_* E_n$ for $n = (p - 1)f$ as a module over the group ring $WC_p$, where $W = W(F_{p^n})$. We will do this more generally, replacing $C_p$ by any finite subgroup $H$ of the (nonextended) Morava stabilizer group $\text{Aut}(F_n)$. We saw earlier that $\pi_* E_n$ is a completed localization of the graded ring $R_n = W[x_0, \ldots, x_{n-1}]$ with $|x_i| = -2$. Its component in degree $-2$ is a free $W$-module of rank $n$, as is our endomorphism ring $\text{End}(F_n)$. This isomorphism defines an action of $H$ on the degree -2 component of $R_n$, which extends to an action on all of $R_n$ and its completed localization by continuous ring homomorphisms.
The main theorem

We will now describe $\pi_* E_n$ for $n = (p - 1)f$ as a module over the group ring $WC_p$, where $W = W(F_p^n)$. We will do this more generally, replacing $C_p$ by any finite subgroup $H$ of the (nonextended) Morava stabilizer group $\text{Aut}(F_n)$ whose $p$-Sylow subgroup is cyclic.
The main theorem

We will now describe $\pi_* E_n$ for $n = (p - 1)f$ as a module over the group ring $WC_p$, where $W = W(\mathbb{F}_p^n)$. We will do this more generally, replacing $C_p$ by any finite subgroup $H$ of the (nonextended) Morava stabilizer group $\text{Aut}(F_n)$ whose $p$-Sylow subgroup is cyclic.

We saw earlier that $\pi_* E_n$ is a completed localization of the graded ring $\mathbb{R}_n$. This isomorphism defines an action of $H$ on the degree -2 component of $\mathbb{R}_n$, which extends to an action on all of $\mathbb{R}_n$ and its completed localization by continuous ring homomorphisms.
The main theorem

We will now describe $\pi_* E_n$ for $n = (p - 1)f$ as a module over the group ring $WC_p$, where $W = W(F_p^n)$. We will do this more generally, replacing $C_p$ by any finite subgroup $H$ of the (nonextended) Morava stabilizer group $\text{Aut}(F_n)$ whose $p$-Sylow subgroup is cyclic.

We saw earlier that $\pi_* E_n$ is a completed localization of the graded ring

$$R_n = W[x_0, \ldots, x_{n-1}]$$

with $|x_i| = -2$. 

A classical example
TMF at $p = 3$
Larger primes
The main theorem

We will now describe $\pi_* E_n$ for $n = (p - 1)f$ as a module over the group ring $WC_p$, where $W = W(\mathbb{F}_{p^n})$. We will do this more generally, replacing $C_p$ by any finite subgroup $H$ of the (nonextended) Morava stabilizer group $\text{Aut}(F_n)$ whose $p$-Sylow subgroup is cyclic.

We saw earlier that $\pi_* E_n$ is a completed localization of the graded ring

$$R_n = W[x_0, \ldots, x_{n-1}] \quad \text{with} \quad |x_i| = -2.$$ 

Its component in degree $-2$ is a free $W$-module of rank $n$,
The main theorem

We will now describe $\pi_* E_n$ for $n = (p - 1)f$ as a module over the group ring $WC_p$, where $W = W(\mathbb{F}_p)$. We will do this more generally, replacing $C_p$ by any finite subgroup $H$ of the (nonextended) Morava stabilizer group $\text{Aut}(F_n)$ whose $p$-Sylow subgroup is cyclic.

We saw earlier that $\pi_* E_n$ is a completed localization of the graded ring

$$R_n = W[x_0, \ldots, x_{n-1}] \quad \text{with} \quad |x_i| = -2.$$ 

Its component in degree $-2$ is a free $W$-module of rank $n$, as is our endomorphism ring $\text{End}(F_n)$. 
The main theorem

We will now describe $\pi_* E_n$ for $n = (p - 1)f$ as a module over the group ring $WC_p$, where $W = W(\mathbb{F}_p)$. We will do this more generally, replacing $C_p$ by any finite subgroup $H$ of the (nonextended) Morava stabilizer group $\text{Aut}(F_n)$ whose $p$-Sylow subgroup is cyclic.

We saw earlier that $\pi_* E_n$ is a completed localization of the graded ring

$$R_n = W[x_0, \ldots, x_{n-1}] \quad \text{with} \quad |x_i| = -2.$$ 

Its component in degree $-2$ is a free $W$-module of rank $n$, as is our endomorphism ring $\text{End}(F_n)$. This isomorphism defines an action of $H$ on the degree -2 component of $R_n$. 
The main theorem

We will now describe $\pi_\ast E_n$ for $n = (p - 1)f$ as a module over the group ring $WC_p$, where $W = W(\mathbb{F}_p^n)$. We will do this more generally, replacing $C_p$ by any finite subgroup $H$ of the (nonextended) Morava stabilizer group $\text{Aut}(F_n)$ whose $p$-Sylow subgroup is cyclic.

We saw earlier that $\pi_\ast E_n$ is a completed localization of the graded ring

$$R_n = W[x_0, \ldots, x_{n-1}] \quad \text{with} \quad |x_i| = -2.$$ 

Its component in degree $-2$ is a free $W$-module of rank $n$, as is our endomorphism ring $\text{End}(F_n)$. This isomorphism defines an action of $H$ on the degree -2 component of $R_n$, which extends to an action on all of $R_n$ and its completed localization by continuous ring homomorphisms.
The main theorem (continued)

For the case $H = C_p$, $R_n$ is isomorphic as a $WC_p$-algebra to
For the case $H = C_p$, $R_n$ is isomorphic as a $W C_p$-algebra to

$$
\tilde{R}_n = W[x_{i,j} : 1 \leq i \leq f, j \in \mathbb{Z}/p] \bigg/ \left( \sum_j x_{i,j} : 1 \leq i \leq f \right)
$$

with $|x_{i,j}| = -2$. 

The main theorem (continued)

For the case $H = C_p$, $R_n$ is isomorphic as a $WC_p$-algebra to

$$\tilde{R}_n = W[x_{i,j} : 1 \leq i \leq f, j \in \mathbb{Z}/p] \bigg/ \left( \sum_j x_{i,j} : 1 \leq i \leq f \right)$$

with $|x_{i,j}| = -2$.

For a generator $\gamma \in C_p$ we have $\gamma x_{i,j} = x_{i,j+1}$, and the trace $Tx_{i,j}$ vanishes.
For the case $H = C_p$, $R_n$ is isomorphic as a $WC_p$-algebra to

$$\tilde{R}_n = W[x_{i,j} : 1 \leq i \leq f, j \in \mathbb{Z}/p] \left/ \left( \sum_j x_{i,j} : 1 \leq i \leq f \right) \right.$$

with $|x_{i,j}| = -2$.

For a generator $\gamma \in C_p$ we have $\gamma x_{i,j} = x_{i,j+1}$, and the trace $Tx_{i,j}$ vanishes. It follows that the degree -2 component of $\tilde{R}_n$ is the direct sum of $f$ copies of $\rho \otimes W$. Thus $\tilde{R}_n$ is the symmetric $W$-algebra

$$\text{Symm}_W \left( \rho \oplus f \right).$$
The main theorem (continued)

\[ \tilde{R}_n = W[x_{i,j} : 1 \leq i \leq f, j \in \mathbb{Z}/p] / \left( \sum_{j \in \mathbb{Z}/p} x_{i,j} : 1 \leq i \leq f \right) \]

with \(|x_{i,j}| = -2\)

\[ \cong \text{Symm}_W \left( \overset{\rho \oplus f}{\rho} \right) . \]
The main theorem (continued)

\[ \tilde{R}_n = W[x_{i,j} : 1 \leq i \leq f, j \in \mathbb{Z}/p] / \left( \sum_{j \in \mathbb{Z}/p} x_{i,j} : 1 \leq i \leq f \right) \]

with \( |x_{i,j}| = -2 \)

\[ \cong \text{Symm}_W \left( \bar{\rho} \oplus f \right) . \]

Even though the \( x_{i,j} \)'s are not linearly independent, we define

\[ \Phi' = \prod_{1 \leq i \leq f} \prod_{0 \leq j < p} x_{i,j} \]
The main theorem (continued)

\[ \widetilde{R}_n = W[x_{i,j} : 1 \leq i \leq f, j \in \mathbb{Z}/p] / \left( \sum_{j \in \mathbb{Z}/p} x_{i,j} : 1 \leq i \leq f \right) \]

with \(|x_{i,j}| = -2\)

\[ \cong \text{Symm}_W \left( \overline{\rho} \oplus f \right) . \]

Even though the \(x_{i,j}\)s are not linearly independent, we define

\[ \Phi' = \prod_{1 \leq i \leq f} \prod_{0 \leq j < p} x_{i,j} \]

and complete \(\widetilde{R}_n[\Phi'^{\pm 1}]\) with respect to the kernel \(\overline{\mathfrak{m}}\) of the map
The main theorem (continued)

\[ \tilde{R}_n = W[x_{i,j} : 1 \leq i \leq f, j \in \mathbb{Z}/p]/\left( \sum_{j \in \mathbb{Z}/p} x_{i,j} : 1 \leq i \leq f \right) \]

with \( |x_{i,j}| = -2 \)

\[ \cong \text{Symm}_W (\rho \oplus f) . \]

Even though the \( x_{i,j} \)'s are not linearly independent, we define

\[ \Phi' = \prod_{1 \leq i \leq f} \prod_{0 \leq j < p} x_{i,j} \]

and complete \( \tilde{R}_n[\Phi'^{\pm 1}] \) with respect to the kernel \( \tilde{m} \) of the map

\[ \tilde{R}_n[\Phi'^{\pm 1}] \to \mathbb{F}_p[n][u^{\pm 1}] \quad \text{with} \quad x_{i,j} \mapsto u \quad \text{and} \quad \gamma u = u. \]
The main theorem (continued)

\[ \tilde{R}_n = W[x_{i,j} : 1 \leq i \leq f, j \in \mathbb{Z}/p] / \left( \sum_{j \in \mathbb{Z}/p} x_{i,j} : 1 \leq i \leq f \right) \]

with \( |x_{i,j}| = -2 \)

\[ \cong \text{Symm}_W \left( \bar{\rho} \oplus f \right). \]

Even though the \( x_{i,j} \)s are not linearly independent, we define

\[ \Phi' = \prod_{1 \leq i \leq f} \prod_{0 \leq j < p} x_{i,j} \]

and complete \( \tilde{R}_n[\Phi'^{\pm 1}] \) with respect to the kernel \( \tilde{m} \) of the map

\[ \tilde{R}_n[\Phi'^{\pm 1}] \to \mathbb{F}_p^n [u^{\pm 1}] \quad \text{with } x_{i,j} \mapsto u \text{ and } \gamma u = u. \]

to obtain

\[ \hat{R}_n := \tilde{R}_n[\Phi'^{\pm 1}]^{\wedge}_{\tilde{m}}. \]
The main theorem (continued)

\[ \hat{R}_n := \hat{R}_n[\Phi^\pm 1]_{\hat{m}_n} \quad \text{and} \quad \hat{R}_n \cong \text{Symm}_W \left( \frac{\rho \oplus f}{C_p} \right). \]
The main theorem (continued)

\[ \hat{R}_n := \tilde{R}_n[\Phi^{\pm 1}]_{\mathfrak{m}} \quad \text{and} \quad \hat{R}_n \cong \text{Symm}_W \left( \overline{\rho}^{\oplus f} \right) . \]

**Theorem**

*For* \( n = (p - 1)f \), *the Lubin-Tate ring* \( E_n \) *is isomorphic to* \( \hat{R}_n \) *as an algebra over* \( W[C_p] \).
The main theorem (continued)

\[ \hat{R}_n := \tilde{R}_n[\Phi'^\pm 1]_{\mathfrak{m}_n} \quad \text{and} \quad \hat{R}_n \cong \text{Symm}_W \left( \frac{\rho \oplus f}{\cdot} \right). \]

**Theorem**

For \( n = (p - 1)f \), the Lubin-Tate ring \( E_n \) is isomorphic to \( \hat{R}_n \) as an algebra over \( W[C_p] \).

This means that \( H^*(C_p; E_n) \) is closely related to \( H^*(C_p; \text{Symm}_W \left( \frac{\rho \oplus f}{\cdot} \right)) \).
The main theorem (continued)

\[ \hat{R}_n := \hat{R}_n[\Phi'^{\pm 1}]_{\mathfrak{m}_n} \quad \text{and} \quad \hat{R}_n \cong \text{Symm}_W \left( \hat{\rho}^{\oplus f} \right). \]

**Theorem**

For \( n = (p - 1)f \), the Lubin-Tate ring \( E_n \) is isomorphic to \( \hat{R}_n \) as an algebra over \( W[C_p] \).

This means that \( H^*(C_p; E_n) \) is closely related to \( H^*(C_p; \text{Symm}_W (\hat{\rho}^{\oplus f})) \). That symmetric algebra is easy to describe modulo free summands over \( W[C_p] \),
The main theorem (continued)

\[ \tilde{R}_n := \tilde{R}_n[\Phi^\pm 1]^\wedge_{\hat{m}_n} \quad \text{and} \quad \tilde{R}_n \cong \text{Symm}_W \left( \rho^\oplus f \right). \]

**Theorem**

For \( n = (p - 1)f \), the Lubin-Tate ring \( E_n \) is isomorphic to \( \tilde{R}_n \) as an algebra over \( W[C_p] \).

This means that \( H^*(C_p; E_n) \) is closely related to \( H^*(C_p; \text{Symm}_W (\rho^\oplus f)) \). That symmetric algebra is easy to describe modulo free summands over \( W[C_p] \), which contribute nothing to cohomology in positive degrees.
The main theorem (continued)

\( \hat{R}_n := \tilde{R}_n[\Phi'^{\pm 1}]_{\tilde{m}_n} \) and \( \tilde{R}_n \cong \text{Symm}_W \left( \overline{\rho} \oplus f \right) \).

**Theorem**

For \( n = (p - 1)f \), the Lubin-Tate ring \( E_n \) is isomorphic to \( \hat{R}_n \) as an algebra over \( W[C_p] \).

This means that \( H^*(C_p; E_n) \) is closely related to \( H^*(C_p; \text{Symm}_W (\overline{\rho} \oplus f)) \). That symmetric algebra is easy to describe modulo free summands over \( W[C_p] \), which contribute nothing to cohomology in positive degrees.

We know that
The main theorem (continued)

\[ \hat{R}_n := \tilde{R}_n[\Phi'\pm 1]_{m_n} \quad \text{and} \quad \hat{R}_n \cong \text{Symm}_W (\overline{\rho}^\oplus f) . \]

**Theorem**

For \( n = (p - 1)f \), the Lubin-Tate ring \( E_n \) is isomorphic to \( \hat{R}_n \) as an algebra over \( W[C_p] \).

This means that \( H^*(C_p; E_n) \) is closely related to \( H^*(C_p; \text{Symm}_W (\overline{\rho}^\oplus f)) \). That symmetric algebra is easy to describe modulo free summands over \( W[C_p] \), which contribute nothing to cohomology in positive degrees.

We know that

\[ \text{Symm}^\ell (\overline{\rho}) \equiv \begin{cases} \mathbb{Z} & \text{for } \ell \equiv 0 \mod p \\ \overline{\rho} & \text{for } \ell \equiv 1 \mod p \\ 0 & \text{otherwise} \end{cases} \]
The main theorem (continued)

\[ \hat{R}_n := \hat{R}_n[\Phi'^{\pm 1}]_{\mathbb{m}} \quad \text{and} \quad \hat{R}_n \cong \text{Symm}_W (\overline{\rho}^{\oplus f}) . \]

**Theorem**

For \( n = (p - 1)f \), the Lubin-Tate ring \( E_n \) is isomorphic to \( \hat{R}_n \) as an algebra over \( W[C_p] \).

This means that \( H^*(C_p; E_n) \) is closely related to \( H^*(C_p; \text{Symm}_W (\overline{\rho}^{\oplus f})) \). That symmetric algebra is easy to describe modulo free summands over \( W[C_p] \), which contribute nothing to cohomology in positive degrees.

We know that

\[
\text{Symm}^\ell (\overline{\rho}) \equiv \begin{cases} 
\mathbb{Z} & \text{for } \ell \equiv 0 \text{ mod } p \\
\overline{\rho} & \text{for } \ell \equiv 1 \text{ mod } p \\
0 & \text{otherwise}
\end{cases}
\]

and that \( \overline{\rho} \otimes \overline{\rho} \equiv \mathbb{Z} \).
A classical example: $p = 2$ and $n = 1$

For $p = 2$, 

• $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 
• The group $G_1$ is the group of 2-adic units, which is isomorphic to $\{±1\} × \mathbb{Z}_2$.
• For a generator $γ \in C_2$ (namely $-1 \in \mathbb{Z} × 2$), we have $γ(u_i) = (-1)^iu_i$.
• The homotopy fixed point spectrum $E_{hC_2}$ is the 2-adic completion of the real K-theory spectrum $KO$. It follows that as $\mathbb{Z}C_2$-modules, $π_{2i}E_1 = \mathbb{Z}_2$ for $i$ even $\mathbb{Z}_2 ⊗ ρ$ for $i$ odd where $ρ$ is isomorphic to the integers with the sign action.
A classical example: $p = 2$ and $n = 1$

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$. 

For $p = 2$,
A classical example: \( p = 2 \) and \( n = 1 \)

For \( p = 2 \),

- \( E_1 \) is the 2-adic completion of complex K-theory spectrum \( K \).
- The group \( G_1 \) is the group of 2-adic units, which is isomorphic to \( \{ \pm 1 \} \times \mathbb{Z}_2 \).
A classical example: $p = 2$ and $n = 1$

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$.

- The group $\mathbb{G}_1$ is the group of 2-adic units, which is isomorphic to $\{\pm 1\} \times \mathbb{Z}_2$.

- For a generator $\gamma \in C_2$ (namely $-1 \in \mathbb{Z}_2^\times$), we have $\gamma(u^i) = (-1)^i u^i$. 

A classical example: $p = 2$ and $n = 1$

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$.
- The group $\mathbb{G}_1$ is the group of 2-adic units, which is isomorphic to $\{\pm 1\} \times \mathbb{Z}_2$.
- For a generator $\gamma \in \mathcal{C}_2$ (namely $-1 \in \mathbb{Z}_2^\times$), we have $\gamma(u^i) = (-1)^i u^i$.
- The homotopy fixed point spectrum $E_1^{h\mathbb{C}_2}$ is the 2-adic completion of the the real K-theory spectrum $KO$.
**A classical example: p = 2 and n = 1**

For \( p = 2 \),

- \( E_1 \) is the 2-adic completion of complex K-theory spectrum \( K \).
- The group \( \mathbb{G}_1 \) is the group of 2-adic units, which is isomorphic to \( \{ \pm 1 \} \times \mathbb{Z}_2 \).
- For a generator \( \gamma \in C_2 \) (namely \( -1 \in \mathbb{Z}_2^\times \)), we have \( \gamma(u^i) = (-1)^i u^i \).
- The homotopy fixed point spectrum \( E_1^{hC_2} \) is the 2-adic completion of the the real K-theory spectrum \( KO \).

It follows that as \( \mathbb{Z} C_2 \)-modules,
A classical example: $p = 2$ and $n = 1$

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$.
- The group $\mathbb{G}_1$ is the group of 2-adic units, which is isomorphic to $\{\pm 1\} \times \mathbb{Z}_2$.
- For a generator $\gamma \in C_2$ (namely $-1 \in \mathbb{Z}_2^\times$), we have $\gamma(u^i) = (-1)^i u^i$.
- The homotopy fixed point spectrum $E_1^{hC_2}$ is the 2-adic completion of the real K-theory spectrum $KO$.

It follows that as $\mathbb{Z}C_2$-modules,

\[
\pi_{2i} E_1 = \begin{cases} 
\mathbb{Z}_2 & \text{for } i \text{ even} \\
\mathbb{Z}_2 \otimes \bar{\rho} & \text{for } i \text{ odd}
\end{cases}
\]
A classical example: $p = 2$ and $n = 1$

For $p = 2$,

- $E_1$ is the 2-adic completion of complex K-theory spectrum $K$.
- The group $\mathbb{G}_1$ is the group of 2-adic units, which is isomorphic to $\{\pm 1\} \times \mathbb{Z}_2$.
- For a generator $\gamma \in C_2$ (namely $-1 \in \mathbb{Z}_2^\times$), we have $\gamma(u^i) = (-1)^i u^i$.
- The homotopy fixed point spectrum $E_1^{hC_2}$ is the 2-adic completion of the real K-theory spectrum $KO$.

It follows that as $\mathbb{Z}C_2$-modules,

$$\pi_{2i} E_1 = \begin{cases} \mathbb{Z}_2 & \text{for } i \text{ even} \\ \mathbb{Z}_2 \otimes \bar{\rho} & \text{for } i \text{ odd} \end{cases}$$

where $\bar{\rho}$ is isomorphic to the integers with the sign action.
As $\mathbb{Z}C_2$-modules,

$$\pi_{2i}E_1 = \begin{cases} 
\mathbb{Z}_2 & \text{for } i \text{ even} \\
\mathbb{Z}_2 \otimes \bar{\rho} & \text{for } i \text{ odd}
\end{cases}$$
A classical example: \( p = 2 \) and \( n = 1 \)

(continued)

As \( \mathbb{Z}C_2 \)-modules,

\[
\pi_{2i}E_1 = \begin{cases} 
\mathbb{Z}_2 & \text{for } i \text{ even} \\
\mathbb{Z}_2 \otimes \bar{\rho} & \text{for } i \text{ odd}
\end{cases}
\]

It follows that the \( E_2 \)-term of the homotopy fixed point spectral sequence is

\[
E_2^{s,t} = H^s(C_2; \pi_tE_2) = \begin{cases} 
\mathbb{Z}_2 & \text{for } s = 0 \text{ and } t \text{ divisible by 4} \\
0 & \text{for } s = 0 \text{ and } t \equiv 2 \mod 4 \\
\mathbb{Z}/2 & \text{for } s > 0, t \text{ even, and } s \equiv t \mod 2 \\
0 & \text{otherwise.}
\end{cases}
\]
The homotopy fixed point spectral sequence for $\pi_* KO$
The homotopy fixed point spectral sequence for $\pi_* KO$

Squares and bullets denote copies of $\mathbb{Z}_2$ and $\mathbb{Z}/2$. 
The homotopy fixed point spectral sequence for $\pi_* KO$

Squares and bullets denote copies of $\mathbb{Z}_2$ and $\mathbb{Z}/2$. The white diagonal lines indicate multiplication by $\eta \in E_2^{1,2}$. 
The homotopy fixed point spectral sequence for $\pi_\ast KO$

Squares and bullets denote copies of $\mathbb{Z}_2$ and $\mathbb{Z}/2$. The white diagonal lines indicate multiplication by $\eta \in E_2^{1,2}$.

The indicated $d_3$s can be established by equivariant methods,
The homotopy fixed point spectral sequence for $\pi_* KO$.

Squares and bullets denote copies of $\mathbb{Z}_2$ and $\mathbb{Z}/2$. The white diagonal lines indicate multiplication by $\eta \in E^1_2$. The indicated $d_3$s can be established by equivariant methods, or by the requirement that the spectral sequence must converge to the known value of $\pi_* KO$. 
Here is the homotopy fixed point spectral sequence for $E_{2}^{hC_{3}}$. 

Squares and bullets denote copies of $W_{9}$ and $F_{9}$. Green and blue lines indicate multiplication by $\alpha_{1} \in E_{1,2}$ and the Massey product operation $\langle \alpha_{1}, \alpha_{1}, - \rangle$. The composite is multiplication by $\beta_{1} \in E_{2,12}$. 

**TMF at $p = 3$**

Doug Ravenel

- Historical introduction
- $K(n)$ localization
- Properties of $E_{n}$ and $G_{n}$
- Finding a root of unity
- Group cohomology
- The main theorem
- A classical example
- **TMF at $p = 3$**
- Larger primes
Here is the homotopy fixed point spectral sequence for $E_2^{hc_3}$ with copies of $WC_3$ in $\pi_* E_2$ omitted.
Here is the homotopy fixed point spectral sequence for $E_{2}^{hC_{3}}$ with copies of $WC_{3}$ in $\pi_{*}E_{2}$ omitted.
Here is the homotopy fixed point spectral sequence for $E_{2}^{hC_{3}}$ with copies of $WC_{3}$ in $\pi_{*}E_{2}$ omitted.

Squares and bullets denote copies of $W(F_{9})$ and $F_{9}$.
Hiking in the Alps: $C_p$-fixed points of Lubin-Tate spectra

Doug Ravenel

Historical introduction
$K(n)$ localization
Properties of $E_n$ and $\mathbb{G}_n$
Finding a root of unity
Group cohomology
The main theorem
A classical example
TMF at $p = 3$
Larger primes

$TMF$ at $p = 3$

Here is the homotopy fixed point spectral sequence for $E^{hC_3}_2$ with copies of $WC_3$ in $\pi_* E_2$ omitted.

Squares and bullets denote copies of $W(\mathbb{F}_9)$ and $\mathbb{F}_9$. Green and blue lines indicate multiplication by $\alpha_1 \in E_2^{1,4}$.
Here is the homotopy fixed point spectral sequence for $E_{2}^{hC_{3}}$ with copies of $WC_{3}$ in $\pi_{\ast}E_{2}$ omitted.

Squares and bullets denote copies of $W(\mathbb{F}_{9})$ and $\mathbb{F}_{9}$. Green and blue lines indicate multiplication by $\alpha_{1} \in E_{2}^{1,4}$ and the Massey product operation $\langle \alpha_{1}, \alpha_{1}, - \rangle$. 
Here is the homotopy fixed point spectral sequence for $E_{2}^{hC_{3}}$ with copies of $WC_{3}$ in $\pi_{*}E_{2}$ omitted.

Squares and bullets denote copies of $W(F_{9})$ and $F_{9}$. Green and blue lines indicate multiplication by $\alpha_{1} \in E_{2}^{1,4}$ and the Massey product operation $\langle \alpha_{1}, \alpha_{1}, - \rangle$. The composite is multiplication by $\beta_{1} \in E_{2}^{2,12}$. 

\[ \begin{array}{cccccccc}
-18 & -12 & -6 & 0 & 6 & 12 & 18 \\
\end{array} \]
This pattern of differentials is 18-periodic. A comparable homotopy fixed point spectral sequence for $\text{TMF}$ is 72-periodic.

The picture above can be "spread out" by enlarging the group $\mathbb{C}_3$ by adjoining the fourth roots of unity in $\mathbb{W}$. Extending by the Galois group converts each copy of $\mathbb{W}$ and $\mathbb{F}_9$ to $\mathbb{Z}_3$ and $\mathbb{F}_3$.

Thus we are extending $\mathbb{C}_3$ by $D_8$, the group dihedral group of order 8 to get a group $G_{24}$.
This pattern of differentials is 18-periodic.
This pattern of differentials is $18$-periodic. A comparable homotopy fixed point spectral sequence for $TMF$ is $72$-periodic.
This pattern of differentials is 18-periodic. A comparable homotopy fixed point spectral sequence for \( TMF \) is 72-periodic. The picture above can be “spread out” by enlarging the group \( C_3 \) by adjoining the fourth roots of unity in \( W \).
This pattern of differentials is 18-periodic. A comparable homotopy fixed point spectral sequence for $TMF$ is 72-periodic. The picture above can be “spread out” by enlarging the group $C_3$ by adjoining the fourth roots of unity in $W$. Extending by the Galois group converts each copy of $W$ and $\mathbb{F}_9$ to $\mathbb{Z}_3$ and $\mathbb{F}_3$. 
This pattern of differentials is 18-periodic. A comparable homotopy fixed point spectral sequence for $TMF$ is 72-periodic. The picture above can be “spread out” by enlarging the group $C_3$ by adjoining the fourth roots of unity in $W$. Extending by the Galois group converts each copy of $W$ and $\mathbb{F}_9$ to $\mathbb{Z}_3$ and $\mathbb{F}_3$. Thus we are extending $C_3$ by $D_8$, the group dihedral group of order 8 to get a group $G_{24}$.
Some group theory

In terms of the algebra $\text{End}(F_2)$ at $p = 3$, let $\omega \in W$ be a primitive 8th root of unity, and $i = \omega^2$. Then we have a cube root of unity $\zeta = -1 - \omega F_2$ with $i \zeta^i - 1 = \zeta - 1 = -1 + \omega F_2$.

Let $\phi \in \text{Gal}(F_9 : F_3)$ be the Frobenius element. Then $\omega \phi$ commutes with $\zeta$ and has order 4. The group $\langle i, \omega \phi \rangle$ is isomorphic to $Q_8$, and the group $C_3 \rtimes Q_8$ is the group of Goerss-Henn-Mahowald-Rezk.
In terms of the algebra $\text{End}(F_2)$ at $p = 3$, let $\omega \in W$ be a primitive 8th root of unity, and $i = \omega^2$. 

Let $\phi \in \text{Gal}(F_9:F_3)$ be the Frobenius element. Then $\omega \phi$ commutes with $\zeta$ and has order 4. The group $\langle i, \omega \phi \rangle$ is isomorphic to $\mathbb{Q}_8$, and the group $C_3 \rtimes \mathbb{Q}_8$ is the group $G_{24}$ of Goerss-Henn-Mahowald-Rezk.
Some group theory

In terms of the algebra $\text{End}(F_2)$ at $p = 3$, let $\omega \in W$ be a primitive 8th root of unity, and $i = \omega^2$. Then we have a cube root of unity

$$\zeta = \frac{-1 - \omega F}{2} \quad \text{with} \quad i\zeta^{-1} = \zeta^{-1} = \frac{-1 + \omega F}{2}.$$
Some group theory

In terms of the algebra $\text{End}(F_2)$ at $p = 3$, let $\omega \in W$ be a primitive 8th root of unity, and $i = \omega^2$. Then we have a cube root of unity

$$\zeta = \frac{-1 - \omega F}{2} \quad \text{with} \quad i \zeta^{-1} = \zeta^{-1} = \frac{-1 + \omega F}{2}.$$

Let $\phi \in \text{Gal}(F_9 : F_3)$ be the Frobenius element.
Some group theory

In terms of the algebra $\text{End}(F_2)$ at $p = 3$, let $\omega \in W$ be a primitive 8th root of unity, and $i = \omega^2$. Then we have a cube root of unity

$$\zeta = \frac{-1 - \omega F}{2} \quad \text{with} \quad i\zeta i^{-1} = \zeta^{-1} = \frac{-1 + \omega F}{2}.$$ 

Let $\phi \in \text{Gal}(F_9 : F_3)$ be the Frobenius element. Then $\omega\phi$ commutes with $\zeta$ and has order 4.
In terms of the algebra $\text{End}(F_2)$ at $p = 3$, let $\omega \in W$ be a primitive 8th root of unity, and $i = \omega^2$. Then we have a cube root of unity

$$\zeta = \frac{-1 - \omega F}{2} \quad \text{with} \quad i\zeta i^{-1} = \zeta^{-1} = \frac{-1 + \omega F}{2}.$$ 

Let $\phi \in \text{Gal}(F_9 : F_3)$ be the Frobenius element. Then $\omega \phi$ commutes with $\zeta$ and has order 4. The group $\langle i, \omega \phi \rangle$ is isomorphic to $Q_8$. 

\[\text{Some group theory}\]
In terms of the algebra $\text{End}(F_2)$ at $p = 3$, let $\omega \in \mathbb{W}$ be a primitive 8th root of unity, and $i = \omega^2$. Then we have a cube root of unity

$$\zeta = \frac{-1 - \omega F}{2} \quad \text{with} \quad i\zeta i^{-1} = \zeta^{-1} = \frac{-1 + \omega F}{2}.$$ 

Let $\phi \in \text{Gal}(F_9 : F_3)$ be the Frobenius element. Then $\omega \phi$ commutes with $\zeta$ and has order 4. The group $\langle i, \omega \phi \rangle$ is isomorphic to $Q_8$, and the group $C_3 \rtimes Q_8$ is the group $G_{24}$ of Goerss-Henn-Mahowald-Rezk.
This is the homotopy fixed point spectral sequence for $E^h_{24} G_{24}$, which is $TMF_{K(2)}$, also known as $EO_3$. 

It is known that the following elements in the Adams-Novikov $E^2$-term have nontrivial images here:

- $\beta_1$,
- $\beta_3 / 3$,
- $\beta_4$,
- $\beta_6 / 3$,
- $\beta_9$,
- $\beta_9 / 3$,
- $\beta_7$,
- $\beta_9 / 3$,
- $\beta_{10}$,
- $|x|_{10}$.
TMF at $p = 3$ (continued)

This is the homotopy fixed point spectral sequence for $E_2^{hG_{24}}$, which is $TMF_{K(2)}$, also known as $EO_3$. 

It is known that the following elements in the Adams-Novikov $E_2$-term have nontrivial images here.

$\beta_1 \beta_3 / 3 \beta_4 \beta_6 / 3 \beta_9$, $9$, $\beta_7 \beta_9 / 3$, $2 \beta_{10}$
This is the homotopy fixed point spectral sequence for $E_2^{hG_{24}}$, which is $TMF_{K(2)}$, also known as $EO_3$.

It is known that the following elements in the Adams-Novikov $E_2$-term have nontrivial images here.
This is the homotopy fixed point spectral sequence for $E_{2}^{hG_{24}}$, which is $TMF_{K(2)}$, also known as $EO_{3}$.

It is known that the following elements in the Adams-Novikov $E_{2}$-term have nontrivial images here.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\beta_{1}$</th>
<th>$\beta_{3/3}$</th>
<th>$\beta_{4}$</th>
<th>$\beta_{6/3}$</th>
<th>$\beta_{9,9}, \beta_{7}$</th>
<th>$\beta_{9/3,2}$</th>
<th>$\beta_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>x</td>
<td>$</td>
<td>10</td>
<td>34</td>
<td>58</td>
<td>82</td>
<td>106</td>
</tr>
</tbody>
</table>
Larger primes

For $p \geq 3$ one has an extension $H$ of $C_p$ by $C_{(p-1)^2}$,
Larger primes

For $p \geq 3$ one has an extension $H$ of $C_p$ by $C_{(p-1)^2}$, where a generator of the quotient acts on $C_p$ by an automorphism of order $p - 1$. 
Larger primes

For $p \geq 3$ one has an extension $H$ of $C_p$ by $C_{(p-1)^2}$, where a generator of the quotient acts on $C_p$ by an automorphism of order $p - 1$. This subgroup of $\mathbb{S}_{p-1}$ can be extended by the Galois group $C_{p-1}$ to give a maximal finite subgroup $G \subseteq \mathbb{G}_{p-1}$ of order $p(p - 1)^3$. 

We define $EO_p := E_{hG_{p-1}}$. In the $E_2$-term of the resulting homotopy fixed point spectral sequence we have $\alpha_1 \in E_{1,2}^{p-2}, \beta_1 \in E_{2,2}^{p-2}, \Delta \in E_{0,2}^{p(p-1)^2}$, with $E_2 = E_2(\alpha_1) \otimes P(\beta_1) \otimes P(\Delta \pm 1)$. Here are the dimensions of these elements for small primes.

| $p$ | $|\alpha_1|$ | $|\beta_1|$ | $|\Delta|$ |
|-----|-------------|-------------|-------------|
| 3   | 3           | 3           | 10          |
| 5   | 5           | 7           | 38          |
| 7   | 7           | 11          | 82          |
| 11  | 11          | 13          | 32          |
| 17  | 17          | 19          | 126         |

$TMF$ at $p = 3$

Larger primes
Larger primes

For $p \geq 3$ one has an extension $H$ of $C_p$ by $C_{(p-1)^2}$, where a generator of the quotient acts on $C_p$ by an automorphism of order $p - 1$. This subgroup of $S_{p-1}$ can be extended by the Galois group $C_{p-1}$ to give a maximal finite subgroup $G \subseteq \mathbb{G}_{p-1}$ of order $p(p - 1)^3$. We define $EO_p := E_{hG}^{p-1}$. 

Here are the dimensions of these elements for small primes.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\alpha_1$</th>
<th>$\beta_1$</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>10</td>
<td>24</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>38</td>
<td>160</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>82</td>
<td>504</td>
</tr>
</tbody>
</table>
Larger primes

For \( p \geq 3 \) one has an extension \( H \) of \( C_p \) by \( C_{(p-1)^2} \), where a generator of the quotient acts on \( C_p \) by an automorphism of order \( p - 1 \). This subgroup of \( S_{p-1} \) can be extended by the Galois group \( C_{p-1} \) to give a maximal finite subgroup \( G \subseteq \mathbb{G}_{p-1} \) of order \( p(p - 1)^3 \). We define \( EO_p := E_{hG}^{p-1} \).

In the \( E_2 \)-term of the resulting homotopy fixed point spectral sequence we have
Larger primes

For $p \geq 3$ one has an extension $H$ of $C_p$ by $C_{(p-1)^2}$, where a generator of the quotient acts on $C_p$ by an automorphism of order $p-1$. This subgroup of $\mathbb{S}_{p-1}$ can be extended by the Galois group $C_{p-1}$ to give a maximal finite subgroup $G \subseteq \mathbb{G}_{p-1}$ of order $p(p-1)^3$. We define $EO_p := E_{p-1}^{hG}$.

In the $E_2$-term of the resulting homotopy fixed point spectral sequence we have

$$\alpha_1 \in E_2^{1,2p-2}, \quad \beta_1 \in E_2^{2,2p^2-2p}, \quad \text{and} \quad \Delta \in E_2^{0,2p(p-1)^2},$$

with $E_2 = E_{p-1}^{hG}$.
Larger primes

For $p \geq 3$ one has an extension $H$ of $C_p$ by $C_{(p-1)^2}$, where a generator of the quotient acts on $C_p$ by an automorphism of order $p - 1$. This subgroup of $S_{p-1}$ can be extended by the Galois group $C_{p-1}$ to give a maximal finite subgroup $G \subseteq \mathbb{G}_{p-1}$ of order $p(p-1)^3$. We define $EO_p := E^{hG}_{p-1}$.

In the $E_2$-term of the resulting homotopy fixed point spectral sequence we have

$$\alpha_1 \in E_2^{1,2p-2}, \quad \beta_1 \in E_2^{2,2p^2-2p}, \quad \text{and} \quad \Delta \in E_2^{0,2p(p-1)^2},$$

with

$$E_2 = E(\alpha_1) \otimes P(\beta_1) \otimes P(\Delta^{\pm 1}).$$
For $p \geq 3$ one has an extension $H$ of $C_p$ by $C_{(p-1)^2}$, where a generator of the quotient acts on $C_p$ by an automorphism of order $p - 1$. This subgroup of $\mathbb{S}_{p-1}$ can be extended by the Galois group $C_{p-1}$ to give a maximal finite subgroup $G \subseteq \mathbb{G}_{p-1}$ of order $p(p - 1)^3$. We define $EO_p := E_{hG}^{p-1}$.

In the $E_2$-term of the resulting homotopy fixed point spectral sequence we have

$$\alpha_1 \in E_2^{1,2p-2}, \quad \beta_1 \in E_2^{2,2p^2-2p}, \quad \text{and} \quad \Delta \in E_2^{0,2p(p-1)^2},$$

with

$$E_2 = E(\alpha_1) \otimes P(\beta_1) \otimes P(\Delta^{\pm 1}).$$

Here are the dimensions of these elements for small primes.

| $p$ | $|\alpha_1|$ | $|\beta_1|$ | $|\Delta|$ |
|-----|--------------|--------------|-------------|
| 3   | 3            | 10           | 24          |
| 5   | 7            | 38           | 160         |
| 7   | 11           | 82           | 504         |
In the homotopy fixed point spectral sequence for $EO_p$ we have

$$E_2 = E(\alpha_1) \otimes P(\beta_1) \otimes P(\Delta^{\pm 1}).$$

with

$$\alpha_1 \in E_2^{1,2p-2}, \quad \beta_1 \in E_2^{2,2p^2-2p}, \quad \text{and} \quad \Delta \in E_2^{0,2p(p-1)^2}.$$

Then there are differentials

$$d_{2p-1} \Delta = \alpha_1 \beta_1^{p-1} \quad \text{and} \quad d_{2(p-1)^2+1}(\alpha_1 \Delta^{p-1}) = \beta_1^{(p-1)^2+1}.$$
Larger primes (continued)

In the homotopy fixed point spectral sequence for $EO_p$ we have

$$E_2 = \{\alpha_1\} \otimes P(\beta_1) \otimes P(\Delta^{\pm 1}).$$

with

$$\alpha_1 \in E_2^{1,2p-2}, \quad \beta_1 \in E_2^{2,2p^2-2p}, \quad \text{and} \quad \Delta \in E_2^{0,2p(p-1)^2}.$$

Then there are differentials

$$d_{2p-1}\Delta = \alpha_1\beta_1^{p-1} \quad \text{and} \quad d_{2(p-1)^2+1}(\alpha_1\Delta^{p-1}) = \beta_1^{(p-1)^2+1}.$$

From the Adams-Novikov $E_2$-term for the sphere spectrum we have

$$\theta_j := \beta_{p^j-1}/p^j \mapsto \beta_1\Delta^{(p^j-1)/(p-1)} \quad \text{for all } j \geq 1,$$
Larger primes (continued)

In the homotopy fixed point spectral sequence for $EO_p$ we have

$$E_2 = E(\alpha_1) \otimes P(\beta_1) \otimes P(\Delta^{\pm 1}).$$

with

$$\alpha_1 \in E_2^{1,2p-2}, \quad \beta_1 \in E_2^{2,2p^2-2p}, \quad \text{and} \quad \Delta \in E_2^{0,2p(p-1)^2}.$$

Then there are differentials

$$d_{2p-1}\Delta = \alpha_1 \beta_1^{p-1} \quad \text{and} \quad d_{2(p-1)^2+1}(\alpha_1 \Delta^{p-1}) = \beta_1^{(p-1)^2+1}.$$

From the Adams-Novikov $E_2$-term for the sphere spectrum we have

$$\theta_j := \beta_{p^{j-1}/p-1} \mapsto \beta_1 \Delta^{(p^{j-1}-1)/(p-1)} \quad \text{for all } j \geq 1,$$

and for $p = 5$ only, we have

$$\gamma_3 \mapsto \alpha_1 \beta_1 \Delta^4 \quad \text{in dimension 685}.$$
THANK YOU

and have a wonderful retirement, Paul!